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## On a mixed problem for Hilfer type differential equation of higher order

The study considers the solvability of a mixed problem for a Hilfer type partial differential equation of the even order with initial value conditions and small positive parameters in mixed derivatives in three-dimensional domain. It studies the solution to this fractional differential equation of higher order in the class of regular functions. The case, when the order of fractional operator is  $1 < \alpha < 2$ , is examined. During this study the authors use the Fourier series method and obtain a countable system of ordinary differential equations. The initial value problem is integrated as an ordinary differential equation and the integrated constants find by the aid of given initial value conditions. Using the Cauchy–Schwarz inequality and the Bessel inequality, it is proved the absolute and uniform convergence of the obtained Fourier series. The stability of the solution to the mixed problem on the given functions is studied.

*Keywords:* fractional order, Hilfer operator, mixed problem, Fourier series, initial value conditions, unique solvability.

### Introduction

The theory of the mixed problems is one of the most important directions of the modern theory of differential equations. A large number of works are devoted to the study of the mixed problems for differential and integro-differential equations (see, for example, [1–12]). Many problems of gas dynamics, theory of elasticity, plates, and shells are described by higher-order partial differential equations.

Fractional calculus plays an important role for the mathematical modeling in many applied problems. In [13], it is considered problems of continuum and statistical mechanics. The work [14] studies the mathematical problems of the Ebola epidemic model. The studies [15] and [16] investigate the fractional model for the dynamics of tuberculosis infection and novel coronavirus (nCoV-2019), respectively. The construction of various models of theoretical physics by the aid of fractional calculus is described in [17, Vol. 4, 5], [18], [19]. Some applications of fractional calculus in solving applied problems are given in [17, Vol. 6–8], [20]. In [21], the solvability of an initial value problem for Hilfer type fractional differential equation with nonlinear maxima is studied. In [22], by analytical method, the unique solvability of boundary value problem for weak nonlinear partial differential equations of mixed type with fractional Hilfer operator is studied. In [23], the solvability of nonlocal problem for a mixed type fourth-order differential equation with Hilfer fractional operator is examined. In [24], it is considered an inverse problem for a mixed type integro-differential equation with fractional order Gerasimov-Caputo operators. The research works [25–34] obtained the results on the direction of applications of fractional derivatives to solve partial differential equations.

Let  $(t_0; T) \subset \mathbb{R}^+ \equiv [0; \infty)$  be an interval on the set of positive real numbers, where  $0 \leq t_0 < T < \infty$ . The Riemann–Liouville  $0 < \alpha$ -order fractional integral of a function  $\eta(t)$  is defined as follows:

$$I_{t_0+}^{\alpha} \eta(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^t (t-s)^{\alpha-1} \eta(s) ds, \quad \alpha > 0, \quad t \in (t_0; T),$$

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where  $\Gamma(\alpha)$  is the Gamma function.

Let  $n - 1 < \alpha \leq n$ ,  $n \in \mathbb{N}$ . The Riemann–Liouville  $\alpha$ -order fractional derivative of a function  $\eta(t)$  is defined as follows:

$$D_{t_0+}^{\alpha} \eta(t) = \frac{d^n}{dt^n} I_{t_0+}^{n-\alpha} \eta(t), \quad t \in (t_0; T).$$

The Hilfer fractional derivatives of  $\alpha$ -order ( $n - 1 < \alpha \leq n$ ,  $n \in \mathbb{N}$ ) and  $\beta$ -type ( $0 \leq \beta \leq 1$ ), are defined by the following composition of three operators:

$$D_{t_0+}^{\alpha, \beta} \eta(t) = I_{t_0+}^{\beta(n-\alpha)} \frac{d^n}{dt^n} I_{t_0+}^{(1-\beta)(n-\alpha)} \eta(t), \quad t \in (t_0; T).$$

Let  $\gamma = \alpha + \beta n - \alpha \beta$ . It is easy to see that  $\alpha \leq \gamma \leq n$ . Then it is convenient to use another designation for the operator  $D^{\alpha, \gamma} \eta(t) = D_{t_0+}^{\alpha, \beta} \eta(t)$ . Hilfer operator is generalization of the Riemann–Liouville operator and was introduced by R. Hilfer based on fractional time evolutions that arise during the transition from the microscopic scale to the macroscopic time scale (see [17]).

In this paper, for the case  $1 < \alpha < 2$  we study the regular solvability of mixed value problem for a Hilfer type partial differential equation of higher even order with positive small parameters. The stability of the solution from the given functions is proved.

In three-dimensional domain  $\Omega = \{(t, x, y) \mid t_0 < t < T, 0 < x, y < l\}$  a higher order partial differential equation of the following form is considered

$$D_{\varepsilon_1, \varepsilon_2}^{\alpha, \gamma} [U] = a(t) b(x, y) \tag{1}$$

with initial value conditions

$$\lim_{t \rightarrow +t_0} J_{t_0+}^{2-\gamma} U(t, x, y) = \varphi_1(x, y), \quad \lim_{t \rightarrow +t_0} \frac{d}{dt} J_{t_0+}^{2-\gamma} U(t, x, y) = \varphi_2(x, y), \tag{2}$$

where  $T$  and  $l$  are given positive real numbers,  $0 \leq t_0 < T$ ,

$$D_{\varepsilon_1, \varepsilon_2}^{\alpha, \gamma} [U] = \left[ D^{\alpha, \gamma} - D^{\alpha, \gamma} \left( \varepsilon_1 \left( \frac{\partial^{2k}}{\partial x^{2k}} + \frac{\partial^{2k}}{\partial y^{2k}} \right) - \varepsilon_2 \left( \frac{\partial^{4k}}{\partial x^{4k}} + \frac{\partial^{4k}}{\partial y^{4k}} \right) \right) - \omega \left( \left( \frac{\partial^{2k}}{\partial x^{2k}} + \frac{\partial^{2k}}{\partial y^{2k}} \right) - \left( \frac{\partial^{4k}}{\partial x^{4k}} + \frac{\partial^{4k}}{\partial y^{4k}} \right) \right) \right] U(t, x, y),$$

$\omega$  is positive parameter,  $\varepsilon_1$  and  $\varepsilon_2$  are positive small parameters,  $1 < \alpha < \gamma \leq 2$ ,  $k$  is given positive integer,  $a(t) \in C(\Omega_T)$ ,  $\Omega_T \equiv [t_0; T]$ ,  $\Omega_l \equiv [0; l]$ ,  $b(x, y) \in C(\Omega_l^2)$  is known function,  $\varphi_1(x, y)$  and  $\varphi_2(x, y)$  are given continuous functions,  $\Omega_l^2 \equiv \Omega_l \times \Omega_l$ . We assume that for given functions the following boundary conditions hold

$$\varphi_i(0, y) = \varphi_i(l, y) = \varphi_i(x, 0) = \varphi_i(x, l) = 0, \quad i = 1, 2,$$

$$b(0, y) = b(l, y) = b(x, 0) = b(x, l) = 0.$$

*Problem Statement.* We find the function  $U(t, x, y)$ , which satisfies differential equation (1), initial value conditions (2), zero boundary value conditions for  $t_0 \leq t \leq T$

$$\begin{aligned} U(t, 0, y) &= U(t, l, y) = U(t, x, 0) = U(t, x, l) = \\ &= \frac{\partial^2}{\partial x^2} U(t, 0, y) = \frac{\partial^2}{\partial x^2} U(t, l, y) = \frac{\partial^2}{\partial x^2} U(t, x, 0) = \frac{\partial^2}{\partial x^2} U(t, x, l) = \\ &= \frac{\partial^2}{\partial y^2} U(t, 0, y) = \frac{\partial^2}{\partial y^2} U(t, l, y) = \frac{\partial^2}{\partial y^2} U(t, x, 0) = \frac{\partial^2}{\partial y^2} U(t, x, l) = \dots = \end{aligned}$$

$$\begin{aligned}
 &= \frac{\partial^{4k-2}}{\partial x^{4k-2}} U(t, 0, y) = \frac{\partial^{4k-2}}{\partial x^{4k-2}} U(t, l, y) = \frac{\partial^{4k-2}}{\partial x^{4k-2}} U(t, x, 0) = \frac{\partial^{4k-2}}{\partial x^{4k-2}} U(t, x, l) = \\
 &= \frac{\partial^{4k-2}}{\partial y^{4k-2}} U(t, 0, y) = \frac{\partial^{4k-2}}{\partial y^{4k-2}} U(t, l, y) = \frac{\partial^{4k-2}}{\partial y^{4k-2}} U(t, x, 0) = \frac{\partial^{4k-2}}{\partial y^{4k-2}} U(t, x, l) = 0, \quad (3)
 \end{aligned}$$

class of functions

$$\left[ \begin{aligned} &(t - t_0)^{2-\gamma} U(t, x, y) \in C(\bar{\Omega}), \\ &D^{\alpha, \gamma} U(t, x, y) \in C_{x, y}^{4k, 4k}(\Omega) \cap C_{x, y}^{4k+0}(\Omega) \cap C_{x, y}^{0+4k}(\Omega), \end{aligned} \right. \quad (4)$$

where  $C_{x, y}^{4k+0}(\Omega)$  is the class of continuous functions  $\frac{\partial^{4k} U(t, x, y)}{\partial x^{4k}}$  on  $\Omega$ , while  $C_{x, y}^{0+4k}(\Omega)$  is the class of continuous functions  $\frac{\partial^{4k} U(t, x, y)}{\partial y^{4k}}$  on  $\Omega$ ,  $\frac{\partial^{4k-2}}{\partial y^{4k-2}} U(t, x, l)$  we understand as  $\frac{\partial^{4k-2}}{\partial y^{4k-2}} U(t, x, y) \Big|_{y=l}$ ,  $\bar{\Omega} = \{(t, x, y) \mid t_0 \leq t \leq T, 0 \leq x, y \leq l\}$ .

### 1 Transform of fractional differential equation

*Lemma.* The solution to the ordinary fractional differential equation

$$D^{\alpha, \gamma} v(t) + \omega v(t) = f(t, v(t)) \quad (5)$$

with initial value condition

$$\lim_{t \rightarrow t_0} J_{t_0+}^{2-\gamma} v(t) = v_0, \quad \lim_{t \rightarrow t_0} \frac{d}{dt} J_{t_0+}^{2-\gamma} v(t) = v_1, \quad (6)$$

is represented as follows

$$\begin{aligned}
 v(t) &= v_0 (t - t_0)^{\gamma-2} E_{\alpha, \gamma-1}(\omega (t - t_0)^\alpha) + v_1 (t - t_0)^{\gamma-1} E_{\alpha, \gamma}(\omega (t - t_0)^\alpha) + \\
 &+ \int_{t_0}^t (t - s)^{\alpha-1} E_{\alpha, \alpha}(-\omega (t - s)^\alpha) f(s, v(s)) ds, \quad (7)
 \end{aligned}$$

where  $E_{\alpha, \gamma}(z)$  is the Mittag-Leffler function and has the form [17, vol. 1, 269–295]

$$E_{\alpha, \gamma}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \gamma)}, \quad z, \alpha, \gamma \in (0; \infty),$$

$f(t, v) \in C(\Omega_1)$ ,  $0 < \omega$  is real parameter,  $v_0, v_1 = \text{const}$ ,  $\Omega_1 \equiv [t_0; T] \times X$ ,  $0 \leq t_0$ ,  $X \subset \mathbb{R} \equiv (-\infty; \infty)$ ,  $X$  is closed set,

$$D^{\alpha, \gamma} = J_{t_0+}^{\gamma-\alpha} \frac{d^2}{dt^2} J_{t_0+}^{2-\gamma}, \quad 1 < \alpha < \gamma \leq 2, \quad \gamma = \alpha + 2\beta - \alpha\beta.$$

*Proof.* We rewrite the differential equation (5) in the form

$$J_{t_0+}^{\gamma-\alpha} D_{t_0+}^{\gamma} v(t) = -\omega v(t) + f(t, v).$$

Applying the operator  $J_{t_0+}^{\alpha}$  to both sides of this equation (5), taking into account the linearity of this operator and the formula [35]

$$J_{t_0+}^{\delta} D_{t_0+}^{\delta} v(t) = v(t) - \sum_{k=0}^{n-1} \frac{(t - t_0)^{\delta+k-n}}{\Gamma(\delta + k + 1 - n)} \lim_{t \rightarrow t_0+} \frac{d^k}{dt^k} J_{t_0+}^{n-\delta} v(t), \quad \delta \in (n - 1; n],$$

we obtain

$$v(t) = -\omega J_{t_0+}^\alpha v(t) \frac{v_0}{\Gamma(\gamma-1)} (t-t_0)^{\gamma-2} + \frac{v_1}{\Gamma(\gamma)} (t-t_0)^{\gamma-1} + J_{t_0+}^\alpha f(t, v(t)). \quad (8)$$

Using the lemma from [26], we represent the solution to equation (8) in the form

$$v(t) = \frac{v_0}{\Gamma(\gamma-1)} (t-t_0)^{\gamma-2} + \frac{v_1}{\Gamma(\gamma)} (t-t_0)^{\gamma-1} + J_{t_0+}^\alpha f(t, v(t)) - \omega \int_{t_0}^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\omega(t-s)^\alpha) \times \left[ \frac{v_0}{\Gamma(\gamma-1)} (s-t_0)^{\gamma-2} + \frac{v_1}{\Gamma(\gamma)} (s-t_0)^{\gamma-1} + J_{t_0+}^\alpha f(s, v(s)) \right] ds. \quad (9)$$

We rewrite the presentation (9) as the sum of two expressions:

$$I_1(t) = \frac{v_0}{\Gamma(\gamma-1)} \left[ (t-t_0)^{\gamma-2} - \omega \int_{t_0}^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\omega(s-t_0)^\alpha) (s-t_0)^{\gamma-2} ds \right] + \frac{v_1}{\Gamma(\gamma)} \left[ (t-t_0)^{\gamma-1} - \omega \int_{t_0}^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\omega(t-s)^\alpha) (s-t_0)^{\gamma-1} ds \right], \quad (10)$$

$$I_2(t) = J_{t_0+}^\alpha f(t, v(t)) - \omega \int_{t_0}^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\omega \cdot (t-s)^\alpha) J_{t_0+}^\alpha f(s, v(s)) ds. \quad (11)$$

We apply the following presentations [17, vol. 1, 269–295]

$$E_{\alpha,\mu}(z) = \frac{1}{\Gamma(\mu)} + z \cdot E_{\alpha,\mu+\alpha}(t), \quad \alpha > 0, \quad \mu > 0, \quad (12)$$

$$\frac{1}{\Gamma(\nu)} \int_0^z (z-t)^{\nu-1} E_{\alpha,\beta}(\lambda t^\alpha) t^{\beta-1} dt = z^{\beta+\nu-1} \cdot E_{\alpha,\beta+\nu}(\lambda z^\alpha), \quad \nu > 0, \quad \beta > 0. \quad (13)$$

Then for the integral (10) we obtain the presentation

$$I_1(t) = v_0 (t-t_0)^{\gamma-2} E_{\alpha,\gamma-1}(-\omega(t-t_0)^\alpha) + v_1 (t-t_0)^{\gamma-1} E_{\alpha,\gamma}(-\omega(t-t_0)^\alpha). \quad (14)$$

The integral in (11) is easily transformed to the form

$$\begin{aligned} & \int_{t_0}^t (t-\xi)^{\alpha-1} E_{\alpha,\alpha}(-\omega \cdot (t-\xi)^\alpha) J_{t_0+}^\alpha f(\xi, v(\xi)) d\xi = \\ & = \frac{1}{\Gamma(\alpha)} \int_{t_0}^t (t-\xi)^{\alpha-1} E_{\alpha,\alpha}(-\omega \cdot (t-\xi)^\alpha) d\xi \int_{t_0}^\xi (\xi-s)^{\alpha-1} f(s, v(s)) ds = \end{aligned}$$

$$= \frac{1}{\Gamma(\alpha)} \int_{t_0}^t f(s, v(s)) ds \int_s^t (t-\xi)^{\alpha-1} (\xi-s)^{\alpha-1} E_{\alpha,\alpha}(-\omega \cdot (t-\xi)^\alpha) d\xi. \tag{15}$$

Taking into account the (13) the second integral in the last equality of (15) can be written as

$$\int_s^t (t-\xi)^{\alpha-1} (\xi-s)^{\alpha-1} E_{\alpha,\alpha}(-\omega \cdot (t-\xi)^\alpha) d\xi = \Gamma(\alpha) (t-\xi)^{2\alpha-1} E_{\alpha,2\alpha}(-\omega \cdot (t-\xi)^\alpha).$$

Then, taking into account (12), we represent (11) in the following form

$$I_2(t) = \int_{t_0}^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\omega \cdot (t-s)^\alpha) f(s, v(s)) ds. \tag{16}$$

Substituting (14) and (16) into the sum  $v(t) = I_1(t) + I_2(t)$ , we obtain (7). The Lemma is proved.

*2 Expansion of the solution into Fourier series*

Nontrivial solutions to the problem are sought as a Fourier series

$$U(t, x, y) = \sum_{n,m=1}^{\infty} u_{n,m}(t) \vartheta_{n,m}(x, y), \tag{17}$$

where

$$u_{n,m}(t) = \int_0^l \int_0^l U(t, x, y) \vartheta_{n,m}(x, y) dx dy, \tag{18}$$

$$\vartheta_{n,m}(x, y) = \frac{2}{l} \sin \frac{\pi n}{l} x \sin \frac{\pi m}{l} y, \quad n, m = 1, 2, \dots$$

We also suppose that the following function is expanded to Fourier series

$$b(x, y) = \sum_{n,m=1}^{\infty} b_{n,m} \vartheta_{n,m}(x, y), \tag{19}$$

where

$$b_{n,m} = \int_0^l \int_0^l b(x, y) \vartheta_{n,m}(x, y) dx dy. \tag{20}$$

Substituting Fourier series (17) and (19) into partial differential equation (1), we obtain the countable system of ordinary fractional differential equations of order:  $1 < \alpha, \gamma < 2$

$$D^{\alpha,\gamma} u_{n,m}(t) + \lambda_{n,m}^{2k}(\varepsilon_1, \varepsilon_2) \omega u_{n,m}(t) = \frac{a(t) b_{n,m}}{1 + \mu_{n,m}^{2k}(\varepsilon_1 + \varepsilon_2 \mu_{n,m}^{2k})}, \tag{21}$$

where

$$\lambda_{n,m}^{2k}(\varepsilon_1, \varepsilon_2) = \frac{\mu_{n,m}^{2k}(1 + \mu_{n,m}^{2k})}{1 + \mu_{n,m}^{2k}(\varepsilon_1 + \varepsilon_2 \mu_{n,m}^{2k})}, \quad \mu_{n,m}^k = \left(\frac{\pi}{l}\right)^k \sqrt{n^{2k} + m^{2k}}.$$

According to the Lemma, the general solution to countable system of differential equations (21) has the form

$$u_{n,m}(t) = C_{1n,m} (t - t_0)^{\gamma-2} E_{\alpha,\gamma-1} \left( -\lambda_{n,m}^{2k} (\varepsilon_1, \varepsilon_2) \omega (t - t_0)^\alpha \right) + C_{2n,m} (t - t_0)^{\gamma-1} E_{\alpha,\gamma} \left( -\lambda_{n,m}^{2k} (\varepsilon_1, \varepsilon_2) \omega (t - t_0)^\alpha \right) + b_{n,m} h_{n,m}(t), \quad (22)$$

where

$$h_{n,m}(t) = \frac{1}{1 + \mu_{n,m}^{2k} (\varepsilon_1 + \varepsilon_2 \mu_{n,m}^{2k})} \int_{t_0}^t (t - s)^{\alpha-1} E_{\alpha,\alpha} \left( -\lambda_{n,m}^{2k} (\varepsilon_1, \varepsilon_2) \omega (t - s)^\alpha \right) a(s) ds,$$

$C_{1n,m}$  and  $C_{2n,m}$  are arbitrary constants.

By Fourier coefficients (18), the initial conditions (2) we rewrite in the forms

$$\begin{aligned} \lim_{t \rightarrow +t_0} J_{t_0+}^{2-\gamma} u_{n,m}(t) &= \int_0^l \int_0^l \lim_{t \rightarrow +t_0} J_{t_0+}^{2-\gamma} U(t, x, y) \vartheta_{n,m}(x, y) dx dy = \\ &= \int_0^l \int_0^l \varphi_{1n,m}(x, y) \vartheta_{n,m}(x, y) dx dy = \varphi_{1n,m}, \end{aligned} \quad (23)$$

$$\begin{aligned} \lim_{t \rightarrow +t_0} \frac{d}{dt} J_{t_0+}^{2-\gamma} u_{n,m}(t) &= \int_0^l \int_0^l \lim_{t \rightarrow +t_0} \frac{d}{dt} J_{t_0+}^{2-\gamma} U(t, x, y) \vartheta_{n,m}(x, y) dx dy = \\ &= \int_0^l \int_0^l \varphi_{2n,m}(x, y) \vartheta_{n,m}(x, y) dx dy = \varphi_{2n,m}. \end{aligned} \quad (24)$$

To find the unknown coefficients  $C_{1n,m}$  and  $C_{2n,m}$  in (22), we use conditions (23) and (24). Then from (22) we have

$$\begin{aligned} u_{n,m}(t) &= \varphi_{1n,m} (t - t_0)^{\gamma-2} E_{\alpha,\gamma-1} \left( -\lambda_{n,m}^{2k} (\varepsilon_1, \varepsilon_2) \omega (t - t_0)^\alpha \right) + \\ &+ \varphi_{2n,m} (t - t_0)^{\gamma-1} E_{\alpha,\gamma} \left( -\lambda_{n,m}^{2k} (\varepsilon_1, \varepsilon_2) \omega (t - t_0)^\alpha \right) + \\ &+ \frac{b_{n,m}}{1 + \mu_{n,m}^{2k} (\varepsilon_1 + \varepsilon_2 \mu_{n,m}^{2k})} \int_{t_0}^t (t - s)^{\alpha-1} E_{\alpha,\alpha} \left( -\lambda_{n,m}^{2k} (\varepsilon_1, \varepsilon_2) \omega (t - s)^\alpha \right) a(s) ds. \end{aligned} \quad (25)$$

Substituting the presentation of the Fourier coefficients (25) of main unknown function into Fourier series (6), we obtain

$$\begin{aligned} U(t, x, y) &= \sum_{n,m=1}^{\infty} \vartheta_{n,m}(x, y) \left[ \varphi_{1n,m} (t - t_0)^{\gamma-2} E_{\alpha,\gamma-1} \left( -\lambda_{n,m}^{2k} (\varepsilon_1, \varepsilon_2) \omega (t - t_0)^\alpha \right) + \right. \\ &+ \varphi_{2n,m} (t - t_0)^{\gamma-1} E_{\alpha,\gamma} \left( -\lambda_{n,m}^{2k} (\varepsilon_1, \varepsilon_2) \omega (t - t_0)^\alpha \right) + \\ &+ \left. \frac{b_{n,m}}{1 + \mu_{n,m}^{2k} (\varepsilon_1 + \varepsilon_2 \mu_{n,m}^{2k})} \int_{t_0}^t (t - s)^{\alpha-1} E_{\alpha,\alpha} \left( -\lambda_{n,m}^{2k} (\varepsilon_1, \varepsilon_2) \omega (t - s)^\alpha \right) a(s) ds \right]. \end{aligned} \quad (26)$$

This Fourier series (26) is a formal solution to the initial value problem (1)–(4).

3 Convergence of the Fourier series (26)

We prove absolute and uniform convergence of the Fourier series (26). We need to use the concepts of the following Banach spaces. Hilbert coordinate space  $\ell_2$  of number sequences  $\{\varphi_{n,m}\}_{n,m=1}^\infty$  with norm

$$\|\varphi\|_{\ell_2} = \sqrt{\sum_{n,m=1}^\infty |\varphi_{n,m}|^2} < \infty.$$

The space  $L_2(\Omega_l^2)$  of square-summable functions on the domain  $\Omega_l^2 = \Omega_l \times \Omega_l$  with norm

$$\|\vartheta(x, y)\|_{L_2(\Omega_l^2)} = \sqrt{\int_0^l \int_0^l |\vartheta(x, y)|^2 dx dy} < \infty.$$

Conditions of smoothness. Let for functions

$$\varphi_i(x, y) \ (i = 1, 2), \ b(x, y) \in C^{4k}(\Omega_l^2)$$

there exist piecewise continuous  $4k + 1$  order derivatives. Then by integrating in parts the functions (20), (23) and (24)  $4k + 1$  times over every variable  $x, y$ , we obtain the following relations

$$|\varphi_{i n, m}| = \left(\frac{l}{\pi}\right)^{8k+2} \frac{|\varphi_{i n, m}^{(8k+2)}|}{n^{4k+1} m^{4k+1}}, \quad |b_{n, m}| = \left(\frac{l}{\pi}\right)^{8k+2} \frac{|b_{n, m}^{(8k+2)}|}{n^{4k+1} m^{4k+1}}, \quad (27)$$

$$\|\varphi_{i n, m}^{(8k+2)}\|_{\ell_2} \leq \frac{2}{l} \left\| \frac{\partial^{8k+2} \varphi_i(x, y)}{\partial x^{4k+1} \partial y^{4k+1}} \right\|_{L_2(\Omega_l^2)}, \quad (28)$$

$$\|b_{n, m}^{(8k+2)}\|_{\ell_2} \leq \frac{2}{l} \left\| \frac{\partial^{8k+2} b(x, y)}{\partial x^{4k+1} \partial y^{4k+1}} \right\|_{L_2(\Omega_l^2)}, \quad (29)$$

where

$$\varphi_{i n, m}^{(8k+2)} = \int_0^l \int_0^l \frac{\partial^{8k+2} \varphi_i(x, y)}{\partial x^{4k+1} \partial y^{4k+1}} \vartheta_{n, m}(x, y) dx dy, \quad i = 1, 2,$$

$$b_{n, m}^{(8k+2)} = \int_0^l \int_0^l \frac{\partial^{8k+2} b(x, y)}{\partial x^{4k+1} \partial y^{4k+1}} \vartheta_{n, m}(x, y) dx dy.$$

To obtain estimates for solution, we use the properties of the Mittag-Leffler function [36]. Let  $\alpha \in (0; 2)$  and  $\gamma \in \mathbb{R}$ . If  $\arg z = \pi$ , then there takes place the following estimate

$$|E_{\alpha, \gamma}(z)| \leq \frac{M_1}{1 + |z|},$$

where  $0 < M_1 = \text{const}$  does not depend from  $z$ .

Therefore, it is easy to see that there exists constant  $M_2$  such that

$$\max_{t_0 \leq t \leq T} \left| E_{\alpha, \gamma-1} \left( -\lambda_{n, m}^{2k} (\varepsilon_1, \varepsilon_2) \omega (t - t_0)^\alpha \right) \right| \leq M_2 < \infty, \quad (30)$$

$$\max_{t_0 \leq t \leq T} \left| E_{\alpha, \gamma} \left( -\lambda_{n, m}^{2k} (\varepsilon_1, \varepsilon_2) \omega (t - t_0)^\alpha \right) \right| \leq M_2 < \infty, \quad (31)$$

$$\max_{t_0 \leq t \leq T} \left| \int_{t_0}^t (t-s)^{\alpha-1} (t-t_0)^{2-\gamma} E_{\alpha, \alpha} \left( -\lambda_{n,m}^{2k} (\varepsilon_1, \varepsilon_2) \omega(t-s)^\alpha \right) a(s) ds \right| \leq M_2 < \infty. \quad (32)$$

*Theorem 1.* Suppose that the conditions of smoothness and estimates (27)–(29) are fulfilled. Then Fourier series (26) convergence is absolute and uniform.

*Proof.* We apply the formulas (27)–(29) and estimates (30)–(32) to estimate the series (26). Using the Cauchy–Schwartz inequality for series (26), we get the estimate

$$\begin{aligned} \left| (t-t_0)^{2-\gamma} U(t, x, y) \right| &\leq M_2 \sum_{n,m=1}^{\infty} |\vartheta_{n,m}(x, y)| \cdot [|\varphi_{1n,m}| + |\varphi_{2n,m}| + |b_{n,m}|] \leq \\ &\leq \frac{2}{l} M_2 \left[ \sum_{n,m=1}^{\infty} |\varphi_{1n,m}| + \sum_{n,m=1}^{\infty} |\varphi_{2n,m}| + \sum_{n,m=1}^{\infty} |b_{n,m}| \right] \leq \\ &\leq \frac{2}{l} \left( \frac{l}{\pi} \right)^{8k+2} M_2 \left[ \sum_{n,m=1}^{\infty} \frac{|\varphi_{1n,m}^{(8k+2)}|}{n^{4k+1} m^{4k+1}} + \sum_{n,m=1}^{\infty} \frac{|\varphi_{2n,m}^{(8k+2)}|}{n^{4k+1} m^{4k+1}} + \sum_{n,m=1}^{\infty} \frac{|b_{n,m}^{(8k+2)}|}{n^{4k+1} m^{4k+1}} \right] \leq \\ &\leq \frac{2}{l} \left( \frac{l}{\pi} \right)^{8k+2} M_2 M_3 \left[ \|\varphi_{1n,m}^{(8k+2)}\|_{\ell_2} + \|\varphi_{2n,m}^{(8k+2)}\|_{\ell_2} + \|b_{n,m}^{(8k+2)}\|_{\ell_2} \right] \leq \\ &\leq \gamma_1 \left[ \sum_{i=1}^2 \left\| \frac{\partial^{8k+2} \varphi_i(x, y)}{\partial x^{4k+1} \partial y^{4k+1}} \right\|_{L_2(\Omega_i^2)} + \left\| \frac{\partial^{8k+2} b(x, y)}{\partial x^{4k+1} \partial y^{4k+1}} \right\|_{L_2(\Omega_i^2)} \right] < \infty, \end{aligned} \quad (33)$$

where

$$\gamma_1 = M_2 M_3 \left( \frac{2}{l} \right)^2 \left( \frac{l}{\pi} \right)^{8k+2}, \quad M_3 = \sqrt{\sum_{n,m=1}^{\infty} \frac{1}{n^{8k+2} m^{8k+2}}} < \infty.$$

From the estimate (33) the absolute and uniform convergence of Fourier series (26) implies. The Theorem 1 is proved.

#### 4 Uniqueness of the solution

To establish the uniqueness of the function  $U(t, x, y)$  we suppose that there are two functions  $U_1$  and  $U_2$  that satisfy the given conditions (1)–(4). Then their difference  $U = U_1 - U_2$  is a solution to differential equation (1), satisfying conditions (2)–(4) with zero functions  $\varphi_1(x, y) = \varphi_2(x, y) = 0$ . By virtue of relations (23) and (24) we have that  $\varphi_{1n,m} = \varphi_{2n,m} = 0$ . Hence, we that obtain from formulas (18) and (26) in the domain  $\Omega$  follows the zero identity

$$\int_0^l \int_0^l (t-t_0)^{2-\gamma} U(t, x, y) \vartheta_{n,m}(x, y) dx dy \equiv 0.$$

Hence, by virtue of the completeness of the systems of eigenfunctions  $\left\{ \sqrt{\frac{2}{l}} \sin \frac{\pi n}{l} x \right\}, \left\{ \sqrt{\frac{2}{l}} \sin \frac{\pi m}{l} y \right\}$  in  $L_2(\Omega_l^2)$  we deduce that  $U(t, x, y) \equiv 0$  for all  $x \in \Omega_l^2 \equiv [0, l]^2$  and  $t \in \Omega_T \equiv [0; T]$ .

Since  $(t-t_0)^{2-\gamma} U(t, x, y) \in C(\bar{\Omega})$ , then  $(t-t_0)^{2-\gamma} U(t, x, y) \equiv 0$  in the domain  $\bar{\Omega}$ . Therefore, the solution to the initial value problem (1)–(4) is unique in the domain  $\bar{\Omega}$ .

5 Term-by-term differentiation possibility

*Theorem 2.* Let the conditions of the Theorem 1 be fulfilled. Then term-by-term differentiation of the series (26) is possible.

*Proof.* The function (26) we differentiate the required number of times

$$\frac{\partial^{4k}}{\partial x^{4k}} (t - t_0)^{2-\gamma} U(t, x, y) = \sum_{n,m=1}^{\infty} \left(\frac{\pi n}{l}\right)^{4k} \vartheta_{n,m}(x, y) (t - t_0)^{2-\gamma} u_{n,m}(t), \tag{34}$$

$$\frac{\partial^{4k}}{\partial y^{4k}} (t - t_0)^{2-\gamma} U(t, x, y) = \sum_{n,m=1}^{\infty} \left(\frac{\pi m}{l}\right)^{4k} \vartheta_{n,m}(x, y) (t - t_0)^{2-\gamma} u_{n,m}(t), \tag{35}$$

where  $u_{n,m}(t)$  is defined from the presentation (25).

The expansion of the following functions into Fourier series are defined in a similar way

$$(t - t_0)^{2-\gamma} D^{\alpha,\gamma} U(t, x, y), \quad \frac{\partial^{4k}}{\partial x^{4k}} (t - t_0)^{2-\gamma} D^{\alpha,\gamma} U(t, x, y), \quad \frac{\partial^{4k}}{\partial y^{4k}} (t - t_0)^{2-\gamma} D^{\alpha,\gamma} U(t, x, y).$$

We show the convergence of series (34) and (35). Analogously to the case of estimate (33), applying the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned} \left| \frac{\partial^{4k}}{\partial x^{4k}} (t - t_0)^{2-\gamma} U(t, x, y) \right| &\leq \sum_{n,m=1}^{\infty} \left(\frac{\pi n}{l}\right)^{4k} \left| (t - t_0)^{2-\gamma} u_{n,m}(t) \right| \cdot \left| \vartheta_{n,m}(x, y) \right| \leq \\ &\leq \frac{2}{l} \left(\frac{\pi}{l}\right)^{4k} M_2 \left[ \sum_{n,m=1}^{\infty} n^{4k} |\varphi_{1n,m}| + \sum_{n,m=1}^{\infty} n^{4k} |\varphi_{2n,m}| + \sum_{n,m=1}^{\infty} n^{4k} |b_{n,m}| \right] \leq \\ &\leq \frac{2}{l} \left(\frac{l}{\pi}\right)^{4k+2} M_2 \left[ \sum_{n,m=1}^{\infty} \frac{|\varphi_{1n,m}^{(8k+2)}|}{n m^{4k+1}} + \sum_{n,m=1}^{\infty} \frac{|\varphi_{2n,m}^{(8k+2)}|}{n m^{4k+1}} + \sum_{n,m=1}^{\infty} \frac{|b_{n,m}^{(8k+2)}|}{n m^{4k+1}} \right] \leq \\ &\leq \frac{2}{l} \left(\frac{l}{\pi}\right)^{4k+2} M_2 M_4 \left[ \left\| \varphi_{1n,m}^{(8k+2)} \right\|_{\ell_2} + \left\| \varphi_{2n,m}^{(8k+2)} \right\|_{\ell_2} + \left\| b_{n,m}^{(8k+2)} \right\|_{\ell_2} \right] \leq \\ &\leq \gamma_2 \left[ \sum_{i=1}^2 \left\| \frac{\partial^{8k+2} \varphi_i(x, y)}{\partial x^{4k+1} \partial y^{4k+1}} \right\|_{L_2(\Omega_i^2)} + \left\| \frac{\partial^{8k+2} b(x, y)}{\partial x^{4k+1} \partial y^{4k+1}} \right\|_{L_2(\Omega_i^2)} \right] < \infty, \tag{36} \end{aligned}$$

where  $\gamma_2 = \left(\frac{2}{l}\right)^2 \left(\frac{l}{\pi}\right)^{4k+2} M_2 M_4$ ,  $M_4 = \sqrt{\sum_{n,m=1}^{\infty} \frac{1}{n m^{8k+2}}} < \infty$ ;

$$\begin{aligned} \left| \frac{\partial^{4k}}{\partial y^{4k}} (t - t_0)^{2-\gamma} U(t, x, y) \right| &\leq \sum_{n,m=1}^{\infty} \left(\frac{\pi m}{l}\right)^{4k} \left| (t - t_0)^{2-\gamma} u_{n,m}(t) \right| \cdot \left| \vartheta_{n,m}(x, y) \right| \leq \\ &\leq \frac{2}{l} \left(\frac{\pi}{l}\right)^{4k} M_2 \left[ \sum_{n,m=1}^{\infty} m^{4k} |\varphi_{1n,m}| + \sum_{n,m=1}^{\infty} m^{4k} |\varphi_{2n,m}| + \sum_{n,m=1}^{\infty} m^{4k} |b_{n,m}| \right] \leq \\ &\leq \frac{2}{l} \left(\frac{l}{\pi}\right)^{4k+2} M_2 \left[ \sum_{n,m=1}^{\infty} \frac{|\varphi_{1n,m}^{(8k+2)}|}{n^{4k+1} m} + \sum_{n,m=1}^{\infty} \frac{|\varphi_{2n,m}^{(8k+2)}|}{n^{4k+1} m} + \sum_{n,m=1}^{\infty} \frac{|b_{n,m}^{(8k+2)}|}{n^{4k+1} m} \right] \leq \end{aligned}$$

$$\begin{aligned} &\leq \frac{2}{l} \left(\frac{l}{\pi}\right)^{4k+2} M_2 M_5 \left[ \left\| \varphi_{1n,m}^{(8k+2)} \right\|_{\ell_2} + \left\| \varphi_{2n,m}^{(8k+2)} \right\|_{\ell_2} + \left\| b_{n,m}^{(8k+2)} \right\|_{\ell_2} \right] \leq \\ &\leq \gamma_3 \left[ \sum_{i=1}^2 \left\| \frac{\partial^{8k+2} \varphi_i(x,y)}{\partial x^{4k+1} \partial y^{4k+1}} \right\|_{L_2(\Omega_l^2)} + \left\| \frac{\partial^{8k+2} b(x,y)}{\partial x^{4k+1} \partial y^{4k+1}} \right\|_{L_2(\Omega_l^2)} \right] < \infty, \end{aligned} \quad (37)$$

where  $\gamma_3 = \left(\frac{2}{l}\right)^2 \left(\frac{l}{\pi}\right)^{4k+2} M_2 M_5$ ,  $M_5 = \sqrt{\sum_{n,m=1}^{\infty} \frac{1}{n^{8k+2m}}} < \infty$ .

The convergence of Fourier series for functions

$$(t-t_0)^{2-\gamma} D^{\alpha,\gamma} U(t,x,y), \quad \frac{\partial^{4k}}{\partial x^{4k}} (t-t_0)^{2-\gamma} D^{\alpha,\gamma} U(t,x,y), \quad \frac{\partial^{4k}}{\partial y^{4k}} (t-t_0)^{2-\gamma} D^{\alpha,\gamma} U(t,x,y)$$

is easy to prove, and the necessary estimates are obtained similarly to the cases of estimates (33), (36), and (37). Therefore, the function  $U(t,x,y)$  belongs to the class of functions (4). Theorem 2 is proved.

### 6 Stability of the solution $U(t,x,y)$ with respect to given functions

*Theorem 3.* Suppose that all the conditions of Theorem 2 are fulfilled. Then, the function  $U(t,x,y)$  as a solution to the problem (1)–(4) is stable with respect to given functions  $\varphi_1(x,y)$ ,  $\varphi_2(x,y)$ .

*Proof.* We show that the solution to the differential equation (1)  $U(t,x,y)$  is stable with respect to a given functions  $\varphi_1(x,y)$ ,  $\varphi_2(x,y)$ . Let  $U_1(t,x,y)$  and  $U_2(t,x,y)$  be two different solutions to the initial value problem (1)–(4), corresponding to two different values of the functions  $\varphi_{11}(x,y)$ ,  $\varphi_{12}(x,y)$  and  $\varphi_{21}(x,y)$ ,  $\varphi_{22}(x,y)$ , respectively.

We put that  $|\varphi_{11n,m} - \varphi_{12n,m}| + |\varphi_{21n,m} - \varphi_{22n,m}| < \delta_{n,m}$ , where  $0 < \delta_{n,m}$  is sufficiently small positive quantity and the series  $\sum_{n,m=1}^{\infty} |\delta_{n,m}|$  is convergent. Then, considering this, by virtue of the conditions of the theorem, from the Fourier series (26) it is easy to obtain that

$$\begin{aligned} &|t^{2-\gamma} [U_1(t,x,y) - U_2(t,x,y)]| \leq \\ &\leq \frac{2}{l} \sigma_3 \sum_{n,m=1}^{\infty} [|\varphi_{11n,m} - \varphi_{12n,m}| + |\varphi_{21n,m} - \varphi_{22n,m}|] < \frac{2}{l} \sigma_3 \sum_{n,m=1}^{\infty} |\delta_{n,m}| < \infty. \end{aligned}$$

If we put  $\varepsilon = \frac{2}{l} \sigma_3 \sum_{n,m=1}^{\infty} |\delta_{n,m}| < \infty$ , then from last estimate we finally obtain assertions about the stability of the solution to the differential equation (1) with respect to a given functions  $\varphi_1(x,y)$ ,  $\varphi_2(x,y)$  in (2). The theorem 3 is proved.

Similarly, it is proved that there holds the following theorem.

*Theorem 4.* Suppose that all the conditions of Theorem 2 are fulfilled. Then, the function  $U(t,x,y)$  as a solution to the problem (1)–(4) is stable with respect to given function  $b(x,y)$  in the right-hand side of the differential equation (1).

### Conclusions

In three-dimensional domain, the solvability of a mixed problem for a Hilfer type partial differential equation (1) of the higher even order with initial value conditions (2) and small positive parameters in mixed derivatives is considered. Suppose that the conditions of smoothness are fulfilled. Then the solution to this fractional differential equation of higher order for  $1 < \alpha < \gamma \leq 2$  is studied in the class of regular functions. The Fourier series method is used and a countable system of ordinary differential

equations is obtained (21). The initial value problem is integrated as an ordinary differential equation. We obtained the presentation for unknown function  $U(t, x, y)$ . Using the Cauchy–Schwarz inequality and the Bessel inequality, we proved the absolute and uniform convergence of the obtained Fourier series (26) for function  $U(t, x, y)$  and its derivatives. It is proved that solution to the problem (1)–(4)  $U(t, x, y)$  is stable with respect to given functions  $b(x, y)$  and  $\varphi_i(x, y)$ ,  $i = 1, 2$ .

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## Жоғары ретті Гильфер типті дифференциалдық теңдеу үшін аралас есеп туралы

Үшөлшемді облыста аралас туындыларда бастапқы шарттары және шағын оң параметрлері бар жүйе ретті Гильфер типті дербес туындылы теңдеу үшін аралас есептің шешілу мүмкіндігі қарастырылған. Бұл жоғары ретті бөлшек дифференциалдық теңдеудің шешімі тұрақты функциялар класында зерттелген. Бөлшек операторының реті  $1 < \alpha < 2$  тең болатын жағдай зерттелді. Фурье қатарларының әдісі қолданылып, қарапайым дифференциалдық теңдеулердің есептелетін жүйесі алынды. Бастапқы есеп қарапайым дифференциалдық теңдеу ретінде интегралданады, ал интегралдық тұрақтылар берілген бастапқы шарттарды пайдалана отырып табылады. Коши-Шварц теңсіздігі мен Бессель теңсіздігін пайдаланып, алынған Фурье қатарының абсолютті және біркелкі жинақтылығы дәлелденді. Берілген функцияларға қатысты есептің шешімінің тұрақтылығы да зерттелді.

*Кілт сөздер:* бөлшек реті, Гильфер операторы, аралас есеп, Фурье қатары, бастапқы шарттар, бірегей шешім.

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## О смешанной задаче для дифференциального уравнения типа Хильфера высшего порядка

В трехмерной области рассмотрена разрешимость смешанной задачи для дифференциального уравнения в частных производных типа Хильфера четного порядка с начальными условиями и малыми положительными параметрами в смешанных производных. Решение этого дробного дифференциального уравнения высшего порядка изучено в классе регулярных функций. Исследован случай, когда порядок дробного оператора равен  $1 < \alpha < 2$ . Применен метод рядов Фурье, и получена счетная система обыкновенных дифференциальных уравнений. Начальная задача интегрируется как обыкновенное дифференциальное уравнение, и интегральные константы находятся с помощью заданных начальных условий. С помощью неравенства Коши–Шварца и неравенства Бесселя доказана абсолютная и равномерная сходимость полученного ряда Фурье. Изучена также устойчивость решения задачи по заданным функциям.

*Ключевые слова:* дробный порядок, оператор Хильфера, смешанная задача, ряды Фурье, начальные условия, однозначная разрешимость.

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