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On the existence and coercive estimates of solutions to the Dirichlet problem for a class of third-order differential equations

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As you know, the third order partial differential equation is one of the basic equations of wave theory. For example, in particular, a linearized Korteweg-de Vries type equation with variable coefficients models ion-acoustic waves into plasma and acoustic waves on a crystal lattice. In this paper, the properties of solutions of a class of the third order degenerate partial differential equations with variable coefficients given in a rectangle were studied. Sufficient conditions for the existence and uniqueness of a strong solution have been established. Note that the solution of the degenerate equation does not retain its smoothness, therefore, these difficulties in turn affect the coercive estimates.

Keywords: resolvent, third order differential equation, Dirichlet problem, coercive estimates.

2020 Mathematics Subject Classification: 39A14.

Introduction

In the rectangle $\overline{\Omega} = \{(x,y) : -\pi \le x \le \pi; 0 \le y \le 1\}$, the problem

$$Lu + \mu u = -k(y)\frac{\partial^3 u}{\partial x^3} - \frac{\partial u^2}{\partial y^2} + a(y)\frac{\partial u}{\partial x} + c(y)u + \mu u = f(x,y) \in L_2(\Omega), \tag{1}$$

$$u_x^{(\alpha)}(-\pi, y) = u_x^{(\alpha)}(\pi, y), \quad \alpha = 0, 1, 2, \tag{2}$$

$$u(x,0) = u(x,1), \tag{3}$$

is considered.

Suppose that the coefficients $k(y)$, $a(y)$, $c(y)$ of equation (1) satisfy the conditions:

1) $k(y) \ge 0$ is a piecewise continuous function on the segment $[0,1]$ and $k(0) = 0$;

2) $a(y) \ge \delta_0$, $c(y) \ge \delta > 0$ are continuous functions on the segment $[0,1]$.

Equation (1) degenerates along the line $y = 0$, i.e. at these points equation (1) changes order. This means that solutions do not retain their smoothness, hence these difficulties in turn affect the coercive estimates of solutions.

Many papers [1–13] and the works cited there are devoted to the study of partial differential equations of the third order. From these works and from a review of literary sources, it follows that previously differential equations without degeneracy were mainly studied.

To present the results obtained regarding this work, we will need the following designations and definitions. By $W_2^1(\Omega)$ we denote the S.L. Sobolev space with norm $\|u\|_{2,1,\Omega} = [\|u_y\|_2^2 + \|u_x\|_2^2 + \|u\|_2^2]^{\frac{1}{2}}$. $C_{0,\pi}^{\infty}(\overline{\Omega})$ is a set consisting of infinitely differentiable functions and satisfying conditions (2)-(3).

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Definition 1. A function $u(x,y) \in L_2(\Omega)$ is called a strong solution to problem (1)–(3) if there exists a sequence of functions $\{u_n\} \subset C^{\infty}_{0,\pi}(\overline{\Omega})$, such that

$$\|u_n - u\|_{L_2(\Omega)} \to 0, \ \ \|Lu_n - f\|_{L_2(\Omega)} \to 0asn \to 0.$$

Theorem 1. Let the conditions 1) - 2) be fulfilled. Then for $\mu \geq 0$, for any $f(x,y) \in L_2(\Omega)$ there is a unique strong solution to the problem (1)–(3).

Theorem 2. Let conditions 1) - 2) be fulfilled. Then for $\mu \geq 0$, for any $f(x,y) \in L_2(\Omega)$ there is a unique strong solution to the problem(1)–(3) such that the coercive estimate

$$\|u\|_{1,2,\Omega} \leq C \, \|(L+\mu I)u\|_2$$

is valid for it, where $C > 0$ is a constant, $\|\cdot\|_2$ is the norm of $L_2(\Omega)$.

1 *Proof of Theorems 1-2*

In what follows, we denote by $(L+\mu I)$ the operator corresponding to problem (1)–(3).

Lemma 1. Let the conditions 1) - 2) be fulfilled. Then the following inequality

$$\|(L+\mu I)u\|_2 \geq (\delta_0 + \lambda) \, \|u\|_2 \, , \tag{4}$$

holds for all $u \in D(L)$, where $\delta_0 > 0$, $\mu \geq 0$. $D(L)$ is the domain of definition of the operator L.

Proof. Consider the functionality $< (L+\mu I)u, u >$, $u \in D(L)$, where $< \cdot, \cdot >$ is scalar product in $L_2(\Omega)$. Integrating in parts, we get an estimate (4). Lemma 1 is proved.

Using the Fourier method, we reduce the problem (1)–(3) to the study of the following differential operator with the parameter n ($n = \pm 0, \pm 1, \pm 2, ...$):

$$(l_n + \mu I)z(y) = -z'(y) + (-ik(y)n^3 + ina(y) + c(y) + \mu)z(y),$$

where $z(y) \in D(l_n)$, $D(l_n)$ is the domain of definition of the operator l_n.

Lemma 2. Let the conditions 1) - 2) be fulfilled. Then the following inequality

$$\|(l_n + \mu I)z\|_2 \geq (\delta_0 + \mu) \, \|z\|_2$$

holds for all $z(y) \in D(l_n + \mu I)$, where $\|\cdot\|_2$ is the norm of the Hilbert space $L_2(0,1)$. $D(l_n)$ is the domain of definition of the operator l_n.

Proof. Let us denote by $C^2_0[0,1]$ the set consisting of doubly differentiable functions and satisfying condition (3). Let $z(y) \in C^2_0[0,1]$ and consider the functional

$$< (l_n + \mu I)z, z >= \int_0^1 [|z'|^2 + (c(y)+\mu)|z|^2 + (in^3k(y) + ina(y))|z|^2]dy. \tag{5}$$

Hence, using the properties of complex numbers, we find that

$$| < (l_n + \mu I)z, z > | \geq \int_0^1 [|z'|^2 + (c(y)+\mu)|z|^2]dy \geq \int_0^1 (|z'|^2 + (\delta+\mu)|z|^2)dy. \tag{6}$$

From the last inequality, using the Cauchy-Bunyakovsky inequality, we have

$$\|(l_n + \mu I)z\|_2 \geq (\delta_0 + \mu) \, \|z\|_2 \, .$$

Hence, and by virtue of the continuity of the norm in $L_2(0,1)$, we will be convinced of the validity of the last estimate for all $z(y) \in D(l_n)$. Lemma 2 is proved.

Lemma 3. Let the conditions 1) - 2) be fulfilled and $\mu \geq 0$. Then for the operator $(l_n + \mu I)$ there is a bounded inverse operator $(l_n + \mu I)^{-1}$ defined on the whole $L_2(0,1)$.

Proof. Lemma 3 is also proved as Lemma 2.3 of [14, 15].

Lemma 4. Let the conditions 1) - 2) be fulfilled and $\mu \geq 0$. Then the following estimates are valid for operators $(l_n + \mu I)^{-1}$ and $\frac{d}{dy}(l_n + \mu I)^{-1}$:

$$\left\|(l_n + \mu I)^{-1}\right\|_{2\to 2} \leq \frac{1}{\delta + \mu}; \tag{7}$$

$$\left\|(l_n + \mu I)^{-1}\right\|_{2\to 2} \leq \frac{1}{|n| \cdot \delta_0}, \quad n \neq 0; \tag{8}$$

$$\left\|\frac{d}{dy}(l_n + \mu I)^{-1}\right\|_{2\to 2} \leq \frac{1}{(\delta + \mu)^{\frac{1}{2}}}, \tag{9}$$

where $\|\cdot\|_{2\to 2}$ is the norm of the operator from $L_2(\Omega)$ to $L_2(\Omega)$.

Proof. From Lemma 2 we have

$$\left\|(l_n + \mu I)^{-1}\right\|_{2\to 2} \leq \frac{1}{\delta + \mu}.$$

Inequality (7) is proved.

Using inequality (5) and properties of complex numbers, we find that

$$< (l_n + \mu I)z, z > \geq \left| \int_0^1 (in^3 k(y) + ina(y))|z|^2 dy \right|. \tag{10}$$

Note that by virtue of condition 1) - 2) the functions $k(y)$ and $a(y)$ do not change signs, therefore, from the inequality (10) we find that

$$| < (l_n + \mu I)z, z > | \geq \int_0^1 |in^3 k(y) + ina(y)| \cdot |z|^2 dy. \tag{11}$$

From (11) and given $a(y) \geq \delta_0 > 0$ we have

$$\|(l_n + \mu I)z\|_2 \geq |n|\delta_0 \, \|z\|_2 \, .$$

Hence, using the definition of the operator norm, we obtain the following estimate:

$$\left\|(l_n + \mu I)^{-1}\right\|_{2\to 2} \leq \frac{1}{|n| \cdot \delta_0}, \quad n \neq 0.$$

Inequality (8) is proved.

Using inequalities (4) and (6) we find that

$$\frac{1}{\delta + \mu} \left\|(l_n + \mu I)^{-1}\right\|_{2\to 2} \geq \left\|z'\right\|_2^2 .$$

Hence, according to the definition of the operator norm, we find

$$\left\|\frac{d}{dy}(l_n + \mu I)^{-1}\right\|_{2\to 2} \leq \frac{1}{(\delta + \mu)^{\frac{1}{2}}}.$$

Inequality (9) is proved. Lemma 4 is proved.

Proof of Theorem 1. Using Lemma 3, we obtain that

$$u_k(x,y) = \sum_{n=-k}^{n=k} (l_n + \mu I)^{-1} f_n(y) \cdot e^{inx}$$

is the solution of the following problem:

$$(L + \mu I) u_k(x,y) = f_k(x,y), \tag{12}$$

$$u_{k,x}^{(\alpha)}(-\pi, y) = u_{k,x}^{(\alpha)}(\pi, y), \quad \alpha = 0, 1, 2, \tag{13}$$

$$u_k(x,0) = u_k(x,1) = 0, \tag{14}$$

where $f_k(x,y) \to f(x,y)$, $f_k(x,y) = \sum\limits_{n=-k}^{k} f_n(y) \cdot e^{inx}$, $i^2 = -1$.

From inequality (4) and using the fundamentality of the sequence $\{f_k(x,y)\}$, we have

$$\|u_k(x,y) + u_m(x,y)\|_2 \le \frac{1}{\delta + \mu} \|f_k(x,y) - f_m(x,y)\|_2 \to 0, \ \ as \ \ k, m \to \infty.$$

From the last inequality and by virtue of the completeness of the Hilbert space $L_2(\Omega)$ we have

$$u_k(x,y) \overset{L_2(\Omega)}{\to} u(x,y). \tag{15}$$

Further, using the equalities (12)–(15) for any $f(x,y) \in L_2(\Omega)$, we obtain that

$$u(x,y) = (L + \mu I)^{-1} f = \sum_{n=-\infty}^{n=\infty} (l_n + \mu I)^{-1} f_n(y) \cdot e^{inx} \tag{16}$$

is a strong solution to the problem (1)–(3). The theorem is proved.

Proof of Theorem 2. From (16) by virtue of the orthonormality of the system $\{e^{inx}\}$

$$\|u\|_2^2 = \left\| \sum_{n=-\infty}^{\infty} (l_n + \mu I)^{-1} f_n(y) \cdot e^{inx} \right\|_{L_2(0,1)}^2 = 2\pi \sum_{n=-\infty}^{\infty} \left\| (l_n + \mu I)^{-1} f_n(y) \cdot e^{inx} \right\|_{L_2(0,1)}^2 \le$$

$$\le 2\pi \sum_{n=-\infty}^{\infty} \left\| (l_n + \mu I)^{-1} \right\|_{2\to 2}^2 \cdot \|f_n(y)\|_{L_2(0,1)}^2 \le \sup_{\{n\}} \left\| (l_n + \mu I)^{-1} \right\|_{2\to 2}^2 2\pi \cdot \sum_{n=-\infty}^{\infty} \|f_n(y)\|_{L_2(0,1)}^2 \le$$

$$\le \sup_{\{n\}} \left\| (l_n + \mu I)^{-1} \right\|_{2\to 2}^2 \cdot \|f(x,y)\|_{L_2(\Omega)}^2 . \tag{17}$$

Here we note that we used by virtue of the orthonormality of the system $\{e^{inx}\}$, i.e.

$$\|f(x,y)\|_2^2 = \left\| \sum_{n=-\infty}^{\infty} f_n(y) \cdot e^{inx} \right\|_{L_2(\Omega)}^2 = 2\pi \cdot \sum_{n=-\infty}^{\infty} \|f_n(y)\|_{L_2(0,1)}^2 .$$

From estimates (17), (4) and (7) we obtain that

$$\|u\|_{L_2(\Omega)}^2 \le (\frac{1}{\delta + \mu})^2 \|f(x,y)\|_{L_2(\Omega)}^2 .$$

From here we finally have

$$\|u\|_2 \leq \|f(x,y)\|_{L_2(\Omega)}\,, \tag{18}$$

where $C_1 = \frac{1}{\delta+\mu}$.

Next, we calculate the norm $\|u_x\|_2$:

$$\|u_x\|^2_{L_2(\Omega)} = \left\| \sum_{n=-\infty}^{\infty} in(l_n+\mu I)^{-1} e^{inx} \right\|^2_{L_2(\Omega)} \leq \sup_{\{n\}} \left\| in(l_n+\mu I)^{-1} \right\|^2_{2\to 2} \cdot 2\pi \cdot \sum_{n=-\infty}^{\infty} \|f_n(y)\|^2_{L_2(0,1)} \leq$$

$$\leq \sup_{\{n\}} |n|^2 \cdot \left\| in(l_n+\mu I)^{-1} \right\|^2_{2\to 2} \cdot \|f(x,y)\|^2_{L_2(\Omega)}\,.$$

Hence and from inequality (8) we have

$$\|u_x\|^2_{L_2(\Omega)} \leq \sup_{\{n\}} |n|^2 \left\| in(l_n+\mu I)^{-1} \right\|^2_{2\to 2} \cdot \|f(x,y)\|^2_{L_2(\Omega)} \leq \sup_{\{n\}} |n|^2 \cdot \frac{1}{|n|^2 \cdot \delta_0^2} \|f(x,y)\|^2_{L_2(\Omega)}\,.$$

Hence

$$\|u_x\|_{L_2(\Omega)} \leq C_2 \|f(x,y)\|_{L_2(\Omega)}\,, \tag{19}$$

where $C_2 = \frac{1}{(\delta+\mu)^{\frac{1}{2}}}$.

Then, repeating the above calculations, we get the following estimate

$$\|u_y\|_{L_2(\Omega)} \leq C_3 \|f(x,y)\|_{L_2(\Omega)}\,, \tag{20}$$

where $C_2 = \frac{1}{\delta_0}$.

Using the equalities (18)–(20), we find that

$$\|u\|_{2,1,\Omega} \leq C \|f(x,y)\|_{L_2(\Omega)}\,,$$

where $C = \max\{C_1, C_2, C_3\}$. The theorem is proved.

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Author Contributions

A.O. Suleimbekova collected and analyzed data, and led manuscript preparation. B.M. Musilimov assisted in data collection and analysis. All authors participated in the revision of the manuscript and approved the final submission.

Conflict of Interest

The authors declare no conflict of interest.

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Үшінші ретті дифференциалдық теңдеулердің бір класы үшін Дирихле есебі шешімдерінің бар болуы және коэрцитивті бағалаулары туралы

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Білетініміздей үшінші ретті дербес туындылы дифференциалдық теңдеулер толқындар теориясының негізгі теңдеулерінің бірі. Мысалы, айнымалы коэффициентті сызықталған Кортевег–де Фриз типті теңдеуі иондық акустикалық толқындарды кристалдық тордағы плазмалық және акустикалық толқындарға модельдейді. Жұмыста тіктөртбұрышта берілген айнымалы коэффициентті үшінші ретті дербес туындылы еселенген теңдеулердің бір класының шешімдерінің қасиеттері зерттелген. Күшті шешімнің бар болуы мен жалғыздығына жеткілікті шарттар алынған. Еселенген теңдеудің шешімі өзінің тегістігін сақтамайтынын ескерсек, бұл қиындықтар өз кезегінде коэрцитивті бағалауға әсер етеді.

Кілт сөздер: резольвента, үшінші ретті дифференциалдық теңдеулер, Дирихле есебі, коэрцитивті бағалаулар.

О существовании и коэрцитивных оценках решений задачи Дирихле для одного класса дифференциальных уравнений третьего порядка

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Как известно, уравнения в частных производных третьего порядка являются одним из основных уравнений теории волн. В частности, линеаризованное уравнение типа Кортевега–де Фриза с переменными коэффициентами моделирует ионно-акустические волны в плазменные и акустические волны на кристаллической решетке. В данной работе исследованы свойства решений одного класса вырождающихся уравнений в частных производных третьего порядка с переменными коэффициентами, заданных в прямоугольнике. Установлены достаточные условия существования и единственности сильного решения. Заметим, что решение вырождающегося уравнения не сохраняет свою гладкость, следовательно, эти трудности, в свою очередь, влияют на коэрцитивные оценки.

Ключевые слова: резольвента, дифференциальные уравнения третьего порядка, задача Дирихле, коэрцитивные оценки.

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