UDC 539.186

PROBABILITY OF THE LEVEL NUMBER IN A GIVEN ENERGY INTERVAL

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The statistics of the energy spectra for random electron systems are investigated near the metal-insulator transition. The fluctuations of the number of one-electron states in a finite energy interval are considered. The total probability of the level number is shown to be scale-invariant. The results are obtained at the critical point with and without time-reversal symmetry. The existence of the critical statistics is a prominent signature of the localization transition.

Keywords: metal-insulator transition, level number, critical statistics, energy spectra, *transition.*

The statistical properties of electron transport in quantum disordered systems have attracted much attention [1-3]. The problem of localization of quantum states in disordered systems has been formulated more than five decades ago by P. Anderson [4]. The important issue has been that depending on the disorder of the potential energy of atoms the quantum solids undergo a phase transition from extended to localized behaviour corresponding to metallic and insulating behaviour at low temperature. This disorder-induced metal-insulator transition, conventionally called the Anderson transition, has been the subject of considerable experimental and theoretical work [5,6]. Main efforts have been concentrating on the critical behaviour at the metal-insulator transition [2] and the statistical properties of the energy levels of the Anderson model [7,8].

In order to analyze the correlations between several consecutive eigenvalues one can study the probability that an energy interval of the width ε centred at a randomly chosen energy contains exactly n levels. The distribution of this probability $Q_n(\varepsilon)$ is given by

$$Q_n(\varepsilon) = \frac{N!}{n!(N-n)!}\int_{-\infty}^{\infty}\!\!\ldots\int_{-\infty}^{\infty} d\varepsilon_1 \ldots d\varepsilon_N \prod_{i=1}^{n}\Theta(\frac{\varepsilon}{2}-|\varepsilon_i|)\prod_{j=n+1}^{N}[1-\Theta(\frac{\varepsilon}{2}-|\varepsilon_j|)]P_n(\varepsilon_1,\ldots,\varepsilon_N). \qquad (1)$$

where $P_N(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N)$ is the joint probability density function of the eigenvalues (or energy levels) of the Hamiltonian under consideration (see Ref. [9] for general definitions). $\Theta(x)$ is the step function. It is intimately related to the *n*-level correlation function

$$R_n(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n) = \frac{N!}{(N-n)!}\int_{-\infty}^{\infty}\!\!\ldots\int_{-\infty}^{\infty} d\varepsilon_{n+1}\ldots d\varepsilon_N P_N(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_N), \qquad (2)$$

The latter provides a more complete description of the level statistics than the two-point correlation function [9] $R_2(\varepsilon_1 - \varepsilon_2)$, which can be expressed in terms of the *n*-level spacing distribution $p_n(s)$, $R_2(s) = \sum_{n=0} p_n(s)$, for practical calculations of $R_2(s)$. On the other hand, the probability of finding no levels ($n = 0$) inside of the interval defines the nearest neighbour spacing distribution, $P(s) = d^2Q_0(s)/ds^2$. The width of the distribution $Q_n(\varepsilon)$ describes the rigidity of the spectrum, which is defined by the variance of the level number n,

$$\left\langle [R(s)\delta n(\varepsilon)]^2 \right\rangle = \sum_{n_l=0}^{\infty}(n-\langle n\rangle)^2 Q_n(\varepsilon), \qquad (3)$$

where $\langle n\rangle = \varepsilon$ is the average number of levels in a given interval ε.

The results of the random-matrix theory for $Q_n(\varepsilon)$, which correspond to the metallic case, have been calculated numerically [9], and were later cast into an explicit analytical form [10] for the three universality classes of random Hamiltonians: orthogonal, unitary and symplectic. In the limit $\delta n \equiv |n - \varepsilon| << \varepsilon$ the distribution for the orthogonal ensemble is approximately Gaussian,

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$$\ln Q_n(\varepsilon) \propto \frac{\pi^2}{4}\frac{\delta n^2}{\ln(8\varepsilon/|\delta n|)+B}, \tag{4}$$

where ε is units of Δ and B depends weakly on δn.

In the insulating regime the sequence of the levels is completely random due to the localization of the electron states, and therefore we have the usual Poissonian process,

$$Q_n(\varepsilon) \propto \frac{\varepsilon^n}{n!}\exp(-\varepsilon), \tag{5}$$

For example, if $n = 0$ the Wigner surmise and the Poisson law provide

$$Q_0(\varepsilon) = \frac{2}{\sqrt{\pi}}Err(\frac{\sqrt{\pi}}{2}\varepsilon), \quad \text{and} \quad Q_0(\varepsilon) = \exp(-\varepsilon), \tag{6}$$

for the metallic and insulating regime, respectively. $Err(x)$ is the error function.

It is of great interest to study how $Q_n(\varepsilon)$ changes from (4) to (5), when increasing the disorder W, and to calculate its shape at the metal-insulator transition as well. Our results suggest the function $Q_n(\varepsilon)$ shows critical behaviour near W_c. In figure 1 we show $Q_n(\varepsilon)$ corresponding to the critical point.

Calculations for different system sizes yield almost the same set of the distributions independently on the number of levels $n = 0, 1, 2,....$ We note that $\ln Q_n(\varepsilon)$ is parabolic in ε and in n in the range of small fluctuations $\delta n <<$ s, but wider than in the metallic regime (4) since the level repulsion becomes less important.

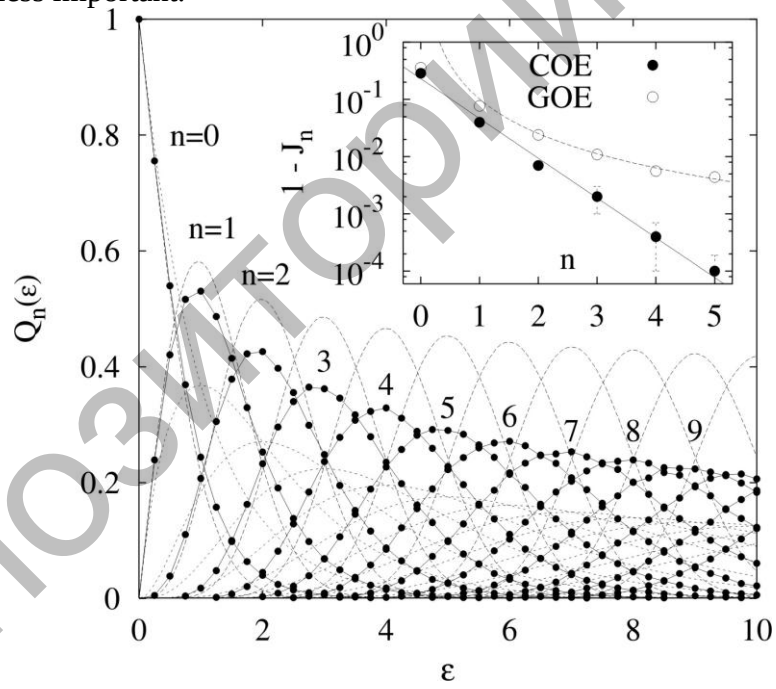

Fig. 1. Joint probability distribution $Q_n(\varepsilon)$ for a 3D system of linear size $L = 32$ at the critical disorder $W_c = 16.5$. Dashed line is the GOE result obtained by diagonalizing 10^4 random matrices of size 100x100. Dotted line is the Poisson process Eq. (5). Inset: The Mehta parameter J_n as a function of index n for the critical orthogonal case shown by solid symbols. Open symbols are the GOE result. Dashed curve is the Dyson conjecture Eq. (9), solid line is the fit to Eq. (11).

Let us consider the quantity $I_n = \int_0^\infty Q_n(\varepsilon)\, d\varepsilon$. For the Gaussian orthogonal ensemble it is known from previous numerical simulations [9] that $I_0 \approx 0.643$, $I_1 \approx 0.922$, and I_n converges to unity in the limit $n \to \infty$. For the Poissonian process (5) it is easy to see that $I_n = 1$ for all n. At the

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MIT, when W = 16.5, we obtained the following values of the integral: $I_1 \approx 0.714$, $I_2 \approx 0.960$, $I_3 \approx 0.998$, and also $\lim_{n\to\infty} I_n = 1$. The latter set of constants is L-independent and characterizes the critical level statistics.

We emphasize that I_n change with W in the vicinity of the critical disorder W_c according to a one-parameter scaling law, $I_n(W,L) = I^c_n + f(L/\xi(W))$, where the function $f(x)$ can be linearized in the vicinity of the critical point: $f(L,W) \sim (W - W_c)L^{1/\nu}$. This allows us to determine the critical exponent ν of the correlation length. Note that in searching ν it is not necessary to choose a certain spacing s_o, as it was earlier done when using $P(s)$ [11,12,13]. One can ask the question how the character of the dependence of the variance $\langle[\delta n(\varepsilon)]^2\rangle$ on the average $\langle n(\varepsilon)\rangle$ change when the delocalized states transform into the localized ones.

An important statistical quantity often used for describing the RMT-to-Possion transition comes from the joint level distribution function $Q_n(\varepsilon)$, which defines the probability to find n energy levels in an interval of given width ε (see Fig. 1).

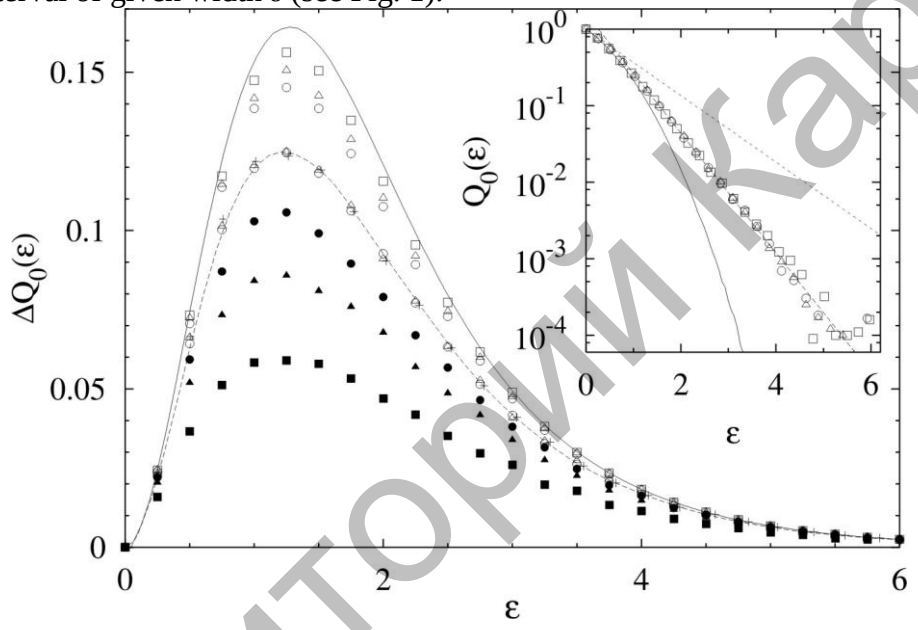

Fig. 2. Deviation $\Delta Q_0(\varepsilon) = -[Q_0(\varepsilon) - exp(-\varepsilon)]$ of $Q_0(\varepsilon)$ from the Poisson process Eq. (10) for a 3D system of linear size L = 6 (circle), 12 (triangle) and 28 (box) at different disorders W = 14 (full symbols), 16.5 and 20 (open symbols). Solid line is the GOE result. Dashed line is guide for eye at the critical disorder, where all data collapse in one curve irrespective of L. Inset: $Q_0(\varepsilon)$ vs. ε for W = 16.5. Dashed line is $A\,exp(-B\varepsilon)$ with $A \approx 1.6$ and $B \approx 1.85$. Dotted line is the Poissonian [Eq. (10)].

This quantity which is referred to as the Mehta parameter

$$J_n = \int_{\varepsilon=0}^{\infty} Q_n(\varepsilon)d\varepsilon, \tag{7}$$

identifies a global statistical measure of the spectrum. For example, J_n=0 is simply proportional to the spacing variance:

$$2J_{n=0} = \int_{s=0}^{\infty} s^2 P(s)ds = \left\langle s^2 \right\rangle = \mathrm{var}\, s + 1, \tag{8}$$

i.e. to the first lowest non-trivial moment of $P(s)$.

The countable set of J_n, where $n \geq 0$ is integer, is known for the three universality classes of the RMT [9] as numerical constants. Dyson has suggested that the Mehta parameter increases with index n according to the power law

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$$J_n = 1 - \frac{1}{\beta\pi^2 n^2}, \qquad n >> 1, \tag{9}$$

saturating to unity at $n \to \infty$. For the Poisson process

$$Q_n(\varepsilon) \propto \frac{\varepsilon^n}{n!}\exp(-\varepsilon), \tag{10}$$

which is valid for the uncorrelated spectrum in the localized regime, there exists no n-dependence: $J_n = 1$ for any n.

Using the data of eigenvalues for systems with T-invariance, one can easily calculate the critical set of Mehta parameters J^c_n for few n and compare with that of both the GOE and the Dyson suggestion Eq. (9). As shown in the inset of figure 1, the critical J^c_n grows with n starting from $J_0 = 0.714 \pm 0.04$ and lies intermediately between J^o_n and 1.

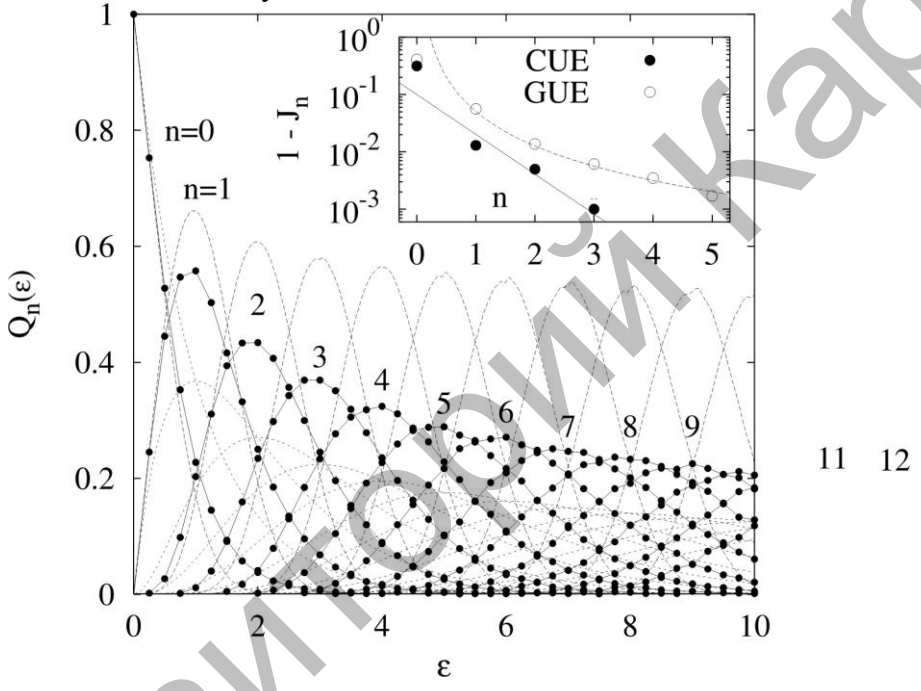

Fig. 3. The same as Fig. 1, but for the unitary case with the maximal 3-component flux $\varphi = 1/4$.

One observes, however, that the RMT power law in Eq. (9) does not hold for the critical J^c_n, but rather follows an exponential behaviour

$$J^c_n = 1 - \varsigma\exp(-Kn), \tag{11}$$

where $\zeta \approx 0.23$ and $K \approx 1.6$ result from the fitting procedure.

In the rest of the paper we investigate the probability $Q_0(\varepsilon)$ to have no level, n=0, within the bin s, which implies simply the level statistics for the case of nearest neighbour spacings. In order to show (L,W)-dependence we plot in Fig. 2 the absolute deviation from the Poisson process $\Delta Q_0(s) = -[Q_0(s) - exp(-s)]$. The data for the metallic phase, i.e. for $W < W_c$, turn out to scale towards the GOE result. The larger the system size, the closer the spectral statistics to this limit. When $W > W_c$, the data of $\Delta Q_0(s)$ approach zero, however with the size effect opposite to the metallic regime. In the very vicinity of the critical disorder W=16.5 all the data within statistical uncertainties start to fall onto a common intermediate curve independent of L. This is the manifestation of critical behaviour characteristic of the localization-delocalization transition. Similar scaling properties are also observed for $Q_{n>0}(s)$.

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Interestingly, the asymptotic decrease of the critical $Q_0(s)$ for large ε is well described by the exponential function $Q_0(s) \propto \exp(-B\varepsilon)$ as depicted in the inset of figure 2. Consequently, the nearest-neighbour level spacing distribution defined as a second derivative

$$P(s) = p_{n=0}(s) = \frac{d^2 Q_0(s)}{ds^2}, \tag{12}$$

should also fall off with increasing s in a similar way. Figure 3 shows the probability $Q_n(\varepsilon)$ to find n energy levels in an interval of given width ε for the system of linear size L=32 and with the maximal 3-component flux φ=1/4 (cf. Fig.1). The result for Gaussian unitary ensemble obtained by diagonalizing 10^4 random hermitian matrices of size NxN=100x100 and the Poisson process [Eq. (10)] are provided for comparison.

References:

K. B. Efetov, Supersymmetry in Disorder and Chaos (Cambridge University Press, Cambridge, 1997).
B. Kramer and A. MacKinnon, Rep. Prog. Phys. -1994, - **Vol. 56**, - P. 1496-1588.
A. D. Mirlin, Phys. Rep. 2000, - **Vol**. 326, - P. 259-377.
P. W. Anderson, Phys. Rev. -1958, -**Vol**. 109, - P. 1492.
P. A. Lee and T. V. Ramakrishnan, Rev. Mod. Phys., - 1985, - **Vol**. 57, - P. 287.
T. Guhr, A. Muller-Groeling, and H. A. Weidenmuller, Phys. Rep. -1998, - **Vol**. 299, - P. 189.
B. I. Shklovskii, B. Shapiro, B. R. Sears, P. Lambrianides, and H. B. Shore, Phys. Rev. B – 1993, **Vol**. 47, -P. 11487-111490.
B. L. Altshuler, I. Kh. Zharekeshev, S. A. Kotochigova, and B. I. Shklovskii, Zh. Eksp. Teor. Fiz. 94, 343 (1988), [Sov. Phys. JETP 67, 625 (1988)].
M. L. Mehta, Random Matrices, Academic Press, Boston, 1991, 523p.
M. M. Fogler and B. I. Shklovskii, Phys. Rev. Lett. – 1995., - **Vol**. **74**, - P. 3312-3315.
E. Hofstetter and M. Schreiber, Phys. Rev. B - 1994, - **Vol**. **49**, - P. 14726-14729.
I. Varga, E. Hofstetter, M. Schreiber, and J. Pipek, Phys. Rev. B – 1997, -**Vol. 52**, - P.7783.
I. Kh. Zharekeshev and B. Kramer, Phys. Rev. B 1995 - **Vol. 51**, - P.17239-17242. Phys. Rev. Lett. – 1997, - **Vol. 79**, - P. 717-720.

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