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Zh.A. Sartabanov[1], G.M. Aitenova[2,*]

[1] K. Zhubanov Aktobe Regional University, Aktobe, Kazakhstan;
[2] M. Utemisov West Kazakhstan University, Uralsk, Kazakhstan
(E-mail: sartabanov-42@mail.ru, gulsezim-88@mail.ru)

Bounded on the semi-axis multiperiodic solution of a linear finite-hereditarity integro-differential equation of parabolic type

The question of the existence of a solution of linear integro-differential systems of parabolic type limited on the semiaxis in a spatial variable and multiperiodic in time variables was considered. Sufficient conditions of multiperiodic oscillations in time variables in a linear homogeneous equation with a boundary condition and in a linear inhomogeneous equation were established. A linear homogeneous and inhomogeneous finite-hereditarity integro-differential equation of convective-diffusion type were investigated.

Keywords: integro-differential, finite-hereditarity, convection, diffusion, parabolic type, differentiation operator, Fourier series.

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Problem statement

It is known [1,2] that many hereditary phenomena in biology and mechanics are described by various types of integro-differential equations. If the state of a phenomenon at the moment τ is determined by the set of states at the moments of the interval $(\tau-\varepsilon,\tau]$, then such a phenomenon is called hereditary with a finite hereditary period $\varepsilon>0$.

In the case of $\varepsilon=+\infty$, the state of the phenomenon at the moment τ depends on its states at moments in the interval $(-\infty,\tau]$. The hereditary of the phenomenon can also be related to the interval $(\tau_0,\tau]$, where τ_0 is some constant.

When the heredity of the phenomenon is bounded by the period $\varepsilon>0$, then a linear phenomenon with bounded hereditarity can be described by an integro-differential equation of the form

$$\frac{du(\tau)}{d\tau}=A(\tau)u(\tau)+\int\limits_{\tau-\varepsilon}^{\tau}K(\tau,s)u(s)ds+f(\tau). \tag{1}$$

In the case of a quasilinear phenomenon of the heredity of the period $\varepsilon>0$ we obtain the equation

$$\frac{du(\tau)}{d\tau}=A(\tau)u(\tau)+\int\limits_{\tau-\varepsilon}^{\tau}K(\tau,s)u(s)ds+f\left(\tau,u(\tau),\int\limits_{\tau-\varepsilon}^{\tau}K(\tau,s)u(s)ds\right).$$

In the linear (1) and quasi-linear equations the functions $A(\tau)$, $K(\tau,s)$ and $f(\tau,u,\overline{v})$ are known. Such equations, along with biological phenomena, describe the processes of elastic deformations, electromagnetism, and other sections of the general dynamics related to the hereditary propagation of thermal, magnetic, light, sound and other waves along the x axis. Propagations of this kind type can also be of a diffusion nature. Propagations of this kind may have a diffusive character also. Then the equation describing this phenomenon takes a form [3, 4]:

$$\frac{\partial u(x,\tau)}{\partial\tau}-a^2\frac{\partial^2u(x,\tau)}{\partial x^2}=a(x,\tau)u(x,\tau)+$$

*Corresponding author.
E-mail: gulsezim-88@mail.ru

$$+\int\limits_{\tau-\varepsilon}^{\tau} b(x,\tau,s)u(x,s)ds + f\left(x,\tau,u(x,\tau)\right). \qquad (2)$$

In the case of multi-frequency waves and fluctuations, to study the processes, following [5–19], it will be necessary to introduce a variable $t=(t_1,...,t_m)$, varying on the vector field $\frac{dt}{d\tau}=c$ and one has to consider the equation

$$D_c u(x,t,\tau) - a^2\frac{\partial^2 u(x,t,\tau)}{\partial x^2} = a(x,t,\tau)u(x,t,\tau)+$$

$$+\int\limits_{\tau-\varepsilon}^{\tau} b(x,t,\tau,t-c\tau+cs,s)u(x,t-c\tau+cs,s)ds + f\left(x,t,\tau,u(x,t,\tau)\right) \qquad (3)$$

with differentiation operator

$$D_c=\frac{\partial}{\partial\tau}+\sum_{j=1}^{m}c_j\frac{\partial}{\partial t_j}$$

in the direction of the vector $c=(c_1,...,c_m)$ with constant coordinates $c_j>0$, $j=\overline{1,m}$, and all the input data of this equation are assumed to be periodic in time variables $(t,\tau)=(t_1,...,t_m,t_0)$, $t_0=\tau$ period-vector $(\omega,\theta)=(\omega_1,...,\omega_m,\omega_0)$, with incommensurable components $\omega_0=\theta$, ω_j, $j=\overline{1,m}$.

Obviously, [5–16] along the $t=c(\tau-\tau_0)$ characteristics vector field operator D_c of the equation (3) turns into the equation (2), and its (ω,θ)-periodic on $(t,\tau)\in R^m\times R$ solutions turn into almost periodic τ solutions of the latter at $x\in R_+$.

Thus, the investigation of multiperiodic by (t,τ) solutions of equation (3) of period (ω,θ) at $x\in R_+$ is of great importance in applied problems of the theory of fluctuations and oscillations.

Note that problem studies in such a formulation are not found in the scientific literature. The research is carried out in the inductive order from the particular to general. In this connection, the problem was studied for various linear cases of equations (3).

It is clear [17, 18] that the problem under consideration and its methods of investigation are closely related to some applied aspects of equations of mathematical physics of parabolic type and analytical problems of the theory of multi-frequency oscillations.

The researchers' interest in the problems for integro-differential equations, started at the end of the XIX century, has not weakened to this day [19, 20]. From various points of view, where the hereditary terms of the equations are described by integrals of Volterra or Fredholm types, and the dynamics of phenomena are characterized by ordinary or partial derivatives of unknowns, developing their theory from equations to inclusions.

1 Multiperiodic zeros of the differentiation operator in the multiperiodic boundary condition

Applying the differentiation operator $\nabla_c=D_c-a^2\frac{\partial^2}{\partial x^2}$ of the variables $x\in R_+=(0,+\infty)$, $\tau=t_0\in R$, $t=(t_1,...,t_m)\in R^m$ to the function $v(x,t,\tau)$ we introduce the equation

$$\nabla_c v(x,t,\tau)=0. \qquad (4)$$

Here D_c the differentiation operator for time variables (t,τ) of the form $D_c=\frac{\partial}{\partial\tau}+\sum\limits_{j=0}^{m}c_j\frac{\partial}{\partial t_j}$, $c_0=1$; $a=conts>0$; ∇_c is the differentiation operator by (x,t,τ). The equation with one-dimensional time t_j of the form $c_j\frac{\partial v_j}{\partial t_j}-a^2\frac{\partial^2 v_j}{\partial x^2}=0$ has solution v_j, depending on $\gamma_j\sqrt{c_j}x+\gamma_j^2a^2t_j$ running waves with

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parameter γ_j, then the solution of equation (4) with multidimensional time (t,τ) can be represented by the relations

$$v(x,t,\tau)=\alpha+\beta e^{\sum\limits_{j=0}^{m}\left(\gamma_j\sqrt{c_j}x+\gamma_j^2a^2t_j\right)} \tag{5}$$

with arbitrary differentiable functions α, β and γ_j, $j=\overline{0,m}$ vector variable $t-c\tau=(t_1-c_1\tau,...,t_m-c_m\tau)$, $c=(c_1,...,c_m)$.

Consequently, relation (5) represents zeros of the operator ∇_c at $x\in R_+$, $(t,\tau)\in R^m\times R$.

In what follows we will deal with bounded zeros of the operator ∇_c. Then by setting x to zero from (5) we obtain the limit function

$$v(x,t,\tau)|_{x=0}=\alpha+\beta e^{a^2\sum\limits_{j=0}^{m}\gamma_j^2t_j}\equiv v^0(t,\tau) \tag{6}$$

and for $x\to+\infty$, in the case of $Re\gamma_j<0$, we have

$$v(x,t,\tau)|_{x=+\infty}=\alpha\equiv v^+(t,\tau). \tag{7}$$

To ensure that the solution (5) for $t_j>0$, by virtue of (6) and (7), the functions α, β, γ_j and along with the condition $Re\gamma_j<0$, the conditions $Im\gamma_j>Re\gamma_j$, $j=\overline{0,m}$ must be bounded.

The main problem is related to the establishment of sufficient conditions for the existence of (ω,θ)-periodic on (t,τ) real-analytic at $t_j\in\Pi_\rho=\left\{t_j:\frac{2\pi}{\omega_j}|Imt_j|<\rho\right\}$, $j=\overline{0,m}$, $\omega_0=\theta$, $\omega=(\omega_1,...,\omega_m)$, $\rho=const>0$, solutions of the equations in question. Therefore in this case we assume that the boundary condition (6) is defined by the function

$$v^0(t+\omega,\tau+\theta)=v^0(t,\tau)\in A_{t,\tau}^{\omega,\theta}\left(\Pi_\rho^m\times\Pi_\rho\right). \tag{8}$$

Here $A_{t,\tau}^{\omega,\theta}\left(\Pi_\rho^m\times\Pi_\rho\right)$ is a class of (ω,θ)-periodic rea-analytic at $(t,\tau)\in\Pi_\rho^m\times\Pi_\rho$ and continuous on closures $\overline{\Pi}_\rho^m\times\overline{\Pi}_\rho$ functions, with $\omega_0=\theta,\omega_1,...,\omega_m$ are rationally incommensurable positive constants, ρ being the bandwidth Π_ρ of the interval $0<\rho<1$.

From the condition (8) we have a Fourier series representation of the function $v^0(t,\tau)$:

$$v^0(t,\tau)=\sum_{k\in Z^{m+1}}v_k^0e^{2\pi i\sum\limits_{j=0}^{m}k_j\nu_jt_j}, \tag{9}$$

where $k=(k_0,k_1,...,k_m)$, $\nu=(\nu_0,\nu_1,...,\nu_m)$, $\nu_j=\omega_j^{-1}$, $j=\overline{0,m}$; v_k^0 - are Fourier coefficients having the properties $\overline{v}_k^0=v_{-k}^0$ and satisfying the estimate

$$|v_k^0|\le||v^0||e^{-\rho|k|} \tag{10}$$

with the norm $||v^0||=\sup\limits_{\Pi_\rho^m\times\Pi_\rho}|v^0(t,\tau)|$ and $|k|=\sum\limits_{j=0}^{m}|k_j|$.

Due to rational incommensurability of frequencies $\nu_j=\omega_j^{-1}$, $j=\overline{0,m}$ parameters α, β, γ_j become constant, for the function depending on the difference $t_j-\tau$ to be ω_j and $\theta=\omega_0$ - periodic as by t_j and so by τ it is necessary and sufficient. Assuming (8) with respect to (6) we find the solution (4), (6) in the form of series

$$v(x,t,\tau)=\sum_{k\in Z^{m+1}}v_ke^{\sum\limits_{j=0}^{m}\left(\gamma_j\sqrt{c_j}x+\gamma_{jk}^2a^2t_j\right)} \tag{11}$$

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with constant coefficients v_k and indicators γ_{jk}, $j=\overline{0,m}$, $k \in Z^{m+1}$.

Obviously, (11) is a generalization of the function (5) to an infinite series, which represents the solution of the equation (4) in general form.

Substituting (11) and (9) into the boundary conditions (6) formally we obtain $v_k = v_k^0$, $\gamma_{jk}^2 a^2 = 2\pi i k_j \nu_j$, $j=\overline{0,m}$, $k_j = Z_+^0$, Z_+^0 is the set of non-negative integers.

Hence $\gamma_{jk} = \pm\frac{\sqrt{2\pi\nu_j k_j}}{a}\cdot\sqrt{i} = \pm\left(\frac{\sqrt{\pi\nu_j k_j}}{a} + i\frac{\sqrt{\pi\nu_j k_j}}{a}\right)$ at $k_j \in Z_+^0$ is the set of positive integers.

Since we are interested in the solution bounded by x in R_+, we have

$$\gamma_{jk} = -\left(\frac{\sqrt{\pi\nu_j k_j}}{a} + i\frac{\sqrt{\pi\nu_j k_j}}{a}\right),\ j=\overline{0,m},\ k_j = Z_+^0. \tag{12}$$

In the case of negative $k_j = -|k_j| < 0$ we have the equation $\gamma_{jk}^2 a^2 = -2\pi i|k_j|\nu_j$, $j=\overline{0,m}$, $k_j = Z_-$ to determine the indicators γ_{jk}. Hence we find $\gamma_{jk} = \pm\sqrt{-1}\sqrt{\frac{2\pi|k_j|\nu_j}{a^2}}\cdot\sqrt{i} = \pm i\frac{\sqrt{2\pi\nu_j|k_j|}}{a}\cdot\frac{1+i}{\sqrt{2}} = \pm\left(-\frac{\sqrt{\pi\nu_j|k_j|}}{a} + i\frac{\sqrt{\pi\nu_j|k_j|}}{a}\right)$.

Hence, to ensure that the solution is bounded by $x \in R_+$ we take the roots with a plus sign:

$$\gamma_{jk} = -\frac{\sqrt{\pi\nu_j|k_j|}}{a} + i\frac{\sqrt{\pi\nu_j|k_j|}}{a},\ j=\overline{0,m},\ k_j = Z_-. \tag{13}$$

Thus, the roots (12) and (13) are mutually conjugate. Hence, combining these formulas we have

$$\gamma_{jk} = -\frac{\sqrt{\pi\nu_j|k_j|}}{a} - sign k_j\frac{\sqrt{\pi\nu_j|k_j|}}{a},\ j=\overline{0,m},\ k_j = Z, \tag{14}$$

where this formula includes the case $k_j = 0$, at which $sign\ 0 = 0$.

Substituting (14) into (11) we obtain the solution

$$v(x,t,\tau) = v_0^0 +$$

$$+\sum_{0\neq k\in Z^{m+1}} v_k^0 exp\left[-\sum_{j=0}^{m}\frac{\sqrt{\pi\nu_j c_j|k_j|}}{a}x + i\left(sign k_j\sum_{j=0}^{m}\frac{\sqrt{\pi\nu_j c_j|k_j|}}{a}x + 2\pi k_j\nu_j t_j\right)\right]. \tag{15}$$

Obviously, the series (15) converges absolutely and uniformly at $x \in \overline{R}_+$ and $(t,\tau) \in R^m \times R$, differentiable by x (a finite number of times), analyticity at (t,τ) is preserved. In support of this claim, we use the evaluation (10) and $\sum\limits_{j=0}^{m}|k_j|^{1/2} \leq \sqrt{m+1}\left(\sum_{j=1}^{m}|k_j|\right)^{1/2}$, which follows from the Bunyakovskii-Schwartz inequality.

The solution (15) is multiperiodic at (t,τ), bounded at $(x,t,\tau) \in \overline{R}_+ \times \overline{\Pi}_\rho^m \times \overline{\Pi}_\rho$ and unique in the class of bounded functions.

Theorem 1. The Problem (4), (6) under the condition (8) has at $(x,t,\tau) \in \overline{R}_+ \times \overline{\Pi}_\rho^m \times \overline{\Pi}_\rho$ the only real-analytic (ω,θ)-periodic on (t,τ) solution $v(x,t,\tau)$ of the form (15) satisfying the

$$|v(x,t,\tau)| \leq c^0||v^0||/\delta^{m+1},\ x \in \overline{R}_+,\ (t,\tau) \in \overline{\Pi}_{\rho-\delta}^m \times \overline{\Pi}_{\rho-\delta} \tag{16}$$

with an arbitrary constant δ from the interval $0 < \delta < \rho < 1$, where $c^0 = c^0(m)$is a constant, independent of δ and v^0.

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The proof of all the positions of the theorem is given above. To complete it, it is necessary to verify the validity of the estimate (16).

Indeed, from (15) we have the series

$$v(x,t,\tau) = \sum_{k \in Z^{m+1}} v_k(x) e^{2\pi i \sum\limits_{j=0}^{m} k_j \nu_j t_j} \tag{17}$$

with coefficients

$$v_k(x) = v_k^0 exp\left[-\sum_{j=0}^{m} \frac{\sqrt{\pi \nu_j c_j |k_j|}}{a} \left(1 + i sign k_j\right) x \right], \tag{18}$$

which satisfy the inequalities

$$|v_k(x)| \le |v_k^0|, \ k \in Z^{m+1}. \tag{19}$$

The case of absence of t is considered in [17; 201–202]. Then, by virtue of (10), from (19) it follows that

$$|v_k(x)| \le ||v^0|| e^{-\rho |k|}. \tag{20}$$

Consequently, according to the properties of the Fourier coefficients of analytic functions [18; 108], the function (17) with coefficients (18) satisfying the evaluation (20) is analytic and obeys the constraint (16).

2 Multiperiodic solution of a linear diffusion equation with a multi-frequency oscillating source

Consider the equation

$$\nabla_c u(x,t,\tau) \equiv D_c u(x,t,\tau) - a^2 \frac{\partial^2 u(x,t,\tau)}{\partial x^2} = f(x,t,\tau). \tag{21}$$

Here $a = const > 0$, the function $f(x,t,\tau)$ is represented as a series

$$f(x,t,\tau) = \sum_{k \in Z^{m+1}} f_k e^{-\gamma_k x + 2\pi i \sum\limits_{j=0}^{m} k_j \nu_j t_j} \tag{22}$$

with constants of $\gamma_k > 0$, f_k, $k = (k_0, k_1, ..., k_m) \in Z^{m+1}$; $\nu_j = \omega_j^{-1}$, $j = \overline{0,m}$ with

$$|f_k| \le ||f|| e^{-\rho |k|}, \tag{23}$$

where $||f|| = \sup\limits_{\overline{R}_+ \times \overline{\Pi}_\rho^m \times \overline{\Pi}_\rho} |f(x,t,\tau)|$.

The multiperiodic solution of the equation (21) will be sought in the form

$$u(x,t,\tau) = \sum_{k \in Z^{m+1}} W_k(t,\tau) e^{-\gamma_k x}. \tag{24}$$

Substituting (22) and (24) in (21) we obtain

$$\sum_{k \in Z^{m+1}} \left[D_c W_k(t,\tau) - a^2 \gamma_k^2 W_k(t,\tau) \right] e^{-\gamma_k x} = \sum_{k \in Z^{m+1}} f_k e^{2\pi i \sum\limits_{j=0}^{m} k_j \nu_j t_j} e^{-\gamma_k x}.$$

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Hence we have equations $D_c W_k(t,\tau) - a^2\gamma_k^2 W_k(t,\tau) = f_k e^{2\pi i \sum\limits_{j=0}^{m} k_j\nu_j t_j}$, $k \in Z^{m+1}$ which have (ω,θ)-periodic by (t,τ) solutions

$$W_k(t,\tau) = \int\limits_{+\infty}^{\tau} f_k e^{2\pi i \sum\limits_{j=0}^{m} k_j\nu_j(t_j - c_j\tau + c_j s) + a^2\gamma_k^2(\tau - s)} ds =$$

$$= \frac{f_k}{-a^2\gamma_k^2 + 2\pi i \sum\limits_{j=0}^{m} k_j\nu_j c_j} e^{2\pi i \sum\limits_{j=0}^{m} k_j\nu_j t_j} = \frac{1}{a_k + ib_k} f_k e^{2\pi i \sum\limits_{j=0}^{m} k_j\nu_j t_j}, \tag{25}$$

since conditions $\Delta_k = a_k + ib_k \neq 0$, where $a_k = -a^2\gamma_k^2$, $b_k = 2\pi \sum\limits_{j=0}^{m} k_j\nu_j c_j$, $k \in Z^{m+1}$ are satisfied. By substituting (25) into (24) we obtain solution

$$u^*(x,t,\tau) = \sum_{k\in Z^{m+1}} \frac{1}{a_k + ib_k} f_k e^{2\pi i \sum\limits_{j=0}^{m} k_j\nu_j t_j - \gamma_k x}. \tag{26}$$

To ensure the convergence of the series (26), we assume that the strong incommensurability condition is fulfilled $\widetilde{\nu}_j = \nu_j c_j$, $j = \overline{0,m}$ of the form

$$|b_k| = 2\pi \left|\sum_{j=0}^{m} k_j \widetilde{\nu}_j\right| \geq \lambda^{-1}|k|^{-1}, \quad |k| = \sum_{j=0}^{m} |k_j| > 0 \tag{27}$$

with constants $\lambda > 0$ and $l \geq m+1$, or the sequence a_k satisfies the condition of boundedness condition of the form

$$|a_k| \geq r, \quad k \in Z^{m+1} \tag{28}$$

with constant $r > 0$.

If one of the conditions (27) and (28), together with estimation (23) is satisfied, the series (26) will converge absolutely and uniformly.

Thus we distinguish two kinds of running waves $\psi_k(x,t,\tau) = 2\pi i \sum\limits_{j=0}^{m} k_j\nu_j t_j - \gamma_k x$, $k \in Z^{m+1}$, for which a) $\Delta_k = a_k + ib_k = 0$ and b) $\Delta_k = a_k + ib_k \neq 0$, $k \in Z^{m+1}$. In the case a) $u_k = e^{\psi_k(x,t,\tau)}$ will turn out to be zeros of the operator ∇_c, and in the case b) $\nabla_c u_k \neq 0$.

Note that a similar result can be obtained when the real function $f(x,t,\tau)$ is defined for complex values $\gamma_k = \alpha_k + i\beta_k$, $\beta_k \neq 0$.

So equation (21) under the conditions (22), (23) and under one of the conditions (27) and (28) admit only (ω,θ)-periodic on (t,τ) solution (26) with values $\Delta_k = a_k + ib_k \neq 0$, $k \in Z^{m+1}$.

In general, equation (21) has an infinite set of (ω,θ)-periodic solutions $u(x,t,\tau)$ by (t,τ), consisting of the sum of the solutions $v(x,t,\tau)$ of the homogeneous equations (4) with $\Delta_k = 0$, $k \in Z^{m+1}$ and the solution $u^*(x,t,\tau)$ of the nonhomogeneous equation (21) with $\Delta_k \neq 0$, $k \in Z^{m+1}$:

$$u(x,t,\tau) = v(x,t,\tau) + u^*(x,t,\tau), \tag{29}$$

where $v(x,t,\tau)$ is defined by the problem (4), (6), and $u^*(x,t,\tau)$ by the relation (26) and satisfies the boundary condition

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$$u(0,t,\tau) = v(0,t,\tau) + u^*(0,t,\tau). \tag{30}$$

The solution (29) of the boundary value problem (21), (30) is singular.

Theorem 2. Under the conditions (22), (23) and (27) or (28) the equations (21) has (ω,θ)-periodic solutions represented in the form (29) with terms (15) and (26).

If for some $k^0 = (k_0^0, k_1^0, ..., k_m^0)$ we have $\Delta_{k_0} = a_{k_0} + ib_{k_0} = 0$, then we exclude the corresponding k^0-subject from relation (26) and introduce a function

$$u^0(x,t,\tau) = \frac{\alpha_0\tau + \alpha_1 t_1 + ... + \alpha_m t_m}{\alpha_0 + \alpha_1 c_1 + ... + \alpha_m c_m} f_{k^0} e^{2\pi i \sum\limits_{j=0}^{m} k_j^0 \nu_j t_j - \gamma_{k^0} x} \tag{31}$$

with an arbitrary constant vector $\alpha = (\alpha_0, \alpha_1, ..., \alpha_m)$ satisfying equation

$$\nabla_c u^0(x,t,\tau) = f_{k^0} e^{2\pi i \sum\limits_{j=0}^{m} k_j^0 \nu_j t_j - \gamma_{k^0} x}. \tag{32}$$

Then, based on (31) and (32), the solution (26) can be represented in the form

$$\widetilde{u}^*(x,t,\tau) = u^0(x,t,\tau) + \sum_{k^0 \neq k \in Z^{m+1}} \frac{f_k}{a_k + ib_k} e^{2\pi i \sum\limits_{j=0}^{m} k_j \nu_j t_j - \gamma_k x}. \tag{33}$$

Theorem 3. If $a_{k^0} + ib_{k^0} = 0$ and $a_k + ib_k \neq 0$ at $k \neq k^0$, then under the conditions of Theorem 2, equation (21) has a solution $u(x,t,\tau) = v(x,t,\tau) + \widetilde{u}^*(x,t,\tau)$, where $v(x,t,\tau)$ is defined by the formula (6), and $\widetilde{u}^*(x,t,\tau)$ by relation (33).

3 Multiperiodic solutions of a linear homogeneous integro-differential parabolic equations with finite hereditarity

Consider (ω,θ)-periodic by (t,τ) equation

$$\nabla_c u(x,t,\tau) \equiv D_c u(x,t,\tau) - a^2 \frac{\partial^2 u(x,t,\tau)}{\partial x^2} =$$

$$= a(x,t,\tau)u(x,t,\tau) + \int\limits_{\tau-\varepsilon}^{\tau} b(x,t,\tau,t-c\tau+cs,s)u(x,t-c\tau+cs,s)ds.$$

This equation describes a multi-frequency phenomenon propagating along the semi-axis R_+ 1) diffusion with constant $a^2 \neq 0$, 2) linearly hereditary with finite period $\varepsilon > 0$ and kernel $b = b(x,t,\tau,\sigma,s)$, 3) at each point $x \in R_+$ it is linearly related to the external environment by the coefficient $a = a(x,t,\tau)$ and 4) flows with speed $D_c u(x,t,\tau)$ defined by differentiation along the direction of vector field of operator $D_c = \frac{\partial}{\partial\tau} + \sum_{j=1}^{m} c_j \frac{\partial}{\partial t_j}$.

An important special case of the process is when its heredity and coupling to the external world do not depend on $x \in R_+$. In this regard, we introduce into consideration the equation

$$\nabla_c u(x,t,\tau) = a(t,\tau)u(x,t,\tau) + \int\limits_{\tau-\varepsilon}^{\tau} b(t,\tau,\sigma,s)u(x,\sigma,s)ds, \tag{34}$$

where the matrices $a(t,\tau)$ and $b(t,\tau,\sigma,s)$ are real-analytic functions.

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Consider the null operator ∇_c, depending on $m+1$ running waves $\sum\limits_{j=0}^{m}(\gamma_j x + 2\pi i t_j \nu_j k_j)$ of the form

$$v_k(x,t,\tau) = be^{2\pi i \sum_{j=0}^{m} t_j\nu_j k_j + \gamma_j x} \tag{35}$$

with constant coefficient $b \neq 0$ and parameter $\gamma_j = \gamma_j(k_j, \nu_j, c_j, a)$.

It's obvious that $v_k(x,t,\tau)$ has the property

$$\nabla_c v_k(x,t,\tau) = 0,$$

$$v_k(x, t - c\tau + cs, s) = b^{-1} v_k(0, -c\tau + cs) v_k(x,t,\tau). \tag{36}$$

Next, by replacing

$$u(x,t,\tau) = U(t,\tau) v_k(x,t,\tau) \tag{37}$$

equation (34) on the basis of (35), (36) is reduced to

$$D_c U(t,\tau) = a(t,\tau)U(t,\tau) + \int\limits_{\tau-\varepsilon}^{\tau} b(t,\tau,\sigma,s) b^{-1} v_k(0,\sigma,s) U(\sigma,s) ds.$$

Under the conditions

$$a(t,\tau) \in A_{t,\tau}^{\omega,\theta}\left(\Pi_\rho^m \times \Pi_\rho\right), \quad b(t,\tau,\sigma,s) \in A_{t,\tau,\sigma,s}^{\omega,\theta,\omega,\theta}\left(\Pi_\rho^m \times \Pi_\rho \times \Pi_\rho^m \times \Pi_\rho\right) \tag{38}$$

it is possible to show the existence of a single solution $U_k(t,\tau,\sigma,s) \equiv U_k(t,\tau,t-c\tau+cs,s)$, satisfying the condition $U_k(t,s,t,s) = E$ at $\tau = s$ and $U_k(t,\tau,\sigma,s) \in A_{t,\tau,\sigma,s}^{\omega,\theta,\omega,\theta}\left(\Pi_\rho^m \times \Pi_\rho \times \Pi_\rho^m \times \Pi_\rho\right)$.

Suppose that $U_k(t,\tau,\sigma,s)$ satisfies the estimate

$$|U_k(t,\tau,t-c\tau+cs,s)| \leq \Lambda e^{-\lambda(\tau-s)} \tag{39}$$

with constants $\Lambda \geq 1$ and $\lambda > 0$ for any $k \in Z^{m+1}$.

Then a solution of the form (37), which is bounded at $x \in \overline{R}_+$, $t \in R^m$ and $\tau \geq s$ and satisfies the evaluation

$$|u(x,t,\tau)| \leq \Lambda e^{-\lambda(\tau-s)} |v_k(x,t,\tau)| \leq u^0 e^{-[\lambda(\tau-s)+\mu x]} \tag{40}$$

with some constant u^0, $\lambda > 0$ and $\mu > 0$. Here $\mu > 0$ is defined on the estimation of the zero (35) operator ∇_c.

Inequality (40) shows that under the condition (39) the homogeneous equation (34) has only a zero bounded (ω,θ)-periodic solution on (t,τ).

Theorem 4. Under the conditions (38) and (39), equation (34) has only zero (ω,θ)-periodic in (t,τ) solution.

4 Multiperiodic solution of a complete linear inhomogeneous integro-differential equation of parabolic type

Let's introduce the equation

$$\nabla_c u(x,t,\tau) = a(t,\tau)u(x,t,\tau) + \int\limits_{\tau-\varepsilon}^{\tau} b(t,\tau,\sigma,s) u(x,\sigma,s) ds +$$

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$$+f(t,\tau)exp\left(\sum_{j=0}^{m}[\gamma_j x+2\pi ik_j\nu_j t_j]\right). \tag{41}$$

Here $v_j(x,t_j)=\gamma_j x+2\pi ik_j\nu_j t_j$ are the travelling waves defined by the equation

$$\nabla_c exp[v_j(x,t_j)]=0, \quad j=\overline{0,m} \tag{42}$$

with unknown parameters γ_j and constants, $\nu_j=\omega_j^{-1}$, $k_j\in Z$ with the condition that $x\to+\infty$ follows

$$expv_j(x,t_j)\to 0. \tag{43}$$

The functions $a(t,\tau)$, $b(t,\tau,\sigma,s)$ and $f(t,\tau)$ are (ω,θ)-periodic by (t,τ) and (σ,s), belong to the class $A^{\omega,\theta,\omega,\theta}_{t,\tau,\sigma,s}\left(\Pi^m_\rho\times\Pi_\rho\times\Pi^m_\rho\times\Pi_\rho\right)$.

From the conditions (42) and (43) we have $a^2\gamma_j^2=\pm 2\pi ik_j\nu_j c_j$, $j=\overline{0,m}$ and $c_0=1$ at $t_0=\tau$. Hence we have $\gamma_j=\pm\frac{\sqrt{2\pi k_j\nu_j c_j}}{a}\frac{1\pm i}{\sqrt{2}}$, $k_j>0$; $\gamma_j=\pm\frac{\sqrt{2\pi|k_j|\nu_j c_j}}{a}\frac{1\mp i}{\sqrt{2}}$, $k_j<0$. To satisfy the condition (43) we choose γ_j as

$$\gamma_j=-\frac{\sqrt{\pi|k_j|\nu_j c_j}}{a}(1-isignk_j). \tag{44}$$

Thus, by virtue of the latter relationship, the function

$$v(x,t,\tau)=exp\left[\sum_{j=0}^{m}v_j(x,t_j)\right] \tag{45}$$

has the property

$$\nabla_c v(x,t,\tau)=0,\ x\in R_+,\ (t,\tau)\in R^m\times R. \tag{46}$$

It can be shown that

$$v(x,t-c\tau+cs,s)=v(x,t,\tau)exp\left[-2\pi i\sum_{j=0}^{m}k_j\nu_j(\tau-s)\right]. \tag{47}$$

Next, enter the replacement

$$u(x,t,\tau)=U(t,\tau)v(x,t,\tau) \tag{48}$$

into the equation (41) and due to (47) we obtain

$$D_cU(t,\tau)v(x,t,\tau)+U(t,\tau)\nabla_c v(x,t,\tau)=a(t,\tau)U(t,\tau)v(x,t,\tau)+$$

$$+\int_{\tau-\varepsilon}^{\tau}b(t,\tau,\sigma,s)exp\left[-2\pi i\sum_{j=0}^{m}k_j\nu_j c_j(\tau-s)\right]U(\sigma,s)v(x,t,\tau)ds+f(t,\tau)v(x,t,\tau).$$

Then, given (46), reducing by $v(x,t,\tau)\neq 0$ we have the equation

$$D_cU(t,\tau)=a(t,\tau)U(t,\tau)+\int_{\tau-\varepsilon}^{\tau}b(t,\tau,\sigma,s)exp\left[-2\pi i\sum_{j=0}^{m}k_j\nu_j c_j(\tau-s)\right]U(\sigma,s)ds+f(t,\tau). \tag{49}$$

The solution $U(t,\tau,\sigma,s)$ of the homogeneous equation corresponding to equation (49) with initial condition $U(t,s,t,s)=E$ satisfies the evaluation (40).

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Then it is easy to show that the inhomogeneous equation (49) admits a single (ω, θ)-periodic by (t, τ) solution

$$U^*(t,\tau) = \int_{-\infty}^{\tau} U(t,\tau,t-c\tau+cs_1,s_1) f(t-c\tau+cs_1,s_1)ds_1. \tag{50}$$

Then by substituting (50) in (48), we obtain a single bounded on $x \in \overline{R}_+$, (ω, θ)-periodic on (t, τ) solution

$$u^*(x,t,\tau) = U^*(t,\tau)v(x,t,\tau) \tag{51}$$

of equations (41).

Theorem 5. Let the functions a, b and f belong to the class $A^{\omega,\theta,\omega,\theta}_{t,\tau,\sigma,s}\left(\Pi^m_\rho \times \Pi_\rho \times \Pi^m_\rho \times \Pi_\rho\right)$. Then under conditions (43), (44) and (40) equation (41) has a unique bounded in $x \in \overline{R}_+$ (ω, θ)-periodic on (t, τ) solution of the form (51) with factors (50) and (45).

By the superposition method, the theorem can be generalised when the free term $f(x, t, \tau)$ equation (41) can be represented as

$$f(x,t,\tau) = \sum_{k \in Z^{m+1}} f_k(t,\tau) exp\left(\sum_{j=0}^{m}\left[\gamma_{k_j}x + 2\pi i k_j \nu_j t_j\right]\right),$$

where γ_{k_j} is a constant from (44).

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Ж.А. Сартабанов[1], Г.М. Айтенова[2]

[1] *Қ.Жұбанов атындағы Ақтөбе өңірлік университеті, Ақтөбе, Қазақстан;*
[2] *М.Өтемісов атындағы Батыс Қазақстан университеті, Орал, Қазақстан*

Параболалық типті ақырлы-эредитарлы сызықты интегралды-дифференциалдық теңдеудің жартылай осьте шектелген көппериодты шешімі

Параболалық типті сызықты интегралды-дифференциалдық теңдеулер жүйесінің кеңістік айнымалысы бойынша жартылай осьте шектелген және уақыт айнымалылары бойынша көппериодты шешімінің бар болуы жөнінде сұрақ қарастырылған. Шекаралық шартты сызықты біртекті теңдеуде және сызықты біртексіз теңдеуде уақыт айнымалысы бойынша көппериодты тербелістердің жеткілікті шарттары анықталған. Конвективті-диффузиялы типті ақырлы-эредитарлы интегралды-дифференциалдық сызықты біртекті және біртексіз теңдеу зерттелген.

Кілт сөздер: интегралды-дифференциалдық, ақырлы-эредитарлы, конвективті, диффузиялы, параболалық типті, дифференциалдық оператор, Фурье қатары.

Ж. А. Сартабанов[1], Г. М. Айтенова[2]

[1] *Актюбинский региональный университет имени К. Жубанова, Актобе, Казахстан;*
[2] *Западно-Казахстанский университет имени М. Утемисова, Уральск, Казахстан*

Ограниченное на полуоси многопериодическое решение линейного конечно-эредитарного интегро-дифференциального уравнения параболического типа

Рассмотрен вопрос о существовании ограниченного на полуоси по пространственной переменной и многопериодического по временным переменным решения линейной интегро-дифференциальной системы параболического типа. Установлены достаточные условия многопериодических колебаний по временным переменным в линейном однородном уравнении с граничным условием и в линейном неоднородном уравнении. Исследованы линейное однородное и неоднородное конечно-эредитарное интегро-дифференциальное уравнения конвективно-диффузионного типа.

Ключевые слова: интегро-дифференциальное, конечно-эредитарное, конвективный, диффузионный, параболический тип, дифференциальный оператор, ряд Фурье.

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