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On outer elements of the noncommutative H_p spaces

In the article let M be a von Neumann algebra equipped with a faithful normal normalized tracial state τ, A be subdiagonal subalgebra of M. We transfer the results of [4] to the case $p<1$.

Key words: subdiagonal algebra, noncommutative Hardy space, inner-outer operators, finite von Neumann algebra.

Let M be a finite von Neumann algebra with a faithful normal tracial state τ. In [1], Arveson introduced the notion of finite, maximal, subdiagonal algebras A of M, as non-commutative analogues of weak-*Dirichlet algebras. Subsequently several authors studied the (non-commutative) H_p spaces associated with such algebras. Blecher and Labuschagne [2] studied outer operators in $H^p(A)(1\le p<\infty)$ (for the case $p<1$, see [3]). In [4] the authors extend their generalized inner-outer factorization theorem in [2] and establish characterizations of outers that are valid even in the case of elements with zero determinant.

In this paper, we will consider extend some results on outer operators in [4] to the case $p<1$, this can be considered as a complement to the work in [4].

This paper is organized as follows. Section 1 contains some preliminary definitions. In section 2, we extend the main results of [4] to the case $0<p<1$.

Preliminaries. Throughout this paper, we denote by M a finite von Neumann algebra on a Hilbert space H with a faithful normal tracial state τ. The closed densely defined linear operator x in H with domain $D(x)$ is said to be affiliated with M if and only if $u^*xu=x$ for all unitary operators u which belong to the commutant M' of M. If x is affiliated with M, then x is said to be τ-measurable if for every $\varepsilon>0$ there exists a projection $e\in M$ such that $e(H)\subseteq D(x)$ and $\tau(e^\perp)<\varepsilon$ (where for any projection e we let $e^\perp=e-1$). The set of all τ-measurable operators will be denoted by $L^0(M;\tau)$ or simply by $L^0(M)$. The set $L^0(M)$ is a *-algebra with sum and product to be the respective closure of the algebraic sum and product.

The measure topology in $L^0(M)$ is given by the system $V(\varepsilon,\delta)=\{x\in L^0(M):\|xe\|_\infty\le\delta$ for some projection $e\in M$ with $\tau(e^\perp)\le\varepsilon\}$, $\varepsilon>0$, $\delta>0$ of neighborhoods of zero.

Given $0<p<\infty$, we define $\|x\|_p=\tau(|x|^p)^{1/p}$, $x\in M$, where $|x|=(x^*x)^{\frac{1}{2}}$. Then $(M,\|\cdot\|_p)$ is a normed (or quasi-normed for $p<1$) space, whose completion is the noncommutative L^p — space associated with (M,τ), satisfying all the expected properties such as duality (see [5, 6]), denoted by $L^p(M,\tau)$ or simply by $L^p(M)$. As usual, we set $L^\infty(M,\tau)=M$ and denote by $\|\cdot\|_\infty(=\|\cdot\|)$ the usual operator norm.

Given a von Neumann subalgebra N of M, an expectation $E:M\to N$ is defined to be a positive linear map which preserves the identity and satisfies $E(xy)=xE(y)$ for all $x\in N$ and $y\in M$. Since E is positive it is hermitian, i.e. $E(x)^*=E(x^*)$ for all $x\in M$. Hence $E(yx)=E(y)x$ for all $x\in N$ and $y\in M$. For a complete study of E, we refer to [1, 7].

Definition 1.1. Let A be a w^*-closed unital subalgebra of M, and let E be a faithful, normal expectation from M onto the diagonal von Neumann algebra $D=A\cap J(A)$. Then A is a finite subdiagonal subalgebra of M with respect to E if:

(*i*) $A+J(A)$ *is w*-dense in* M;

(*ii*) $E(xy)=E(x)E(y)$, $\forall x,y\in A$;

$$(iii)\ \ \tau \circ E = \tau.$$

It is proved by Exel [8] that a finite subdiagonal algebra A is automatically maximal in the sense that if B is another subdiagonal algebra with respect to E containing A, then $B = A$. This maximality yields the following useful characterization of A:

$$A = \{x \in M : \tau(xa) = 0, \forall a \in A_0\},$$

where $A_0 = A \cap \ker E$ (see [1]).

For $p < \infty$ we define $H^p(A)$ to be the closure of A in $L^p(M)$, and for $p = \infty$ we simply set $H^\infty(A) = A$ for convenience. These are the so-called Hardy spaces associated with A. Let K be a subset of $L^p(M)$. We set $J(K) = \{x^* : x \in K\}$ and denote the closed linear span of K in $L^p(M)$ by $[K]_p$. We will keep this notation throughout the paper.

Definition 1.2. Let $0 < p \le \infty$. An operator $h \in H^p(A)$ is called left outer, right outer or bilaterally outer according to $[hA]_p = H^p(A)$, $[Ah]_p = H^p(A)$ or $[AhA]_p = H^p(A)$.

Recall that the Fuglede-Kadison determinant $\Delta(x)$ of an operator $x \in L^p(M)$ $(0 < p \le \infty)$ can be defined by $\Delta(x) = \exp(\tau(\log|x|)) = \exp(\int_0^\infty \log t dv_{|x|}(t))$, where $dv_{|x|}$ denotes the probability measure on R_+ which is obtained by composing the spectral measure of $|x|$ with the trace τ. It is easy to check that $\Delta(x) = \lim\limits_{p \to 0} \|x\|_p$.

As the usual determinant of matrices, Δ is also multiplicative: $\Delta(xy) = \Delta(x)\Delta(y)$. We refer the reader for information on determinant to [1, 2, 9–20].

Outers

Definition 2.1. Let $H^0(A)$ be the closure of A in the topology of convergence in measure. We say that $h \in H^0(A)$ is outer in $H^0(A)$ if hA is dense in $H^0(A)$ with respect to the topology of convergence in measure.

We say that an element $h \in H^0(A)$ is uniform outer in $H^0(A)$ if there is a sequence $a_n \in A$ such that $\{ha_n\}$ is a uniformly bounded sequence in A in operator norm, which converges to 1 in measure.

The following is the extension to the case $p < 1$ of [4] Proposition 2.3.

Proposition 2.2. Let $0 < p < \infty$ and $h \in H^p(A)$. Then h is outer in $H^0(A)$ in the sense above if h is outer in $H^p(A)$.

Proof. The proof is the same as that of [4] Proposition 2.3.

By [3] Theorem 2.1, an argument similar to that of [4] Proposition 2.4 and 2.5, we have the following results.

Proposition 2.3. Let $0 < p < \infty$. Then $h \in H^p(A)$ is outer if and only if $E(h)$ is outer in $L^p(D)$ and $[hA_0]_p = H_0^p(A)$, where $H_0^p(A) = [A_0]_p$.

Proposition 2.4. Let $0 < p < \infty$. Then $h \in H^p(A)$ is outer if and only if $E(h)$ is outer in $L^p(D)$ and $E(h) - h \in [hA_0]_p$.

Definition 2.5. (i) We say that an element $h \in H^p(A)$ is uniform outer in $H^p(A)$ if there exists a sequence $a_n \in A$ such that $\{ha_n\}$ is a uniformly bounded sequence in A in operator norm, and $ha_n \to 1$ in p-norm.

(ii) We say that $h \in H^0(A)$ is uniform outer in $H^0(A)$ if there is a sequence $a_n \in A$ such that $\{ha_n\}$ is a uniformly bounded sequence in A in operator norm, which converges to 1 in measure.

Theorem 2.6. Let $0 < p < \infty$. Suppose that h is outer in $H^p(A)$ and $\Delta(h) \ne 0$. Then h is uniform outer in $H^p(A)$.

Proof. We will use the case $p \ge 1$ already proved in [4]. Thus assume $p < 1$. Choose an integer m such that $np \ge 1$. By [3] Theorem 3.4, there exist $h_1, ..., h_n \in H^{np}(A)$ such that $h = h_1 \cdots h_n$ and

$h_k^{-1} \in A, k = 2,3,\cdots,n.$ Since $\Delta(h_1) \neq 0,$ by [4] Theorem 2.8, there exists a sequence $a_m \in A$ such that $\{h_1 a_m\}$ is a uniformly bounded sequence in A in operator norm, and $h_1 a_m \to 1$ in p-norm. Set $b_m = h_n^{-1} ... h_2^{-1} a_m$. Now $b_m \in A$ such that $\{hb_m\}$ is a uniformly bounded sequence in A in operator norm, and $hb_m \to 1$ in p -norm. Consequently, h is uniform outer.

Lemma 2.7. Let $0 < p < \infty$, $h \in H^p(A)$. If $E(h)$ is outer in $L^p(D)$, then h is of the form $h = ug$ where $g \in H^p(A)$ is outer and $u \in A$ is a unitary. If $\Delta(E(h)) > 0$, then $\Delta(g) > 0$.

Proof. This result is proved in [4] for $p \geq 1$. Let $\frac{1}{p} = \frac{1}{r} + \frac{1}{q}$ and $1 \leq r < \infty$. By [3] Theorem 3.4, there exist $h_1 \in H^r(A)$ and $h_2 \in H^q(A)$ such that $h = h_1 h_2$ and $h_2^{-1} \in A$. Then $E(h) = E(h_1)E(h_2)$ and $E(h_2)^{-1} = E(h_2^{-1})$. Hence $E(h_1)$ is outer in $L^p(D)$. By [4] Lemma 4.1, there are outer $g_1 \in H^r(A)$ and unitary $u \in A$ such that $h_1 = ug_1$. Set $g = g_1 h_2$. Then $g \in H^p(A)$ is outer and $h = ug$. The second part is trivial.

An argument similar to that of [4] Lemma 4.3, we have the following result.

Lemma 2.8. Let $h \in L^p(M)$ be given, where $0 < p < \infty$, and suppose that $\|ah\|_p = \|h\|_p$ for a contraction $a \in M$. Then $h = a^* ah$. If in addition the left support of h is 1, then a is a unitary.

Using Lemma 2.8, [3] Theorem 2.1 and an argument to that of [4] Theorem 4.4, we obtain the following.

Theorem 2.9. Let $h \in L^p(A)$ be given, where $0 < p < \infty$, and let P be the canonical quotient map from $[hA]_p$ to $[hA]_p = [hA_0]_p$. Then h will be outer if and only if $E(h)$ is outer in $L^p(D)$ and $\|E(h)\|_p = \|P(h)\|$.

Theorem 2.10. Let $f \in L^p(M)$ $(0 < P < \infty)$. Then the following conditions are equivalent:

(i) f is of the form $f = uh$ for some outer $h \in H^p(A)$ and a unitary $u \in M$.

(ii) The map $D \to [fA]_p / [fA_0]_p : d \mapsto P(fd)$ is injective, where P is the quotient map $P : [fA]_p \to [fA]_p / [fA_0]_p$.

(iii) $fe \notin [fA_0]_p$ for every nonzero projection e in D.

Proof. $(i) \Rightarrow (ii)$. Let f be of the form $f = uh$ for some outer $h \in H^p(A)$ and a unitary $u \in M$. Then $[fA]_p = u[hA]_p = uH^p(A)$ and $[fA_0]_p = u[hA_0]_p = u[A_0]_p$. Thus the $[fA]_p / [fA_0]_p = (u[A]_p)/(u[A_0]_p) = = uL^p(D)$, which ensures the validity of (ii).

$(ii) \Rightarrow (iii)$. It is trivial.

$(iii) \Rightarrow (i)$. This result is proved in [4] for $p \geq 1$. Let $\frac{1}{p} = \frac{1}{r} + \frac{1}{q}$ and $1 \leq r < \infty$. Then there exist $f_1 \in L^r(M)$ and $f_2 \in L^q(M)$ such that $f^* = f_1^* f_2^*$ and $f_2^{-1} \in M$, so $f = f_2 f_1$. It is clear that $f_1 e \notin [f_1 A_0]_p$ for every nonzero projection e in D. Hence, by [4] Theorem 4.6, there are outer $h_1 \in H^r(A)$ and unitary $v \in M$ such that $f_1 = vh_1$. Let $g_2 = f_2 v$, then $g_2 \in L^q(M)$ and $g_2^{-1} \in M$. By [3] Theorem 3.1, there are $h_2 \in H^q(A)$ and unitary $u \in M$ such that $g_2 = uh_2$ and $h_2^{-1} \in A$. Hence $h_2 h_1$ is outer and $f = uh$.

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Коммутативті емес H_p кеңістігінің сыртқы элементтері

Мақалада тура нормаланған кеңістіктегі және ішкі алгебраның ішкі диагоналі болатын фон Нейман алгебрасы қарастырылды. Авторлар алдыңғы жұмыстарда алынған нәтижелерді қолданды.

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Внешние элементы некоммутативных H_p пространств

В статье рассмотрена алгебра фон Неймана, оснащенная точным нормальным нормированным пространством и являющаяся поддиагональю подалгебры. Авторами были использованы ранее полученные результаты.