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Some estimates for the viscoelastic incompressible Kelvin-Voigt medium

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In this paper, we consider the application of the method of fictitious domains to a viscoelastic incompressible medium based on the Kelvin-Voigt model. Application of the method of fictitious domains allows solving the original problem in regions with complex geometric configuration. This makes it easier to automate the construction of a consistent difference mesh, and to solve the problem in areas of standard shape. Estimates for the proximity of the auxiliary problem's solution are obtained. The auxiliary problem is constructed by the method of fictitious domains. These estimates refer to the solution of the original problem. The original problem describes a viscoelastic incompressible medium. Convergence follows from the estimates of the proximity of the solutions of the original and auxiliary problems. Further, on the basis of the method of fictitious domains, two-sided estimates on a small parameter for the difference between the solution of the original problem and the solution of the auxiliary problem constructed by the method of fictitious domains are obtained. Moreover, the solution to the auxiliary problem is expanded as a series in powers of the small parameter. This is possible because that solution is represented as a functional series that converges absolutely in the original domain.

Keywords: Kelvin-Voigt medium, Kronecker symbol, approximate solution, fictitious area method, two-sided estimates, small parameter, stress, velocity, strain, displacement.

2020 Mathematics Subject Classification: 39A05.

Introduction and problem statement

This paper presents two-sided estimates for a small parameter of the solution of the initial problem. Since a priori estimates help to determine the interval or band in which the solution lies, this is relevant. In addition, since the initial problem does not have an analytical solution, two-sided estimates allow us to determine the initial approximation for finding an approximate solution to the problem, which is an important step in the process of finding a solution.

We consider the application of the method of fictitious domains for incompressible Kelvin-Voigt medium. Two-sided estimates of the convergence of the approximate solution to the exact solution by a small parameter α are obtained. We consider the formulation of a dynamic viscoelastic incompressible medium based on the Kelvin-Voigt model in a cylinder $Q = \{D \times [0 \le t \le t_1]\}$, where $D \subset R^3$ is a bounded singly connected region with a sufficiently smooth boundary γ. Let us introduce the notation $\gamma_t = \gamma \times [0, t_1]$, the strain and stress vector-functions $\overrightarrow{\varepsilon} = \{\varepsilon_{11},\ \varepsilon_{22},\ \varepsilon_{33}, 2\varepsilon_{12}, 2\varepsilon_{13},\ 2\varepsilon_{23}\}^T$, $\overrightarrow{\sigma} = \{\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{13}, \sigma_{23}\}^T$, here the symbol T is transpose, the displacement and velocity vector functions $\overrightarrow{U} = \{U_1, U_2, U_3\}^T$, $\overrightarrow{\vartheta} = \{\vartheta_1, \vartheta_2, \vartheta_3\}^T$. Following the work of [1] we consider the velocity-stress formulation. We find the solution satisfying the following relations:

$$\frac{\partial \overrightarrow{\vartheta}}{\partial t} + R\overrightarrow{\sigma} = \overrightarrow{f}, \quad (x,t) \in Q \tag{1}$$

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which is an equation of motion,

$$\frac{\partial \overrightarrow{\varepsilon}}{\partial t} - R\overrightarrow{\vartheta} = 0, \tag{2}$$

displacement-strain ratio,

$$B\overrightarrow{\sigma} = J\overrightarrow{\varepsilon} + D\frac{\partial \overrightarrow{\varepsilon}}{\partial t}. \tag{3}$$

The equation of state for the Kelvin-Voigt medium is

$$div\,\overrightarrow{u} = 0. \tag{4}$$

The condition of incompressibility of the medium, taking into account the stresses and the pressure function p, is given by the relationship

$$\sigma_{ik} = -\delta_{ik}p + 2\mu\varepsilon_{ik}, \quad i,k = 1,2,3. \tag{5}$$

Here δ_{ik} is the Kronecker symbol, $\overrightarrow{f}$ is the vector of mass forces, $B = B^T$, $C = C^T$ are symmetric positive-definite matrices depending on the Lamé constants and viscosity coefficient, J is an diagonal matrix, their form is given in [2].

R is a linear matrix-differential operator:

$$R = \begin{pmatrix} \nabla_1 & 0 & 0 & \nabla_2 & \nabla_3 & 0 \\ 0 & \nabla_2 & 0 & \nabla_1 & 0 & \nabla_3 \\ 0 & 0 & \nabla_3 & 0 & \nabla_1 & \nabla_2 \end{pmatrix}^T, \quad R^* = -R^T, \quad \nabla_i = \frac{\partial}{\partial x_i}, \quad i = 1,2,3.$$

The system of equations (1)–(5) transform to the following form, a vector function $\overrightarrow{\sigma}(x,t)$ that satisfies the following relations

$$B\frac{\partial^2 \overrightarrow{\sigma}}{\partial t^2} = A\overrightarrow{\sigma} + DA\frac{\partial \overrightarrow{\sigma}}{\partial t} + \overrightarrow{F}, \tag{6}$$

$A = -RR^*$, $\overrightarrow{F} = R\overrightarrow{f}$, satisfies the initial conditions:

$$\overrightarrow{\sigma}(x,0) = \overrightarrow{q}(x), \quad \frac{\partial \overrightarrow{\sigma}}{\partial t}(x,0) = \overrightarrow{g}(x) \tag{7}$$

and boundary conditions

$$\sum_{k=1}^{3} \sigma_{ik}(x,t)\, n_k = 0, \quad (x,t) \in \gamma_t; \tag{8}$$

here $n_k = (n_1, n_2, n_3)^T$ is the vector of normal to γ, $\gamma_t = \gamma \times [0, t_1]$. Let us denote the problem (6)–(8) by the problem I.

Main provisions

In [2], we show the stationalization of the solution of the problem I to the solution of the static elastic problem.

$$R^*\overrightarrow{\sigma}^y(x) + \overrightarrow{F}(x) = 0, \quad \overrightarrow{\sigma}^y(x) = R\varepsilon^y(x),$$
$$\sum_{k=1}^{3} \sigma_{ik}^y(x)\, n_k = 0, \quad x \in \gamma. \tag{9}$$

In [2], the closeness estimate of the solution of problem I and problem (9) is obtained:

$$\left\|\overrightarrow{\sigma} - \overrightarrow{\sigma}^y\right\| \le e^{-\beta t} \cdot \left\|\overrightarrow{\sigma}(x,0) - \overrightarrow{\sigma}^y(x)\right\|,$$

where $\beta>0$ is a constant.

The following theorem is true for problem I:

Theorem 1. Let

$$\overrightarrow{\sigma}(x,t) \in W_2^{2,1}(Q), \quad \overrightarrow{g}(x) \in \dot{W}_2^1(D), \quad \overrightarrow{q}(x) \in L_2(D), \quad \overrightarrow{F}(x,t) \in L_2(Q),$$

then there exists a unique solution to the problem I and the following estimation is true

$$\|\overrightarrow{\sigma}(x,t)\|_{W_2^{2,1}(Q)} \le C_1(\left\|\overrightarrow{F}\right\|_{L_2(Q)} + \|\overrightarrow{q}\|_{L_2(D)} + \int_0^{t_1} \|\overrightarrow{g}\|_{\dot{W}_2^1(D)} d\tau).$$

The proof is similar to the proof of Theorem 1 in [3].

According to the method of fictitious domains [3–6], we augment the original region D with the region D_1 to a composite region $D_0 = D \cup D_1$, with boundary $\Gamma, \Gamma_t = \Gamma \times [0,t_1]$, $+Q_1 = D_1 \times [0,t_1]$ and construct the auxiliary problem

$$\begin{gathered} L_\alpha \overrightarrow{\sigma}^\alpha = \overrightarrow{F}, \ (x,t) \in Q, \ \ L_\alpha \overrightarrow{\sigma}^\alpha = 0, \ \ (x,t) \in Q_1, \\ \sum_{k=1}^{3} (\overrightarrow{\sigma}^\alpha)_{ik} n_k = 0, \ \ (x,t) \in \gamma_t, \ \ \overrightarrow{\sigma}^\alpha(x,0) = 0, \ \ x \in D_1, \\ \overrightarrow{\sigma}^\alpha(x,0) = \overrightarrow{q}(x), \ \ x \in D, \ \frac{\partial \overrightarrow{\sigma}^\alpha}{\partial t}(x,0) = 0, \ \ x \in D_1 \\ \left.\frac{\partial \overrightarrow{\sigma}^\alpha}{\partial t}\right|_t = \overrightarrow{g}(x), \ \ x \in D, \ \sum_{k=1}^{3} \sigma_{ik}^\alpha n_k = 0, \ \ (x,t) \in \Gamma_t, \end{gathered} \tag{10}$$

where $L_\alpha \overrightarrow{\sigma}^\alpha = B \frac{\partial^2 \overrightarrow{\sigma}^\alpha}{\partial t^2} - a^\alpha A \overrightarrow{\sigma}^\alpha + JA \frac{\partial \overrightarrow{\sigma}^\alpha}{\partial t}$, $a^\alpha = \begin{cases} 1, & x \in D, \\ -\alpha^2, & x \in D_1, \end{cases}$ $\alpha > 0$ is a small parameter.

On the coefficient gap curve γ_t, we set the following matching conditions

$$\left.\overrightarrow{\sigma}^\alpha\right|_{\gamma_t}^{+} = \left.\overrightarrow{\sigma}^\alpha\right|_{\gamma_t}^{-}, \quad \left.\frac{\partial \overrightarrow{\sigma}^\alpha}{\partial N}\right|_{\gamma_t}^{+} = \frac{M}{\alpha} \left.\frac{\partial \overrightarrow{\sigma}^\alpha}{\partial n}\right|_{\gamma_t}^{-}.$$

Signs "+" or "–" mean convergence to the limit value of the function from inside or outside to the boundary γ_t. The parameter M takes the values –1 or +1 [6, 7].

Let us introduce the following series into consideration:

$$S_1 = \sum_{k=0}^{\infty} \alpha^k \overrightarrow{V_k}, \ \text{on } Q, \quad S_2 = \sum_{k=1}^{\infty} \alpha^k \overrightarrow{W_k}, \ \text{on } Q_1. \tag{11}$$

Putting (11) into (10), we obtain relations for determining $\overrightarrow{V_k}$ and $\overrightarrow{W_k}$:

$$\begin{gathered} L_\alpha \overrightarrow{V_0} = \overrightarrow{F}, \ \ (x,t) \in QL_\alpha, \quad \overrightarrow{W_1} = 0, \ \ (x,t) \in Q_1, \\ \sum_{k=1}^{3} (V_0)_{ik} n_k = 0, \ \ (x,t) \in \gamma_t, \quad \overrightarrow{W_1}(x,0) = 0, \quad \frac{\partial \overrightarrow{W_1}}{\partial t}(x,0) = 0, \quad x \in D_1. \\ \overrightarrow{V_0}(x,0) = \overrightarrow{q}(x), \ \frac{\partial \overrightarrow{V_0}}{\partial t}(x,0) = \overrightarrow{g}(x), \ x \in D, \quad \frac{\partial \overrightarrow{W_1}}{\partial n} = M \frac{\partial \overrightarrow{V_0}}{\partial N}, \ (x,t) \in \gamma_t. \end{gathered} \tag{12}$$

$$\sum_{k=1}^{3} (\overrightarrow{W_1})_{ik} n_k = 0, \qquad (x,t) \in \gamma_t.$$

And for k≥1

$$L_\alpha \overrightarrow{V_k} = \overrightarrow{F}, \quad (x,t) \in Q, \quad L_\alpha \overrightarrow{W}_{k+1} = 0, \quad (x,t) \in Q_1,$$

$$\sum_{k=1}^{3} (V_k)_{ik} n_k = 0, \quad (x,t) \in \gamma_t, \quad \sum_{k=1}^{3} (W_{k+1})_{ik} n_k = 0, \quad (x,t) \in \gamma_t,$$
$$\overrightarrow{V_k}(x,0) = 0, \qquad x \in D, \quad \overrightarrow{W}_{k+1}(x,0) = 0, \quad x \in D_1,$$
$$\frac{\partial \overrightarrow{V_k}}{\partial t}(x,0) = 0, \quad x \in D, \quad \frac{\partial \overrightarrow{W}_{k+1}}{\partial t}(x,0) = 0, \quad x \in D_1,$$
$$\overrightarrow{V_k} = \overrightarrow{W_k}, \quad (x,t) \in \gamma_t.$$

Functions $\overrightarrow{V_k} \in W_2^{2,1}(Q)\,,\ k = 0,1,\ldots,\ \overrightarrow{W_k} \in W_2^{2,1}(Q_1)\,,\ k = 0,1,\ldots$

We obtain estimates of the convergence of the solution of the auxiliary problem to the solution of the original problem with respect to the small parameter α.

Theorem 2. If α_0 is such that $0 < \alpha < \alpha_0$, then the series S_1, S_2 absolutely converge to $W_2^{2,1}(Q)$ and $W_2^{2,1}(Q_1)$, and the following equalities are true

$$\overrightarrow{\sigma}^{\alpha} = S_1, \quad (x,t) \in Q, \quad \overrightarrow{\sigma}^{\alpha} = S_2, \quad (x,t) \in Q_1,$$

where $\overrightarrow{\sigma}^{\alpha}$ is the solution of problem (10).

Proof. We search obvious priori estimates [7, 8].

$$\left\|\overrightarrow{W_k}\right\|_{W_2^{2,1}(Q_1)} \le C_1 \left\|\frac{\partial \overrightarrow{W_k}}{\partial n}\right\|_{W_2^{\frac{1}{2}}(\gamma_t)} \le C_2 \left\|\frac{\partial \overrightarrow{V}_{k-1}}{\partial N}\right\|_{W_2^{\frac{1}{2},1}(\gamma_t)} \le C_1 C_2 \left\|\overrightarrow{V}_{k-1}\right\|_{W_2^{\frac{1}{2}}(\gamma_t)}, \tag{13}$$

where $C_1,\ C_2$ are constants depending on the regions D, D_1 and not depending on α.

Now we show the convergence of the series S_1 to $W_2^{2,1}(Q_1)$ and S_2 to $W_2^{2,1}(Q_1)$, we have

$$\left\|\overrightarrow{V_k}\right\|_{W_2^{2,1}(Q)} \le C_3 \left\|\overrightarrow{V_k}\right\|_{W_2^{\frac{3}{2},\,1}(\gamma_t)} = C_3 \left\|\overrightarrow{W_k}\right\|_{W_2^{\frac{3}{2},1}(\gamma_t)} \le C_3 C_4 \left\|\overrightarrow{W_k}\right\|_{W_2^{2,1}(Q_1)},$$

and using (8), (13), we obtain

$$\left\|\overrightarrow{V_k}\right\|_{W_2^{2,1}(Q)} \le C_5 \left\|\overrightarrow{V}_{k-1}\right\|_{W_2^{\frac{3}{2},1}(Q)}, \quad k \ge 1,$$

then

$$\left\|\overrightarrow{V_0}\right\|_{W_2^{2,1}(Q)} \le C_6 \left(\left\|\overrightarrow{F}\right\|_{L_2(Q)} + \|\overrightarrow{q}\|_{L_2(D)} + \int_0^{t_1} \|\overrightarrow{g}\|_{\dot{W}_2^1(D)} dt\right),$$

where $C_6 = C_1 C_2 C_3 C_4 C_5$.

Assuming $\alpha < \alpha_0 = C_6^{-1}$, we obtain the series S_1, is absolutely convergent to $W_2^{2,1}(Q)$ and correspondingly the series S_2 is absolutely convergent to $W_2^{2,1}(Q_1)$.

Multiplying (12) for $\overrightarrow{V_k}$ and $\overrightarrow{W_k}$ by α_k, and summing over k, we have

$$LS_1 = \overrightarrow{F}, \quad (x,t) \in Q, \quad L_\alpha S_2 = 0, \quad (x,t) \in Q_1,$$
$$S_1(x,0) = \overrightarrow{q}(x)\,, \quad x \in DS_2(x,0) = 0, \quad x \in D_1, \tag{14}$$

$$\frac{\partial S_1}{\partial t}(x,0) = \overrightarrow{g}(x), \quad x \in D, \quad \frac{\partial S_2}{\partial t}(x,0) = 0, \quad x \in D_1.$$

$$S_2(x,t) = 0, \quad (x,t) \in \gamma_t,$$

$$S_1 = S_2, \quad (x,t) \in \Gamma_t, \quad \frac{\partial S_2}{\partial n} = \frac{M}{\alpha}\frac{\partial S_1}{\partial N}, \quad (x,t) \in \gamma_t,$$

$$L\overrightarrow{\sigma} = B\,\frac{\partial^2 \overrightarrow{\sigma}}{\partial t^2} - A\overrightarrow{\sigma} - JA\frac{\partial \overrightarrow{\sigma}}{\partial t}.$$

Hence, (14) we obtain that $\overrightarrow{\sigma}^{\alpha} = S_1$ in Q and $\overrightarrow{\sigma}^{\alpha} = S_2$ in Q_1, if the condition $0 < \alpha < \alpha_0$ is met. From the proof of Theorem 2 implies the following statement

$$\left\|\overrightarrow{\sigma} - \overrightarrow{\sigma}^{\alpha}_{+}\right\|_{W_2^{2,1}(Q)} \leq C_7\alpha, \quad \left\|\overrightarrow{\sigma} - \overrightarrow{\sigma}^{\alpha}_{-}\right\|_{W_2^{2,1}(Q)} \leq C_8\alpha, \tag{15}$$

here $\overrightarrow{\sigma}^{\alpha}_{+} = \overrightarrow{\sigma}^{\alpha}$, at $M = 1$, $\overrightarrow{\sigma}^{\alpha}_{-} = \overrightarrow{\sigma}^{\alpha}$, at $M = -1$. $\overrightarrow{\sigma}$ is the solution to the problem I. C_7, C_8 are constants depend on the areas D, D_1 and are independent of α.

Next, we can formulate a theorem giving two-sided estimates on α [9].

Theorem 3. If $0 < \alpha < \alpha_0$, $\overrightarrow{\sigma}$ is a solution of problem I, $\overrightarrow{\sigma}^{\alpha}_{+}$, $\overrightarrow{\sigma}^{\alpha}_{-}$ is a solution of problem (10) at $M = 1$ and $M = -1$, then the following estimation (16) is true

$$\left\|\overrightarrow{\sigma} - \frac{1}{2}\left(\overrightarrow{\sigma}^{\alpha}_{+} + \overrightarrow{\sigma}^{\alpha}_{-}\right)\right\|_{W_2^{2,1}(Q)} \leq C_9\alpha^2, \tag{16}$$

where

$$\overrightarrow{\sigma}^{\alpha} = S_1, \quad (x,t) \in Q, \quad \overrightarrow{\sigma}^{\alpha} = S_2, \quad (x,t) \in Q_1. \tag{17}$$

Proof. By virtue of Theorem 2, we have

$$\overrightarrow{\sigma}^{\alpha}_{+} = \sum_{k=0}^{\infty} \alpha^k \overrightarrow{V}^{+}_{k}, \quad (x,t) \in Q, \quad \overrightarrow{\sigma}^{\alpha}_{+} = \sum_{k=1}^{\infty} \alpha^k \overrightarrow{W}^{+}_{k}, \quad (x,t) \in Q_1, \tag{18}$$

here $\overrightarrow{V}^{+}_{k}$, $\overrightarrow{W}^{+}_{k}$ are solutions of (10) at $M = 1$, moreover

$$\overrightarrow{\sigma}^{\alpha}_{-} = \sum_{k=0}^{\infty} \alpha^k \overrightarrow{V}^{-}_{k}, \quad (x,t) \in Q, \quad \overrightarrow{\sigma}^{\alpha}_{-} = \sum_{k=1}^{\infty} \alpha^k \overrightarrow{W}^{-}_{k}, \quad (x,t) \in Q_1, \tag{19}$$

here $\overrightarrow{V}^{-}_{k}$, $\overrightarrow{W}^{-}_{k}$ are solutions of (10) at $M = -1$.

We obtain $\overrightarrow{V}^{+}_{0} \equiv \overrightarrow{V}^{-}_{0} \equiv \overrightarrow{\sigma}$, it is a solution of problem I.

We introduce the notation $\overrightarrow{W_1} = \overrightarrow{W}^{+}_{1} + \overrightarrow{W}^{-}_{1}$, the function $\overrightarrow{W_1}$ satisfies the following problem

$$L_\alpha \overrightarrow{W_1} = 0, \quad (x,t) \in Q_1, \quad \frac{\partial \overrightarrow{W_1}}{\partial n} = 0, \quad (x,t) \in \gamma_t,$$

$$\overrightarrow{W_1}(x,0) = 0, \quad \frac{\partial \overrightarrow{W_1}}{\partial t}(x,0) = 0, \quad x \in D_1, \quad \overrightarrow{W_1}(x,t) = 0, \quad (x,t) \in \Gamma_t,$$

hence, we obtain that $\overrightarrow{W_1} = 0$, or $\overrightarrow{W}^{+}_{1} = -\overrightarrow{W}^{-}_{1}$.

Further we introduce $\overrightarrow{V_1} = \overrightarrow{V}^{+}_{1} + \overrightarrow{V}^{-}_{1}$, the function $\overrightarrow{V_1}$ satisfies the problem

$$L_\alpha \overrightarrow{V_1} = 0, \quad (x,t) \in Q, \quad \frac{\partial \overrightarrow{V_1}}{\partial t}(x,0) = 0, \quad x \in D,$$

$$\overrightarrow{V_1}(x,0)=0, \quad (x,t)\in\Gamma_t, \quad \overrightarrow{V_1}(x,t)=0, \quad (x,t)\in\gamma_t,$$

from which we obtain $\overrightarrow{V_1}=0$, or $\overrightarrow{V}_1^+=-\overrightarrow{V}_1^-$. By sequentially introducing

$$\overrightarrow{W_2}=\overrightarrow{W}_2^+ + \overrightarrow{W}_2^-, \quad \overrightarrow{V_2}=\overrightarrow{V}_2^+ + \overrightarrow{V}_2^-,$$

we get

$$\overrightarrow{W_2}=\overrightarrow{W}_2^-, \quad \overrightarrow{V}_2^+=\overrightarrow{V}_2^-,$$

continuing this process, we get

$$\overrightarrow{V}_k^+=\overrightarrow{V}_k^-, \text{ if } k \text{ is even,}$$
$$\overrightarrow{V}_k^+=-\overrightarrow{V}_k^-, \text{ if } k \text{ is odd.}$$

Substituting (20) into (18), (19), we have

$$\overrightarrow{\sigma}_+^\alpha=\overrightarrow{\sigma}+\sigma\overrightarrow{V}_1^+ + \sigma^2\overrightarrow{V}_2^+ + \ldots$$

$$\overrightarrow{\sigma}_-^\alpha=\overrightarrow{\sigma}-\sigma\overrightarrow{V}_1^+ + \sigma^2\overrightarrow{V}_2^+ + \ldots$$

Applying decomposition (20), as well as estimation (17) at $0<\alpha<\alpha_0$, we obtain

$$\left\|\overrightarrow{\sigma}-\frac{1}{2}\left(\overrightarrow{\sigma}_+^\alpha+\overrightarrow{\sigma}_-^\alpha\right)\right\|_{W_2^{2,1}(Q)} \le \alpha^2\left\|\overrightarrow{V}_2^+ + \alpha^2\overrightarrow{V}_4^+ + \ldots\right\|_{W_2^{2,1}(Q)} \le C_8\alpha^2\left\|\overrightarrow{V}_0^+\right\|_{W_2^{2,1}(Q)} \le C_9\alpha^2;$$

here $C_8=C_\alpha^2$, so for $x\in D$, $0<\alpha<\alpha_0$, we have a two-sided estimation

$$O\left(\alpha^2\right)+\min\left(\overrightarrow{\sigma}_+^\alpha\,,\ \overrightarrow{\sigma}_-^\alpha\right) \le \overrightarrow{\sigma} \le \max\left(\overrightarrow{\sigma}_+^\alpha\,,\ \overrightarrow{\sigma}_-^\alpha\right)+O(\alpha^2).$$

Thus, a two-sided estimate in terms of the small parameter of the solution to the original problem has been obtained through the solution of the auxiliary problem, where the parameter values $M=-1$, $M=1$ corresponds to $\overrightarrow{\sigma}_+^\alpha$.

Conclusion

The obtained estimate is essential for the application of the numerical solution of the auxiliary problem, and it is not considered in the works [10–16]. Continuation by the lowest coefficient in the fictitious region method leads to the same estimates. In works [10–16], the numerical implementation of the Kelvin-Voigt model in equivalent formulations is considered.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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