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A remark on Schottky representations and Reidemeister torsion

The present paper establishes a formula of Reidemeister torsion for Schottky representations. The theoretical results are applied to 3–manifolds with boundary consisting orientable surfaces with genus at least 2.

Keywords: Schottky representations, Reidemeister torsion, representation varieties, Atiyah-Bott-Goldman symplectic form, Thurston symplectic form.

Introduction

It is well-known that the representation varieties are important in many branches of mathematics and physics. For instance, let Σ be a compact Riemann surface of genus at least 2, Teichmüller space $\text{Teich}(\Sigma)$ of Σ is the space of deformation classes of complex structures on it. By the uniformization Theorem, it is the space of hyperbolic metrics, namely Riemannian metrics on Σ with Gaussian curvature constant (-1) . Furthermore, Teichmüller space of Σ can be interpreted as discrete faithful representations of the fundamental group $\pi_1(\Sigma)$ of the surface to $\text{PSL}(2, \mathbb{R})$. It is well-known that some certain geometric structures on Σ can also be identified as certain surface group variety [1–6] and the references therein.

Representation varieties have a large number of applications in many branches of mathematics and physics such as in 3–manifold topology (in Bass-Culler-Shalen theory [7, 8], in A-polynomial [9], in hyperbolic geometry [10], in Casson invariant theory [11]), in Yang-Mills and Chern-Simons quantum field theories [12, 13], in skein theory of quantum invariants of 3-manifolds [14, 15], in the moduli spaces of flat connections, holomorphic bundles, and Higgs bundles [16].

Reidemeister torsion (R-torsion) is a topological invariant and was introduced by K. Reidemeister [17]. Using this invariant, he classified 3–dimensional lens spaces. W. Franz extended the R-torsion and classified the higher dimensional lens spaces [18]. R-torsion has many applications in several branches of mathematics and theoretical physics such as topology [19], differential geometry [20], representation spaces [21] dynamical systems [22], 3-dimensional Seiberg-Witten theory [23], algebraic K-theory [24], Chern-Simon theory [13], knot theory [24], theoretical physics and quantum field theory [13]. See Refs. [25] and [26] and the references therein for further information.

Real symplectic chain complex is an algebraic topological instrument and was introduced by E. Witten [21]. Combining this and R-torsion, he evaluated the volume of several moduli space of $\text{Rep}(\Sigma, G)$, which is the set of all conjugacy classes of homomorphisms from the fundamental group $\pi_1(\Sigma)$ of a Riemann surface Σ to the compact gauge group $G \in \{\text{SU}(2), \text{SO}(3)\}$.

In paper [27], we considered the set $\text{Rep}(\Sigma, G)$ of G –valued representations from the fundamental group $\pi_1(\Sigma)$ of the surface Σ to the exceptional groups G_2, F_4 , and E_6 . We proved the well-definiteness of R-torsion of such representations. We also established a formula for computing R-torsion of such representations in terms of the well known symplectic structure on $\text{Rep}(\Sigma, G)$, namely, Atiyah-Bott-Goldman symplectic form for the Lie group G . Then, we applied to G –valued Hitchin representations.

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In paper [28], we investigated G -valued representations of free or surface group with genus > 1 for $G \in \{GL(n, \mathbb{C}), SL(n, \mathbb{C})\}$. We also established a formula for computing R-torsion of such representations in terms of Atiyah-Bott-Goldman symplectic form for G . Moreover, we applied the obtained results to hyperbolic 3-manifolds.

In the present paper, we prove a formula of R-torsion for Schottky representations. The theoretical results are applied to 3-manifolds with boundary consisting orientable surfaces with genus at least 2.

1 Preliminaries

In this section, we provide the necessary definition and basic facts about the topological invariant R-torsion and the symplectic chain complex. For further information the reader is referred to [21, 25, 26, 29] and the references therein.

Let $C_* = (0 \rightarrow C_n \xrightarrow{\partial_n} C_{n-1} \rightarrow \dots \rightarrow C_1 \xrightarrow{\partial_1} C_0 \rightarrow 0)$ be a chain complex of finite dimensional vector spaces over the field \mathbb{C} of complex numbers. For $p = 0, \dots, n$, we denote the kernel of ∂_p , the image of ∂_{p+1} , and the p th homology group of the chain complex C_* by $Z_p(C_*)$, $B_p(C_*)$, and $H_p(C_*)$, respectively. From the definition of $Z_p(C_*)$, $B_p(C_*)$, and $H_p(C_*)$ it follows

$$0 \longrightarrow Z_p(C_*) \hookrightarrow C_p \twoheadrightarrow B_{p-1}(C_*) \longrightarrow 0$$

and

$$0 \longrightarrow B_p(C_*) \hookrightarrow Z_p(C_*) \twoheadrightarrow H_p(C_*) \longrightarrow 0.$$

For $p = 0, \dots, n$, if \mathbf{c}_p , \mathbf{b}_p , and \mathbf{h}_p are bases of C_p , $B_p(C_*)$, and $H_p(C_*)$, respectively and if $\ell_p : H_p(C_*) \rightarrow Z_p(C_*)$, $s_p : B_{p-1}(C_*) \rightarrow C_p$ are sections of $Z_p(C_*) \rightarrow H_p(C_*)$, $C_p \rightarrow B_{p-1}(C_*)$, respectively, then with the help of above short-exact sequences we have the basis $\mathbf{b}_p \sqcup \ell_p(\mathbf{h}_p) \sqcup s_p(\mathbf{b}_{p-1})$ of C_p . Here, \sqcup denotes the disjoint union.

Let \mathbf{c}_p , \mathbf{b}_p , \mathbf{h}_p , ℓ_p , and s_p be as above. Then, R -torsion of the chain complex C_* with respect to bases $\{\mathbf{c}_p\}_{p=0}^n$, $\{\mathbf{h}_p\}_{p=0}^n$ is defined by

$$\mathbb{T}(C_*, \{\mathbf{c}_p\}_0^n, \{\mathbf{h}_p\}_0^n) = \prod_{p=0}^n [\mathbf{b}_p \sqcup \ell_p(\mathbf{h}_p) \sqcup s_p(\mathbf{b}_{p-1}), \mathbf{c}_p]^{(-1)^{(p+1)}},$$

where $[\mathbf{e}_p, \mathbf{f}_p]$ denotes determinant of the change-base-matrix from basis \mathbf{f}_p to \mathbf{e}_p of C_p .

R-torsion does not depend on the bases \mathbf{b}_p and sections s_p, ℓ_p [24].

Let \mathbf{c}'_p , \mathbf{h}'_p be also bases of C_p , $H_p(C_*)$, respectively. Then, the following change-base-formula is valid [24]:

$$\mathbb{T}(C_*, \{\mathbf{c}'_p\}_0^n, \{\mathbf{h}'_p\}_0^n) = \prod_{p=0}^n \left(\frac{[\mathbf{c}'_p, \mathbf{c}_p]}{[\mathbf{h}'_p, \mathbf{h}_p]} \right)^{(-1)^p} \mathbb{T}(C_*, \{\mathbf{c}_p\}_0^n, \{\mathbf{h}_p\}_0^n).$$

Let

$$0 \longrightarrow A_* \xrightarrow{i} B_* \xrightarrow{j} D_* \longrightarrow 0 \tag{1}$$

be a short-exact sequence of chain complexes, and let \mathbf{c}_p^A , \mathbf{c}_p^B , \mathbf{c}_p^D , \mathbf{h}_p^A , \mathbf{h}_p^B , and \mathbf{h}_p^D are bases of A_p , B_p , D_p , $H_p(A_*)$, $H_p(B_*)$, and $H_p(D_*)$, respectively. Let us consider the corresponding Mayer-Vietoris long-exact sequence of vector spaces

$$C_* : \dots \longrightarrow H_p(A_*) \xrightarrow{i_p} H_p(B_*) \xrightarrow{j_p} H_p(D_*) \xrightarrow{\delta_p} H_{p-1}(A_*) \longrightarrow \dots$$

associated to short-exact sequence (1). Note that $C_{3p} = H_p(D_*)$, $C_{3p+1} = H_p(A_*)$, and $C_{3p+2} = H_p(B_*)$ then we can consider the bases \mathbf{h}_p^D , \mathbf{h}_p^A , and \mathbf{h}_p^B for C_{3p} , C_{3p+1} , and C_{3p+2} , respectively.

Theorem 1. [24] Suppose $\mathbf{c}_p^A, \mathbf{c}_p^B, \mathbf{c}_p^D, \mathbf{h}_p^A, \mathbf{h}_p^B$, and \mathbf{h}_p^D are as above. Suppose also $[\mathbf{c}_p^B, \mathbf{c}_p^A \oplus \widetilde{\mathbf{c}}_p^D] = \pm 1$, where $j(\widetilde{\mathbf{c}}_p^D) = \mathbf{c}_p^D$. Then, it follows

$$\begin{aligned} & \mathbb{T}\left(B_*, \{\mathbf{c}_p^B\}_0^n, \{\mathbf{h}_p^B\}_0^n\right) = \mathbb{T}\left(A_*, \{\mathbf{c}_p^A\}_0^n, \{\mathbf{h}_p^A\}_0^n\right) \\ & \times \mathbb{T}\left(D_*, \{\mathbf{c}_p^D\}_{p=0}^n, \{\mathbf{h}_p^D\}_0^n\right) \mathbb{T}\left(C_*, \{\mathbf{c}_{3p}\}_0^{3n+2}, \{0\}_0^{3n+2}\right). \end{aligned}$$

Theorem 1 yields the sum-lemma.

Lemma 1. Assume A_*, D_* are chain complexes of vector spaces and $\mathbf{c}_p^A, \mathbf{c}_p^D, \mathbf{h}_p^A$, and \mathbf{h}_p^D are bases of $A_p, D_p, H_p(A_*)$, and $H_p(D_*)$, respectively. Then, the following equality

$$\mathbb{T}(A_* \oplus D_*, \{\mathbf{c}_p^A \sqcup \mathbf{c}_p^D\}_0^n, \{\mathbf{h}_p^A \sqcup \mathbf{h}_p^D\}_0^n) = \mathbb{T}(A_*, \{\mathbf{c}_p^A\}_0^n, \{\mathbf{h}_p^A\}_0^n) \mathbb{T}(D_*, \{\mathbf{c}_p^D\}_0^n, \{\mathbf{h}_p^D\}_0^n)$$

is valid.

The proof of Lemma 1 can also be found in [30].

$(C_*, \partial_*, \{\omega_{*,q-*}\})$ is said to be \mathbb{C} -symplectic chain complex of length q , if

1 $C_* : 0 \rightarrow C_q \xrightarrow{\partial_q} C_{q-1} \rightarrow \cdots \rightarrow C_{q/2} \rightarrow \cdots \rightarrow C_1 \xrightarrow{\partial_1} C_0 \rightarrow 0$ is a chain complex of length q , where $q \equiv 2 \pmod{4}$,

2 for $p = 0, \dots, q$, $\omega_{p,q-p} : C_p \times C_{q-p} \rightarrow \mathbb{C}$ is a ∂ -compatible non-degenerate anti-symmetric bilinear form. Namely,

$$\omega_{p,q-p}(\partial_{p+1}a, b) = (-1)^{p+1} \omega_{p+1,q-(p+1)}(a, \partial_{q-p}b)$$

and

$$\omega_{p,q-p}(a, b) = (-1)^{p(q-p)} \omega_{q-p,p}(b, a).$$

From the fact that $q \equiv 2 \pmod{4}$ we have $\omega_{p,q-p}(a, b)$ is $(-1)^p \omega_{q-p,p}(b, a)$. From ∂ -compatibility of $\omega_{p,q-p}$ we obtain the non-degenerate pairing $[\omega_{p,q-p}] : H_p(C_*) \times H_{q-p}(C_*) \rightarrow \mathbb{C}$.

For the rest of the paper, if the \mathbb{C} -symplectic chain complex $(C_*, \partial_*, \{\omega_{*,q-*}\})$ is clear, then $\Delta(\mathbf{h}_p, \mathbf{h}_{q-p})$ is the determinant of the matrix of the non-degenerate pairing

$$[\omega_{p,q-p}] : H_p(C_*) \times H_{q-p}(C_*) \rightarrow \mathbb{C}$$

in the bases $\mathbf{h}_p, \mathbf{h}_{q-p}$.

Assume C_* is a \mathbb{C} -symplectic chain complex of length q and $\mathbf{c}_p, \mathbf{c}_{q-p}$ are bases of C_p, C_{q-p} , respectively. We say ω -compatible, if the matrix of $\omega_{p,q-p}$ in $\mathbf{c}_p, \mathbf{c}_{q-p}$ is equal to the $k \times k$ identity matrix $\text{Id}_{k \times k}$ when $p \neq q/2$ and $\begin{pmatrix} 0_{l \times l} & \text{Id}_{l \times l} \\ -\text{Id}_{l \times l} & 0_{l \times l} \end{pmatrix}$ when $p = q/2$, where $k = \dim C_p = \dim C_{q-p}$ and $2l = \dim C_{q/2}$.

For computing R-torsion in terms of intersections pairings, we have the following result suggests a formula. Namely,

Theorem 2. [31] If $(C_*, \partial_*, \{\omega_{*,q-*}\})$ is a \mathbb{C} -symplectic chain complex with the ω -compatible bases $\mathbf{c}_p, p = 0, \dots, q$ and if \mathbf{h}_p is a basis of $H_p(C_*)$, $p = 0, \dots, q$, then the following formula holds:

$$|\mathbb{T}(C_*, \{\mathbf{c}_p\}_0^q, \{\mathbf{h}_p\}_0^q)| = \prod_{p=0}^{(q/2)-1} |\Delta(\mathbf{h}_p, \mathbf{h}_{q-p})|^{(-1)^p} \sqrt{|\Delta(\mathbf{h}_{q/2}, \mathbf{h}_{q/2})|}^{(-1)^{q/2}}. \quad (2)$$

In case $\mathbf{h}_p = \mathbf{h}_{q-p} = 0$, the convention $0 = 1.0$ is used and hence $\Delta(\mathbf{h}_p, \mathbf{h}_{q-p}) = 1$. Let us also note that equation (2) can be improved as:

$$\mathbb{T}(C_*, \{\mathbf{c}_p\}_0^q, \{\mathbf{h}_p\}_0^q) = \prod_{p=0}^{(q/2)-1} \Delta(\mathbf{h}_p, \mathbf{h}_{q-p})^{(-1)^p} \sqrt{\Delta(\mathbf{h}_{q/2}, \mathbf{h}_{q/2})}^{(-1)^{q/2}}. \tag{3}$$

For details of (3), we refer the reader to [28; Remark 2.4]. See [27, 28, 30], for further applications of Theorem 2.

2 Main results

Let Σ be a closed orientable surface of genus at least 2 with the universal covering $\tilde{\Sigma}$. Let G be the Lie group $\text{PSL}(2, \mathbb{C})$ and \mathcal{G} be the Lie algebra of G with the non-degenerate symmetric bilinear form B . Here, B is the Killing form.

Assume $\varrho : \pi_1(\Sigma) \rightarrow G$ is a homomorphism from the fundamental group $\pi_1(\Sigma)$ of Σ to G . Let $E_\varrho = \tilde{\Sigma} \times \mathcal{G} / \sim$ be the corresponding adjoint bundle over Σ . Here, $(x_1, t_1) \sim (x_2, t_2)$, if $(x_2, t_2) = (\gamma \cdot x_1, \gamma \cdot t_1)$ for some $\gamma \in \pi_1(\Sigma)$, the action of γ in the first component by deck transformation ($\gamma \cdot x_1 = \gamma(x_1)$) and in the second component by the adjoint action ($\gamma \cdot t_1 = \text{Ad}_{\varrho(\gamma)}(t_1) = \varrho(\gamma) t_1 \varrho(\gamma)^{-1}$).

Let K be a cell-decomposition of Σ for which the adjoint bundle E_ϱ is trivial over each cell and \tilde{K} be the lift of K to the $\tilde{\Sigma}$. Denote by $\mathbb{Z}[\pi_1(\Sigma)]$ the integral group ring. Let $C_*(K; \mathcal{G}_{\text{Ad}_\varrho}) = C_*(\tilde{K}; \mathbb{Z}) \otimes \mathcal{G} / \sim$, where for all $\gamma \in \pi_1(\Sigma)$, $\sigma \otimes t \sim \gamma \cdot \sigma \otimes \gamma \cdot t$, the action of γ by the first component is by deck transformation and in the second is by adjoint action. We have the following chain complex:

$$0 \longrightarrow C_2(K; \mathcal{G}_{\text{Ad}_\varrho}) \xrightarrow{\partial_2 \otimes \text{id}} C_1(K; \mathcal{G}_{\text{Ad}_\varrho}) \xrightarrow{\partial_1 \otimes \text{id}} C_0(K; \mathcal{G}_{\text{Ad}_\varrho}) \longrightarrow 0. \tag{4}$$

Here, ∂_p denotes the usual boundary operator. Denote by $H_*(K; \mathcal{G}_{\text{Ad}_\varrho})$ and $H^*(K; \mathcal{G}_{\text{Ad}_\varrho})$ the homologies and cohomologies of the chain complex (4), respectively, where $C^*(K; \mathcal{G}_{\text{Ad}_\varrho})$ denotes the set of $\mathbb{Z}[\pi_1(\Sigma)]$ -module homomorphisms from $C_*(\tilde{K}; \mathbb{Z})$ to \mathcal{G} . See [25] for details and unexplained subjects.

Clearly, for conjugate $\varrho, \varrho' : \pi_1(\Sigma) \rightarrow G$ i.e. $\varrho'(\cdot) = A\varrho(\cdot)A^{-1}$ for some $A \in G$, we have isomorphic $C_*(K; \mathcal{G}_{\text{Ad}_\varrho})$ and $C_*(K; \mathcal{G}_{\text{Ad}_{\varrho'}})$. Similarly, the corresponding cochains $C^*(K; \mathcal{G}_{\text{Ad}_\varrho})$ and $C^*(K; \mathcal{G}_{\text{Ad}_{\varrho'}})$ are isomorphic.

Consider chain complex (4). Assume $\{e_j^p\}_{j=1}^{m_p}$ is a basis of $C_p(K; \mathbb{Z})$. For $j = 1, \dots, m_p$, fix a lift \tilde{e}_j^p of e_j^p . Then, $\mathbf{c}_p = \{\tilde{e}_j^p\}_{j=1}^{m_p}$ of $C_p(\tilde{K}; \mathbb{Z})$ is a $\mathbb{Z}[\pi_1(\Sigma)]$ -basis. Assume $\mathcal{A} = \{a_k\}_{k=1}^{\dim \mathcal{G}}$ is a B -orthonormal basis of the Lie algebra \mathcal{G} . Namely, the matrix of the form B equals to the identity matrix of size $\dim \mathcal{G}$. Hence, we obtain a \mathbb{C} -basis $\mathbf{c}_p = c_p \otimes_\varrho \mathcal{A}$ of $C_p(K; \mathcal{G}_{\text{Ad}_\varrho})$. We call such a basis a *geometric basis* for $C_p(K; \mathcal{G}_{\text{Ad}_\varrho})$.

If $\mathbf{c}_p = c_p \otimes_\varrho \mathcal{A}$ and \mathbf{h}_p are respectively the geometric basis of $C_p(K; \mathcal{G}_{\text{Ad}_\varrho})$ and a basis of $H_p(\Sigma; \mathcal{G}_{\text{Ad}_\varrho})$, then $\mathbb{T}(C_*(K; \mathcal{G}_{\text{Ad}_\varrho}), \{c_p \otimes_\varrho \mathcal{A}\}_{p=0}^2, \{\mathbf{h}_p\}_{p=0}^2)$ is said to be the *R-torsion* of the triple K, Ad_ϱ , and $\{\mathbf{h}_p\}_{p=0}^2$.

Theorem 3. [28; Theorem 3.1] If $\Sigma, K, \varrho, \mathbf{c}_p = c_p \otimes_\varrho \mathcal{A}$, and $\mathbf{h}_p, p = 0, 1, 2$, are as above, then $\mathbb{T}(C_*(K; \mathcal{G}_{\text{Ad}_\varrho}), \{c_p \otimes_\varrho \mathcal{A}\}_{p=0}^2, \{\mathbf{h}_p\}_{p=0}^2)$ does not depend on the basis \mathcal{A} , lifts \tilde{e}_j^p , conjugacy class of ϱ , and the cell-decomposition K .

From Theorem 3, we have the well-definiteness of R-torsion of such representations, and hence we write $\mathbb{T}(\Sigma, \{\mathbf{h}_p\}_{p=0}^2)$ rather than $\mathbb{T}(C_*(K; \mathcal{G}_{\text{Ad}_\varrho}), \{c_p \otimes_\varrho \mathcal{A}\}_{p=0}^2, \{\mathbf{h}_p\}_{p=0}^2)$.

Assume $\Sigma, K, G, \mathcal{G}, \varrho, \mathbf{c}_p = c_p \otimes_{\varrho} \mathcal{A}$ are as above. Let us consider the dual cell-decomposition K' of Σ corresponding to the cell-decomposition K . Consider the lifts \tilde{K} and \tilde{K}' of K and K' , respectively. For $i = 0, 1, 2$, we have the intersection form

$$(\cdot, \cdot)_{i,2-i} : C_i(K; \mathcal{G}_{\text{Ad}_{\varrho}}) \times C_{2-i}(K'; \mathcal{G}_{\text{Ad}_{\varrho}}) \longrightarrow \mathbb{C} \tag{5}$$

defined by $(\sigma_1 \otimes t_1, \sigma_2 \otimes t_2)_{i,2-i} = \sum_{\gamma \in \pi_1(\Sigma)} \sigma_1 \cdot (\gamma \bullet \sigma_2) \cdot B(t_1, \gamma \bullet t_2)$. Here, “ \cdot ” denotes the intersection number pairing, the action of γ on σ_2 by deck transformation and on t_2 is by the adjoint action.

Using the anti-symmetric, ∂ -compatible $(\cdot, \cdot)_{i,2-i}$, we have the non-degenerate anti-symmetric form

$$[\cdot, \cdot]_{i,2-i} : H_i(\Sigma; \mathcal{G}_{\text{Ad}_{\varrho}}) \times H_{2-i}(\Sigma; \mathcal{G}_{\text{Ad}_{\varrho}}) \longrightarrow \mathbb{C}. \tag{6}$$

Note that if $D_i = C_i(K; \mathcal{G}_{\text{Ad}_{\varrho}}) \oplus C_i(K'; \mathcal{G}_{\text{Ad}_{\varrho}})$, and if we consider the bilinear form $\omega_{i,2-i} : D_i \times D_{2-i} \rightarrow \mathbb{C}$ defined by extending the intersection form (5) zero on $C_i(K; \mathcal{G}_{\text{Ad}_{\varrho}}) \times C_{2-i}(K; \mathcal{G}_{\text{Ad}_{\varrho}})$ and $C_i(K'; \mathcal{G}_{\text{Ad}_{\varrho}}) \times C_{2-i}(K'; \mathcal{G}_{\text{Ad}_{\varrho}})$, then D_* becomes a \mathbb{C} -symplectic chain complex. Note also that the bases c_i of $C_i(\tilde{K}; \mathbb{Z})$ and c'_i of $C_i(\tilde{K}'; \mathbb{Z})$ corresponding to c_i result an ω -compatible basis for D_* .

Kronecker pairing $\langle \cdot, \cdot \rangle : C^i(K; \mathcal{G}_{\text{Ad}_{\varrho}}) \times C_i(K; \mathcal{G}_{\text{Ad}_{\varrho}}) \rightarrow \mathbb{C}$ is defined by $\langle \theta, \sigma \otimes_{\varrho} t \rangle = B(t, \theta(\sigma))$. It has natural extended to $\langle \cdot, \cdot \rangle : H^i(\Sigma; \mathcal{G}_{\text{Ad}_{\varrho}}) \times H_i(\Sigma; \mathcal{G}_{\text{Ad}_{\varrho}}) \rightarrow \mathbb{C}$.

Recall the cup product $\cup : C^i(K; \mathcal{G}_{\text{Ad}_{\varrho}}) \times C^j(K; \mathcal{G}_{\text{Ad}_{\varrho}}) \rightarrow C^{i+j}(\tilde{\Sigma}; \mathbb{C})$ is defined by $(\theta_i \cup \theta_j)(\sigma_{i+j}) = B(\theta_i((\sigma_{i+j})_{\text{front}}), \theta_j((\sigma_{i+j})_{\text{back}}))$. Here, σ_{i+j} is in $C_{i+j}(\tilde{K}; \mathbb{Z})$ and \tilde{K} denotes the lift of K to $\tilde{\Sigma}$ $\theta_i : C_i(\tilde{K}; \mathbb{Z}) \rightarrow \mathcal{G}, \theta_j : C_j(\tilde{K}; \mathbb{Z}) \rightarrow \mathcal{G}$ are $\mathbb{Z}[\pi_1(\Sigma)]$ -module homomorphisms. This yields the cup product

$$\smile_B : C^i(K; \mathcal{G}_{\text{Ad}_{\varrho}}) \times C^j(K; \mathcal{G}_{\text{Ad}_{\varrho}}) \longrightarrow C^{i+j}(K; \mathbb{C})$$

with natural extension

$$\smile_B : H^i(\Sigma; \mathcal{G}_{\text{Ad}_{\varrho}}) \times H^j(\Sigma; \mathcal{G}_{\text{Ad}_{\varrho}}) \longrightarrow H^{i+j}(\Sigma; \mathbb{C}),$$

where $[\theta_i] \smile_B [\theta_j] = [\theta_i \smile_B \theta_j]$.

Using the isomorphisms by (6) and the Kronecker pairing, we get the Poincare duality isomorphisms

$$\text{PD} : H_i(\Sigma; \mathcal{G}_{\text{Ad}_{\varrho}}) \cong H_{2-i}(\Sigma; \mathcal{G}_{\text{Ad}_{\varrho}})^* \cong H^{2-i}(\Sigma; \mathcal{G}_{\text{Ad}_{\varrho}}).$$

For $i = 0, 1, 2$ we have the

$$\begin{array}{ccc} H^{2-i}(\Sigma; \mathcal{G}_{\text{Ad}_{\varrho}}) \times H^i(\Sigma; \mathcal{G}_{\text{Ad}_{\varrho}}) & \xrightarrow{\smile_B} & H^2(\Sigma; \mathbb{C}) \\ \uparrow \text{PD} & \uparrow \text{PD} & \uparrow \\ H_i(\Sigma; \mathcal{G}_{\text{Ad}_{\varrho}}) \times H_{2-i}(\Sigma; \mathcal{G}_{\text{Ad}_{\varrho}}) & \xrightarrow{[\cdot, \cdot]_{i,2-i}} & \mathbb{C}. \end{array}$$

Here, $\mathbb{C} \rightarrow H^2(\Sigma; \mathbb{C})$ sends $1 \in \mathbb{C}$ to the fundamental class of $H^2(\Sigma; \mathbb{C})$ and the inverse of this is integration over Σ .

Clearly, we have the following pairing

$$\Omega_{i,2-i} : H^i(\Sigma; \mathcal{G}_{\text{Ad}_{\varrho}}) \times H^{2-i}(\Sigma; \mathcal{G}_{\text{Ad}_{\varrho}}) \xrightarrow{\smile_B} H^2(\Sigma; \mathbb{C}) \xrightarrow{\int_{\Sigma}} \mathbb{C}. \tag{7}$$

$\Omega_{1,1}$ is called Atiyah-Bott-Goldman symplectic form for G on the representation variety $\text{Rep}(\Sigma, G)$.

In [28], we established a formula for computing Reidemeister torsion of representations in terms of $\Omega_{1,1}$ Atiyah-Bott-Goldman symplectic form for the Lie group G . More precisely,

Theorem 4. [28; Theorem 3.2] Let Σ, K, K', ϱ be as above. Let \mathbf{c}_p and \mathbf{c}'_p be the corresponding geometric bases of $C_p(K; \mathcal{G}_{\text{Ad}_\varrho})$ and $C_p(K'; \mathcal{G}_{\text{Ad}_\varrho})$, respectively, $p = 0, 1, 2$. If \mathbf{h}_p is a basis of $H_p(\Sigma; \mathcal{G}_{\text{Ad}_\varrho})$, $p = 0, 1, 2$, then the following formulas are valid

- i. $\mathbb{T}(\Sigma, \{\mathbf{h}_p\}_{p=0}^2) = ie^{\frac{i\theta}{2}} \frac{\Delta(\mathbf{h}_0, \mathbf{h}_2)}{\sqrt{\Delta(\mathbf{h}_1, \mathbf{h}_1)}}$,
- ii. $\mathbb{T}(\Sigma, \{\mathbf{h}_p\}_{p=0}^2) = ie^{\frac{i\theta}{2}} \frac{\sqrt{\delta(\mathbf{h}^1, \mathbf{h}^1)}}{\delta(\mathbf{h}^2, \mathbf{h}^0)}$.

Here, $\Delta(\mathbf{h}_p, \mathbf{h}_{2-p})$ is the determinant of the matrix of (6) in \mathbf{h}_p and \mathbf{h}_{2-p} , $\Delta(\mathbf{h}_0, \mathbf{h}_2) = |\Delta(\mathbf{h}_0, \mathbf{h}_2)| e^{i\theta}$, where $i = \sqrt{-1}$ and $-\pi < \theta \leq \pi$. $\delta(\mathbf{h}^{2-p}, \mathbf{h}^p)$ is the determinant of the matrix of (7) in \mathbf{h}^p and \mathbf{h}^{2-p} , and \mathbf{h}^p denotes the Poincare dual basis of $H^p(\Sigma; \mathcal{G}_{\text{Ad}_\varrho})$ corresponding to \mathbf{h}_p of $H_p(\Sigma; \mathcal{G}_{\text{Ad}_\varrho})$, $p = 0, 1, 2$.

Note that in case $H_0(\Sigma; \mathcal{G}_{\text{Ad}_\varrho})$ and thus $H_2(\Sigma; \mathcal{G}_{\text{Ad}_\varrho})$ are zero, by Theorem 4 we get

$$\mathbb{T}(\Sigma, \{0, \mathbf{h}_1, 0\}) = i \sqrt{\Delta(\mathbf{h}_1, \mathbf{h}_1)}^{(-1)} = i \sqrt{\delta(\mathbf{h}^1, \mathbf{h}^1)}.$$

3 Applications

Schottky representation and Thurston symplectic form

Before stating our application, let us recall Thurston symplectic form. For more information and unexplained subjects, we refer [32] and the references therein.

Let $\Sigma_g, g \geq 2$, be a closed orientable surface. We say that $\lambda \subset \Sigma_g$ is a *geodesic lamination*, if it is closed and also consists of disjoint complete geodesics without any self-intersection points, called *leaves* of λ (see Figure 1 (a)). We say that the geodesic lamination λ is *maximal*, if the complement $\Sigma_g - \lambda$ consists of finitely many ideal triangles, that is, triangles with vertices at infinity (see Figure 1 (b)).

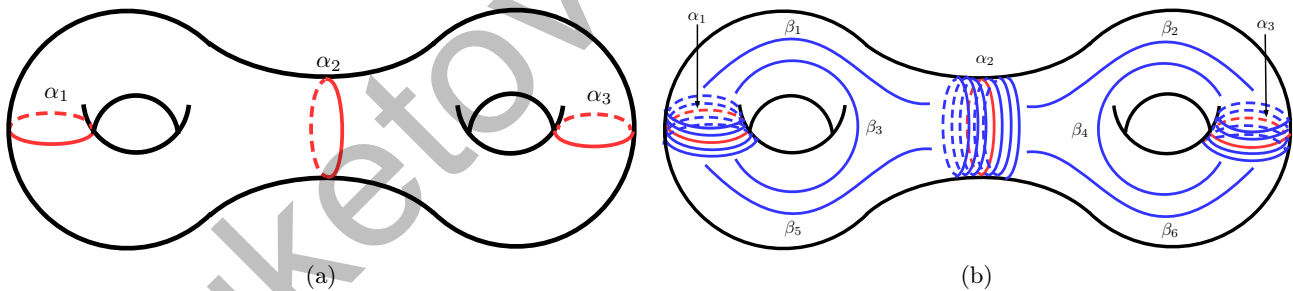


Figure 1. (a) Geodesic lamination with 3 closed leaves (b) Maximal geodesic lamination with 3 closed leaves and 6 infinite leaves spiraling towards closed leaves.

Let $\lambda \subset \Sigma_g$ be a geodesic lamination and G be an abelian group. A G -valued *transverse cocycle* σ for λ is a function from the set of all transverse arcs to the leaves of λ to G so that σ is finitely additive and invariant under the homotopy of arcs transverse to λ . To be more precise, $\sigma(k) = \sigma(k_1) + \sigma(k_2)$, when the arc k transverse to leaves of λ is decomposed into two subarcs k_1, k_2 with disjoint interiors, and $\sigma(k) = \sigma(k')$ when the transverse arc k is deformed to arc k' through arcs transverse to the leaves of the geodesic lamination λ (Fig. 2). Let us denote the group of G -valued transverse cocycles for λ by $\mathcal{H}(\lambda; G)$. In the case λ is a maximal geodesic lamination and $G = \mathbb{R}, \mathbb{C}$, or $\mathbb{R}/2\pi\mathbb{Z}$, $\mathcal{H}(\lambda; G)$ is isomorphic to G^{6g-6} [33]. For example, by using a (fattened) train-track $\Phi \subset \Sigma_g$ carrying the lamination λ , one gets the isomorphism $\mathcal{H}(\lambda; \mathbb{R}) \cong \mathbb{R}^{6g-6}$.

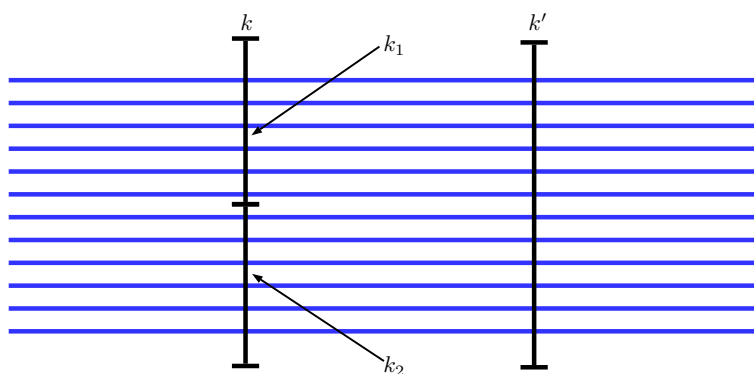


Figure 2. The arcs k and k' are transverse to the leaves of lamination λ . The arc k is deformed to k' through arcs transverse to the leaves of the geodesic lamination. Moreover, k is splitted into two transverse subarcs k_1, k_2 with disjoint interiors.

Recall that a *train-track* $\Phi \subset \Sigma_g$ is composed of finitely many “long” rectangles e_1, \dots, e_n , called *edges* of Φ , foliated by arcs parallel to the “short” sides and meeting only along arcs (possibly reduced to a point) lying in their short sides. Furthermore, each point of the “short” side of a rectangle is also contained in another rectangle, each component of the union of the short sides of all rectangles is an arc, as opposed to a closed curve, and finally since the closure $\overline{\Sigma_g - \Phi}$ of the complement $\Sigma_g - \Phi$ has a certain number of “spikes”, corresponding to the points where at least 3 rectangles meet, it is also required that no component of $\overline{\Sigma_g - \Phi}$ be a disc with 0, 1 or 2 spikes or an annulus with no spike.

Note that foliating the edges of the train-track Φ by using the short sides, we get a foliation of Φ , and the leaves are called the *ties* of Φ . The finitely many ties where several edges meet are said to be the *switches* of Φ . If a tie is not a switch, then it is called a *generic tie*. If λ lies entirely in the interior of Φ and if, moreover, the leaves of λ are transverse to the ties of Φ , then λ is said to be *carried* by Φ (Fig. 3). We refer [34] for constructions of a train-track.

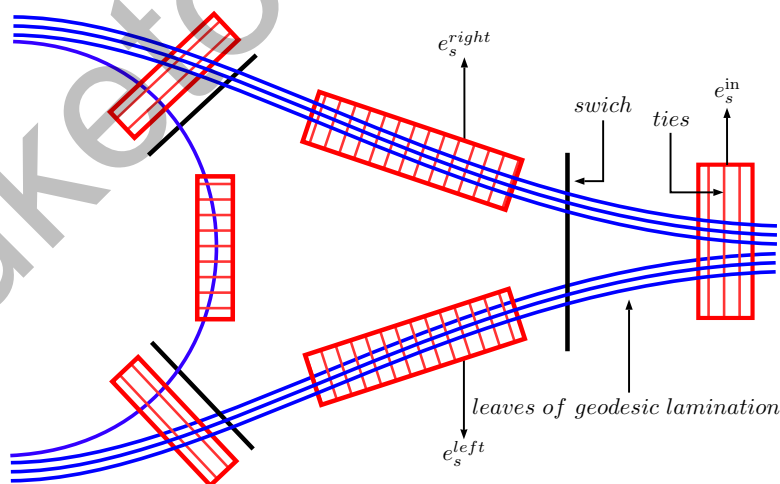


Figure 3. Locally a train-track carries a geodesic lamination.

Suppose $\Phi \subset \Sigma_g$ is a train-track. A real-valued function from the set of edges of Φ is called an *edge*

weight system for Φ , if it satisfies the *switch relation*. Namely, for each switch s of Φ , let e_1, \dots, e_p be the edges adjacent to one side of s and let e_{p+1}, \dots, e_{p+q} be the edges adjacent to the other side, we have $\sum_{i=1}^p a(e_i) = \sum_{j=p+1}^{p+q} a(e_j)$. Let us denote the real vector space of all edge weight systems for Φ by $\mathcal{W}(\Phi; \mathbb{R})$.

Let $\lambda \subset \Sigma_g$ be a geodesic lamination carried by the train-track Φ . Consider the injective map associating each transverse cocycle $\sigma \in \mathcal{H}(\lambda; \mathbb{R})$ to the edge weight system $a_\sigma \in \mathcal{W}(\Phi; \mathbb{R})$ defined by $a_\sigma(e) = \sigma(k_e)$. Here, k_e is a tie of e . In the case of maximal lamination λ , the map is an isomorphism $\mathcal{H}(\lambda; \mathbb{R}) \cong \mathcal{W}(\Phi; \mathbb{R})$ [33].

One can arrange the train-track Φ so that at each switch s of Φ , there are one incoming edge e_s^{in} touching the switch s on one side and two outgoing edges $e_s^{\text{left}}, e_s^{\text{right}}$ touching s on the other side, where as seen from the incoming edge e_s^{in} and for the orientation of the surface Σ_g , e_s^{left} branches out to the left and e_s^{right} branches out to the right. *Thurston symplectic form* on $\mathcal{W}(\Phi)$ is the anti-symmetric bilinear form $\omega_{\text{Thurston}} : \mathcal{W}(\Phi; \mathbb{R}) \times \mathcal{W}(\Phi; \mathbb{R}) \rightarrow \mathbb{R}$ defined by

$$\omega_{\text{Thurston}}(a, b) = \frac{1}{2} \sum_s \det \begin{bmatrix} a(e_s^{\text{left}}) & a(e_s^{\text{right}}) \\ b(e_s^{\text{left}}) & b(e_s^{\text{right}}) \end{bmatrix},$$

where the summation is over all switches of Φ .

By using the isomorphism $\mathcal{H}(\lambda; \mathbb{R}) \cong \mathcal{W}(\Phi; \mathbb{R})$, we have the *Thurston symplectic form* $\omega_{\text{Thurston}} : \mathcal{H}(\lambda; \mathbb{R}) \times \mathcal{H}(\lambda; \mathbb{R}) \rightarrow \mathbb{R}$. As is well known that ω_{Thurston} is an algebraic intersection number and is independent of Φ [32, 34].

Recall that Teichmüller space $\text{Teich}(\Sigma_g)$ of the surface Σ_g is the space of isotopy classes of complex structures on Σ_g . By The Uniformization Theorem, it is the space of isotopy classes of Riemannian metrics with constant Gaussian curvature (-1) , that is, hyperbolic metrics on Σ_g . One can also identify it with the space of conjugacy classes of all discrete faithful homomorphisms from the fundamental group $\pi_1(\Sigma_g)$ to $\text{PSL}(2, \mathbb{R})$. With the help of a maximal geodesic lamination $\lambda \subset \Sigma_g$ and sending to each hyperbolic metric $m \in \text{Teich}(\Sigma_g)$ the corresponding *shearing cocycle* $\sigma_m \in \mathcal{H}(\lambda; \mathbb{R})$, F. Bonahon embedded $\text{Teich}(\Sigma_g)$ as an open cone $\mathcal{C}(\lambda) \subset \mathcal{H}(\lambda; \mathbb{R})$ [32]. If k is an arc transverse to λ , the shearing cocycle $\sigma_m(k)$ measures the “shift to the left” between the two ideal triangles in $\mathbb{H}^2/\varrho_m(\pi_1(\Sigma_g))$ corresponding to the components of $\Sigma_g - \lambda$ containing the endpoints of k . Here, $\varrho_m : \pi_1(\Sigma_g) \rightarrow \text{PSL}(2, \mathbb{R})$ is the discrete faithful representation associated to m .

Recall that for a homomorphism $\varrho : \pi_1(\Sigma_g) \rightarrow \text{PSL}(2, \mathbb{C})$, there is the following commutative diagram

$$\begin{array}{ccc} H^1(\Sigma_g; \mathfrak{sl}(2, \mathbb{C})_{\text{Ad}_\varrho}) \times H^1(\Sigma_g; \mathfrak{sl}(2, \mathbb{C})_{\text{Ad}_\varrho}) & \xrightarrow{\sim_{\mathbb{R}}} & H^2(\Sigma_g; \mathbb{C}) \\ \uparrow \text{PD} & & \uparrow \circlearrowleft \\ H_1(\Sigma_g; \mathfrak{sl}(2, \mathbb{C})_{\text{Ad}_\varrho}) \times H_1(\Sigma_g; \mathfrak{sl}(2, \mathbb{C})_{\text{Ad}_\varrho}) & \xrightarrow{[\cdot, \cdot]_{1,1}} & \mathbb{C}. \end{array} \tag{8}$$

Here, $\mathbb{C} \rightarrow H^2(\Sigma_g; \mathbb{C})$ is the isomorphism sending $1 \in \mathbb{C}$ to the fundamental class of $H^2(\Sigma_g; \mathbb{C})$.

Recall also that

$$\omega_{\text{PSL}(2, \mathbb{C})} : H^1(\Sigma_g; \mathfrak{sl}(2, \mathbb{C})_{\text{Ad}_\varrho}) \times H^1(\Sigma_g; \mathfrak{sl}(2, \mathbb{C})_{\text{Ad}_\varrho}) \xrightarrow{\sim_{\mathbb{R}}} H^2(\Sigma_g; \mathbb{C}) \xrightarrow{\int_{\Sigma_g}} \mathbb{C}$$

is called Atiyah-Bott-Goldman symplectic form for $\text{PSL}(2, \mathbb{C})$ [35]. It is known that $\omega_{\text{PSL}(2, \mathbb{C})}$ is related with the Goldman symplectic form on $\text{Teich}(\Sigma_g)$

$$\omega_{\text{Goldman}} : H^1(\Sigma_g; \mathfrak{sl}(2, \mathbb{R})_{\text{Ad}_\varrho}) \times H^1(\Sigma_g; \mathfrak{sl}(2, \mathbb{R})_{\text{Ad}_\varrho}) \xrightarrow{\sim_{\mathbb{R}}} H^2(\Sigma_g; \mathbb{R}) \xrightarrow{\int_{\Sigma_g}} \mathbb{R}.$$

Here, $B_{\mathbb{R}}$ is the Killing form of the set $\mathfrak{sl}(2, \mathbb{R})$, which is 2×2 trace zero matrices over \mathbb{R} .

In [31], considering the isomorphism $T_{\varrho} \text{Teich}(\Sigma_g) \cong \mathcal{H}(\lambda; \mathbb{R})$, which is obtained by the real-analytical parameterization of F. Bonahon [32] and complexifying ω_{Thurston} , it was proved that

$$\omega_{\text{PSL}(2, \mathbb{C})} = 2\omega_{\text{T}}. \tag{9}$$

Here,

$$\omega_{\text{T}} : \mathcal{H}(\lambda; \mathbb{C}) \times \mathcal{H}(\lambda; \mathbb{C}) \rightarrow \mathbb{C} \tag{10}$$

is the complexified Thurston symplectic form.

For more information and unexplained subjects, we refer the reader to [31] and the references therein.

For a fixed $g \geq 2$, let us consider the free group F_g with generators $X = \{x_1, \dots, x_g\}$. The set $\text{Hom}(F_g, \text{PSL}(2, \mathbb{C}))$ of all homomorphisms from F_g to $\text{PSL}(2, \mathbb{C})$ can be identified with $\text{PSL}(2, \mathbb{C})^g$ by considering the map $\varrho \mapsto (\varrho(x_1), \dots, \varrho(x_g))$.

Let $\chi(F_g, \text{PSL}(2, \mathbb{C}))$ be the quotient $\text{Hom}(F_g, G) // G$. As is well known that $\chi(F_g, \text{PSL}(2, \mathbb{C}))$ naturally has the structure of an algebraic variety and it differs from the set theoretical quotient $\text{Hom}(F_g, \text{PSL}(2, \mathbb{C})) / \text{PSL}(2, \mathbb{C})$ only at reducible points, namely, representations whose images fix a point on $\widehat{\mathbb{C}}$ [36]. Let $\mathcal{D}(F_g, \text{PSL}(2, \mathbb{C}))$ and $\mathcal{E}(F_g, \text{PSL}(2, \mathbb{C}))$ denote respectively the set of all discrete, faithful representations and those of representations with dense image in $\text{PSL}(2, \mathbb{C})$. It is well known $\mathcal{E}(F_g, \text{PSL}(2, \mathbb{C}))$ is not empty and open, $\mathcal{D}(F_g, \text{PSL}(2, \mathbb{C}))$ is closed and outside of these representations in $\chi(F_g, \text{PSL}(2, \mathbb{C}))$ has measure zero [37] and the references therein.

Let $A_i, B_i, i = 1, \dots, g$, be $2g$ disjoint closed (topological) disks in $\partial\mathbb{H}^3$ and let $\gamma_1, \dots, \gamma_g \in \text{PSL}(2, \mathbb{C})$ be the Möbius transformations of the Riemann sphere $\widehat{\mathbb{C}}$ so that $\gamma_i(A_i)$ is the closure of the complement of B_i . The set $\{\gamma_1, \dots, \gamma_g\}$ generate a free discrete group of rank g , called a *Schottky group*. The representation ϱ obtained by $x_i \mapsto \gamma_i$ is in $\mathcal{D}(F_g, \text{PSL}(2, \mathbb{C}))$. Let $\mathcal{S}(F_g, \text{PSL}(2, \mathbb{C}))$ be the set of Schottky representations. As is well known that $\mathcal{S}(F_g, \text{PSL}(2, \mathbb{C}))$ lies in the interior of $\mathcal{D}(F_g, \text{PSL}(2, \mathbb{C}))$ [38].

In [39], Y. Minsky proved the existence of an open set $\mathcal{M}(F_g, \text{PSL}(2, \mathbb{C}))$ of $\chi(F_g, \text{PSL}(2, \mathbb{C}))$ which is strictly larger than $\mathcal{S}(F_g, \text{PSL}(2, \mathbb{C}))$ and on which $\text{Out}(F_g)$ acts properly discontinuously. We have

Theorem 5. Let F_g denote the fundamental group $\pi_1(H_g)$ of handle body H_g of genus $g \geq 2$ with boundary Σ_g , and let M denote the double of H_g . Suppose $\lambda \subset \Sigma_g$ is a fixed maximal geodesic lamination and $\varrho \in \mathcal{M}(F_g, \text{PSL}(2, \mathbb{C}))$ is such that $\varrho \circ r \in \text{Teich}(\Sigma_g)$. Let $\mathbf{h}_i^{F_g}$ be bases for $H_i(F_g; \mathfrak{sl}(2, \mathbb{C})_{Ad_{\varrho}})$, $i = 0, 1, 2, 3$. Then, there exist basis \mathbf{h}_j^M and $\mathbf{h}_k^{\Sigma_g}$ of $H_j(M; \mathfrak{sl}(2, \mathbb{C})_{Ad_{\varrho}})$ and $H_k(\Sigma_g; \mathfrak{sl}(2, \mathbb{C})_{Ad_{\varrho \circ r}})$, $j = 0, 1, 2, 3, k = 0, 1, 2$, respectively so that Reidemeister torsion of the corresponding Mayer-Vietoris long exact sequence \mathcal{H}_{\star} in these bases is 1. In addition, the following formula holds:

$$\mathbb{T}\left(F_g, \left\{\mathbf{h}_i^{F_g}\right\}_0^3\right) = e^{\frac{\sqrt{-1}}{4}(-\beta_0 + \pi - \theta_1)} 2^{\frac{\chi(\Sigma_g; \mathfrak{sl}(2, \mathbb{C}))}{4}} \sqrt[4]{\Omega_{\text{T}}}.$$

Here, $\beta_0 = \dim H_0(M; \mathfrak{sl}(2, \mathbb{C})_{Ad_{\varrho}})$, $\mathbf{h}_{1,1}^0$ is a basis of $H_1(\Sigma_g; \mathfrak{sl}(2, \mathbb{C})_{Ad_{\varrho \circ r}}) \oplus H_1(\Sigma_g; \mathfrak{sl}(2, \mathbb{C})_{Ad_{\varrho \circ r}})$ such that $\mathbb{T}(C_{\star}(\Sigma_g; \mathfrak{sl}(2, \mathbb{C})_{Ad_{\varrho \circ r}}) \oplus C_{\star}(\Sigma_g; \mathfrak{sl}(2, \mathbb{C})_{Ad_{\varrho \circ r}}), \{\mathbf{h}_{1,1}^0\})$ is equal to 1, $[\mathbf{h}_{1,1}^0, \mathbf{h}_1^{\Sigma_g} \oplus \mathbf{h}_1^{\Sigma_g}] = \left|[\mathbf{h}_{1,1}^0, \mathbf{h}_1^{\Sigma_g} \oplus \mathbf{h}_1^{\Sigma_g}]\right| e^{\sqrt{-1}\theta_1}$. Here, $\chi(\Sigma_g; \mathfrak{sl}(2, \mathbb{C}))$ is $\chi(\Sigma_g) \dim_{\mathbb{C}} \mathfrak{sl}(2, \mathbb{C})$, Ω_{T} is determinant of the matrix of the symplectic form (10) in the basis $\mathfrak{h} \oplus \sqrt{-1}\mathfrak{h}$, \mathfrak{h} is the basis of $\mathcal{H}(\lambda; \mathbb{R})$ associated with the isomorphism obtained by the embedding $\text{Teich}(\Sigma_g) \hookrightarrow \mathcal{H}(\lambda; \mathbb{R})$ [32], and \mathbf{h}^1 is the Poincare dual

basis of $H^1\left(\Sigma_g; \mathfrak{sl}(2, \mathbb{C})_{Ad_{\varrho_{or}}}\right)$ corresponding to $\mathbf{h}_1^{\Sigma_g}$. Here, $r : \pi_1(\Sigma_g) \rightarrow \pi_1(F_g)$ is the homomorphism obtained by the embedding $\Sigma_g \hookrightarrow F_g$.

The proof of Theorem 5 is based on combining Theorem 4 and [28; Theorem 4.2], and the above results, using the commutative diagram (8), Eq. (9), and the definition of $\omega_{\text{PSL}(2, \mathbb{C})}$.

Let us now apply [28; Theorem 4.3]. As is well known that for a compact orientable 3-manifold H , the holonomy representation of the complete hyperbolic structure $\text{Hol} : \pi_1(H) \rightarrow \text{Isom}^+ \mathbb{H}^3 \cong \text{PSL}(2, \mathbb{C})$ can be lifted to a representation $\widetilde{\text{Hol}} : \pi_1(H) \rightarrow \text{SL}(2, \mathbb{C})$ [40]. It is also well known that there is a one-to-one correspondence between the lifts and spin structures on H . Considering one of the lifts and composing one of a finite dimensional representation V of $\text{SL}(2, \mathbb{C})$, we get a representation $\varrho : \pi_1(H) \rightarrow \text{SL}(V)$. Recall that for every positive integer n there is a unique irreducible representation V_n of $\text{SL}(2, \mathbb{C})$ of dimension n , namely, $(n - 1)$ -th symmetric power of the standard representation $V_2 = \mathbb{C}^2$. Considering V_n and all above, we get $\varrho_n : \pi_1(H) \rightarrow \text{SL}(n, \mathbb{C})$.

Let H be a compact orientable non-elementary hyperbolic 3-manifold with a boundary consisting of ℓ surfaces $\Sigma_{g_1}, \dots, \Sigma_{g_\ell}$ of genus at least 2, and $n \geq 2$. Recall that H is non-elementary if its holonomy is an irreducible representation in $\text{PSL}(2, \mathbb{C})$.

In [40; Theorem 0.1], P. Menal-Ferrer and J. Porti prove that the inclusion $\partial H \subset H$ induces an injection, $H^1\left(H; \mathfrak{sl}(n, \mathbb{C})_{Ad_{\varrho_n}}\right) \hookrightarrow H^1\left(\partial H; \mathfrak{sl}(n, \mathbb{C})_{Ad_{\varrho_n}}\right)$ with $\dim H^1\left(H; \mathfrak{sl}(n, \mathbb{C})_{Ad_{\varrho_n}}\right) = (1/2) \dim H^1\left(\partial H; \mathfrak{sl}(n, \mathbb{C})_{Ad_{\varrho_n}}\right)$, and an isomorphism $H^2\left(H; \mathfrak{sl}(n, \mathbb{C})_{Ad_{\varrho_n}}\right) \cong H^2\left(\partial H; \mathfrak{sl}(n, \mathbb{C})_{Ad_{\varrho_n}}\right)$.

Theorem 6. Assume Σ_{g_i} , H , M , G , \mathcal{G} , ϱ , \mathbf{h}_k^H , \mathbf{h}_k^M , and $\mathbf{h}_j^{\Sigma_{g_i}}$ are as above. Then, the following formula is valid:

$$\begin{aligned} \mathbb{T}\left(H, \{\mathbf{h}_k^H\}_0^3\right) &= e^{\frac{\sqrt{-1}}{4}(-\beta_0 + \ell\pi - \sum_{i=1}^{\ell} \theta_1^{\Sigma_{g_i}})} \prod_{i=1}^{\ell} \Delta\left(\mathbf{h}_1^{\Sigma_{g_i}}, \mathbf{h}_1^{\Sigma_{g_i}}\right)^{-1/4} \\ &= e^{\frac{\sqrt{-1}}{4}(-\beta_0 + \ell\pi - \sum_{i=1}^{\ell} \theta_1^{\Sigma_{g_i}})} \prod_{i=1}^{\ell} \sqrt[4]{\delta\left(\mathbf{h}^{1,i}, \mathbf{h}^{1,i}\right)}. \end{aligned}$$

Here, $\left[\mathbf{h}_{1,1}^{0, \Sigma_{g_i}}, \mathbf{h}_1^{\Sigma_{g_i}} \oplus \mathbf{h}_1^{\Sigma_{g_i}}\right] = \left| \left[\mathbf{h}_{1,1}^{0, \Sigma_{g_i}}, \mathbf{h}_1^{\Sigma_{g_i}} \oplus \mathbf{h}_1^{\Sigma_{g_i}} \right] \right| e^{\sqrt{-1} \theta_1^{\Sigma_{g_i}}}$, $r_i : \pi_1(\Sigma_{g_i}) \rightarrow \pi_1(H)$ denotes the homomorphism obtained by the embedding $\Sigma_{g_i} \hookrightarrow H$, $\beta_0 = \dim H_0(M; \mathcal{G}_{Ad_{\varrho}})$ and $\mathbf{h}_{1,1}^{0, \Sigma_{g_i}}$ is a basis of $H_1\left(\Sigma_{g_i}; \mathcal{G}_{Ad_{\varrho_{or_i}}}\right) \oplus H_1\left(\Sigma_{g_i}; \mathcal{G}_{Ad_{\varrho_{or_i}}}\right)$ so that $\mathbb{T}\left(C_*\left(\Sigma_{g_i}; \mathcal{G}_{Ad_{\varrho_{or_i}}}\right) \oplus C_*\left(\Sigma_{g_i}; \mathcal{G}_{Ad_{\varrho_{or_i}}}\right), \{\mathbf{h}_{1,1}^{0, \Sigma_{g_i}}\}\right) = 1$, $\mathbf{h}^{j,i}$ is the Poincare dual basis of $H^j\left(\Sigma_{g_i}; \mathcal{G}_{Ad_{\varrho_{or_i}}}\right)$ corresponding to the basis $\mathbf{h}_j^{\Sigma_{g_i}}$ of $H_j\left(\Sigma_{g_i}; \mathcal{G}_{Ad_{\varrho_{or_i}}}\right)$.

The proof of Theorem 6 is based on considering the short-exact sequence

$$0 \rightarrow \bigoplus_{i=1}^{\ell} C_*\left(\Sigma_{g_i}; \mathcal{G}_{Ad_{\varrho_{or_i}}}\right) \rightarrow C_*\left(H; \mathcal{G}_{Ad_{\varrho}}\right) \oplus C_*\left(H; \mathcal{G}_{Ad_{\varrho}}\right) \rightarrow C_*\left(M; \mathcal{G}_{Ad_{\varrho}}\right) \rightarrow 0$$

and combining [28; Theorem 4.1] and [28; Theorem 4.3].

Combining these and Theorem 6, we have

Theorem 7. Considering $n = 2$ and for $i = 1, \dots, \ell$, fixing a maximal geodesic lamination $\lambda_i \subset \Sigma_{g_i}$, if $\varrho_2 : \pi_1(H) \rightarrow \text{SL}(2, \mathbb{C})$ is such that $\varrho_2 \circ r_i \in \text{Teich}\left(\Sigma_{g_i}\right)$, $i = 1, \dots, \ell$, applying (ii) of Theorem 5, and using the notation there, we get

$$\mathbb{T}\left(H, \{\mathbf{h}_k^H\}_0^3\right) = e^{\frac{\sqrt{-1}}{4}(-\beta_0 + \ell\pi - \sum_{i=1}^{\ell} \theta_1^{\Sigma_{g_i}})} 2^{\frac{1}{4} \sum_{i=1}^{\ell} \chi(\Sigma_{g_i}; \mathfrak{sl}(2, \mathbb{C}))} \prod_{i=1}^{\ell} \sqrt[4]{\Omega_{T,i}}.$$

Here, $\Omega_{T,i}$ is the matrix of the complex Thurston symplectic form $\omega_T : \mathcal{H}(\lambda_i; \mathbb{C}) \times \mathcal{H}(\lambda_i; \mathbb{C}) \rightarrow \mathbb{C}$ in the basis $\mathfrak{h}^i \oplus \sqrt{-1} \mathfrak{h}^i$, and $\mathfrak{h}^{j,i}$ is the Poincaré dual basis of $H^j(\Sigma_{g_i}; \mathfrak{sl}(2, \mathbb{C})_{Ad_{\rho_{r_i}}})$ corresponding to $\mathfrak{h}_j^{\Sigma_{g_i}}$, and \mathfrak{h}^i is the basis of $\mathcal{H}(\lambda_i; \mathbb{R})$ associated with the isomorphism obtained by the real analytical embedding $\text{Teich}(\Sigma_{g_i}) \hookrightarrow \mathcal{H}(\lambda_i; \mathbb{R})$ [32]. Here, $r_i : \pi_1(\Sigma_{g_i}) \rightarrow \pi_1(\mathbb{H})$ is the homomorphism obtained by the embedding $\Sigma_{g_i} \hookrightarrow \mathbb{H}$.

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Шоттки көрсетілімі мен Рейдмейстер бұралуы жайында ескерту

Мақалада Шоттки көрсетілімі үшін Рейдмейстердің бұралу формуласы анықталған. Теориялық нәтижелер 2-ден кем емес текті бағдарланған беттерден тұратын жиекті 3-көпбейнелерге қолданылады.

Клт сөздер: Шоттки көрсетілімі, Рейдмейстер бұралуы, көрсетілімнің көпбейнелері, Атьи-Ботта-Голдман симплектикалық формасы, Терстонның симплектикалық формасы.

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Замечание о представлениях Шоттки и кручении Рейдемейстера

В статье установлена формула кручения Рейдемейстера для представлений Шоттки. Теоретические результаты применены к 3-многообразиям с краем, состоящим из ориентируемых поверхностей рода не менее 2.

Ключевые слова: представления Шоттки, кручение Рейдемейстера, многообразие представлений, симплектическая форма Атьи-Ботта-Голдмана, симплектическая форма Терстона.