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Separability of the third-order differential operator given on the whole plane

In this paper, in the space $L_2(R^2)$, we study a third-order differential operator with continuous coefficients in $R(-\infty,+\infty)$. Here, these coefficients can be unlimited functions at infinity. In addition under some restrictions on the coefficients, the bounded invertibility of the given operator is proved and a coercive estimate is obtained, i.e. separability is proved.

Keywords: resolvent, third-order differential operator, separability.

1 Introduction

Third-order partial differential equations are the basis of mathematical models of many phenomena and processes. Significant literature is devoted to the solvability of boundary value problems for third-order differential equations [1–6] and cited papers there.

Consider the differential operator

$$Lu + \lambda u = \frac{\partial u}{\partial y} + R_2(y)\frac{\partial^3 u}{\partial x^3} + R_1(y)\frac{\partial u}{\partial x} + R_0(y)u + \lambda u \qquad (1)$$

initially defined on $C^{\infty}_{0,\pi}(R^2)$, $\lambda \geq 0$.

$C^{\infty}_{0,\pi}$ is a set consisting of infinitely differentiable finite functions in R^2.

We assume that the coefficients of operator (1) $R_0(y), R_1(y), R_2(y)$ satisfy the conditions:

i) $R_0(y) \geq \delta_0) > 0, R_1(y) \geq \delta_1 > 0, -R_2(y) \geq \delta_2 > 0$ are continuous functions in $R(-\infty,+\infty)$;

ii) $\mu_0 = \sup\limits_{|y-t|\leq 1} \frac{R_0(y)}{R_0(t)} < \infty, \mu_1 = \sup\limits_{|y-t|\leq 1} \frac{R_1(y)}{R_1(t)} < \infty, \mu_2 = \sup\limits_{|y-t|\leq 1} \frac{R_2(y)}{R_2(t)} < \infty.$

It is easy to verify that the operator $L + \lambda I$ admits closure in $L_2(R^2)$, which we also denote by $L + \lambda I$.

It should be noted that the issue of the existence of a bounded operator $(L + \lambda I)^2$ of a closed operator $L + \lambda I$ in $L_2(R^2)$ is equivalent to the following problem: Find a unique solution of $(L + \lambda I)u = f(x, y) \in L_2(R^2)$ belonging to $L_2(R^2)$, i.e. $u \in L_2(R^2)$. In this case, the closed operator $L + \lambda I$ generates a problem without initial conditions ([7], Chapter III, Section 4).

Recently, there has been an increased interest in differential operators with unbounded coefficients [8–14].

In [15], the linearized Korteweg-de Vries operator was studied, which generates the so-called periodic problem without initial conditions on the strip.

In contrast to [15], we study the separability of the third-order differential operator defined on the whole plane.

Theorem 1. Let the condition i) be fulfilled. Then the operator $L + \lambda I$ is continuously invertible in $L_2(R^2)$ for $\lambda \geq 0$.

Following the papers [8, 9], we introduce the following definition.

Definition 1. We called the operator L is separable in $L_2(R^2)$ if the estimate

$$\left\|\frac{\partial u}{\partial y}\right\|_2 + \left\|R_2(y)\frac{\partial^3 u}{\partial x^3}\right\|_2 + \left\|R_1(y)\frac{\partial u}{\partial x}\right\|_2 + \|R_0(y)u\|_2 \leq C(\|Lu\|_2 + \|u\|_2),$$

holds for $u \in D(L)$, where C is independent of $u(x, y)$, $\|\cdot\|_2$ is the norm of $L_2(R^2)$.

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Theorem 2. Let conditions i) - ii) be fulfilled. Then the operator L is separable.

Example. Let $R_0(y) = |y|^2 + 1$, $R_1(y) = e^{100|y|}$, $R_2(y) = -e^{1000|y|}$, $-\infty < y < \infty$. It is easy to verify that all the conditions of Theorem 2 are satisfied. Consequently, the operator L is separable, i.e.

$$\left\|\frac{\partial u}{\partial y}\right\|_{L_2(R^2)} + \left\|-e^{1000|y|}\frac{\partial^3 u}{\partial x^3}\right\|_{L_2(R^2)} + \left\|e^{100|y|}\frac{\partial u}{\partial x}\right\|_{L_2(R^2)} + \left\|(|y|^2+1)u\right\|_{L_2(R^2)} \leq C(\|Lu\|_{L_2(R^2)} + \|u\|_{L_2(R^2)}),$$

where C is a constant.

2 Auxiliary lemmas and inequalities

Lemma 2.1. Let the condition i) be fulfilled and $\lambda \geq 0$. Then the inequality

$$\|(L+\lambda I)u\|_{L_2(R^2)} \geq (\delta_0 + \lambda)\|u\|_{L_2(R^2)}, \tag{2}$$

holds for all $u \in D(L)$, where $\delta_0 > 0$.

The proof follows from the functional $< (L+\lambda I)u, u >$, where $< \cdot, \cdot >$ is the scalar product in $L_2(R^2)$, $u \in D(L)$.

Consider the operator

$$(l_{t,j} + \lambda I)z = z'(y) + (-it^3 R_{2,j}(y) + itR_{1,j}(y) + R_{0,j}(y))z(y), \ (-\infty < t < \infty)$$

where $R_{2,j}(y)$, $R_{1,j}(y)$, $R_{0,j}(y)$ are bounded periodic functions of the same period $\Delta_j = (j-1, j+1)$, $j = 0, \pm 1, \pm 2$, $z(y) \in C_0^\infty(R)$, $-\infty < t < \infty$, $z(y) = u(y) + i\vartheta(y)$.

It is easy to verify that the operator $l_{t,j}$ admits closure in $L_2(R)$, which we also denote by $l_{t,j}$.

Lemma 2.2. Let the condition i) be fulfilled. Then the estimate

$$\|(l_{t,j} + \lambda I)z\|_2 \geq (\delta_0 + \lambda)\|z\|_2 \tag{3}$$

holds for all $z(y) \in D(l_{t,j} + \lambda I)$, $\|\cdot\|_2$ is the norm of $L_2(R)$.

Proof. Lemma 2.2 is proved in the same way as estimate (2) of Lemma 2.1.

Lemma 2.3. Let the condition i) be fulfilled. Then the operator $(l_{t,j} + \lambda I)$ has a continuous inverse operator $(l_{t,j} + \lambda I)^{-1}$ defined on the whole $L_2(R)$.

Proof. By the estimate (3) it suffices to show that the range is dense in $L_2(R)$.

Let us prove it by contradiction. Let us assume that the range is not dense in $L_2(R)$. Then there exists an element $\vartheta \in L_2(R)$ such that $< (l_{t,j} + \lambda I)u, \vartheta >= 0$ for all $u \in D(l_{t,j})$. This follows that

$$(l_{t,j} + \lambda I)^*\vartheta = -\vartheta' + (it^3 R_{2,j}(y) - itR_{1,j}(y) + R_{0,j}(y))\vartheta = 0. \tag{4}$$

in the sense of the theory of generalized functions. Now, using the periodicity of the functions $R_0(y)$, $R_1(y)$, $R_2(y)$, we have that $(it^3 R_{2,j}(y) - itR_{1,j}(y) + R_{0,j}(y))\vartheta \in L_2(R)$. Given this and from (4) it follows that $\vartheta \in W_2^1(R)$, where $W_2^1(R)$ is the Sobolev space. The general theory of the embedding theorems implies that

$$\lim_{|y|\to\infty} \vartheta(y) = 0. \tag{5}$$

Taking into account equality (5) and repeating the arguments used in the proof of the estimate (3), we obtain

$$\|(l_{t,j} + \lambda I)^*\vartheta\|_2 \geq \delta_0 \|\vartheta\|_2. \tag{6}$$

From estimates (4), (6) it follows that $\vartheta = 0$. Lemma 2.4 is proved.

Let $\{\varphi_j\}_{j=-\infty}^{\infty} \in C_0^\infty(R)$ is a set of functions such that $\varphi_j(y) \geq 0$, $\sup p\varphi_j \subseteq \Delta_j (j \in Z)$, $\sum\limits_{j=-\infty}^{\infty} \varphi_j^2(y) = 1$.

Here we note that any point $y \in R$ can belong to no more than three segments from the system of segments $\{supp\varphi_j\}$ [9, 10].

Assume that

$$K_\lambda f = \sum_{j=-\infty}^{\infty} \varphi_j(y)(l_{t,j} + \lambda I)^{-1}\varphi_j f,$$

$$B_\lambda f = \sum_{j=-\infty}^{\infty} \varphi_j'(y)(l_{t,j}+\lambda I)^{-1}\varphi_j f, \;\; f \in C_0^\infty(R), \;\; \lambda \ge 0.$$

It is easy to verify that

$$(l_t+\lambda I)K_\lambda f = f + \sum_j \varphi_j'(y)(l_{t,j}+\lambda I)^{-1}\varphi_j f, \tag{7}$$

where

$$(l_t+\lambda I)z = -z'(y) + (-it^3R_2(y) + itR_1(y) + R_0(y))z, \; z \in D(l_t).$$

Lemma 2.4. Let the condition i) be fulfilled. Then there exists a number $\lambda_0 > 0$ such that $\|B_\lambda\|_{2\to 2} < 1$ for all $\lambda \ge \lambda_0$.

Proof. Only functions φ_{j-1}, φ_j, φ_{j+1} are nonzero in the interval $\overline{\Delta_j}(j \in Z)$, consequently

$$\|B_\lambda f\|^2_{L_2(R)} = \int_{-\infty}^{\infty} |\sum_{j=-\infty}^{\infty} \varphi_j'(y)(l_{t,j}+\lambda I)^{-1}\varphi_j f|^2 dy \le \sum_{j=-\infty}^{\infty} \int_{\Delta_j} |\sum_{k=j-1}^{j+1} [\varphi_k'(y)(l_{t,k}+\lambda I)^{-1}\varphi_k f]|^2 dy.$$

From the last inequality and by using the obvious inequality $(a+b+c)^2 \le 3(a^2+b^2+c^2)$ and estimate (3), we have

$$\|B_\lambda f\|^2_{L_2(R)} \le \sum_{j=-\infty}^{\infty} \int_{\Delta_j} |\sum_{k=j-1}^{j+1} [\varphi_k'(l_{t,k}+\lambda I)^{-1}\varphi_k f]|^2 dy \le 9 \sum_{j=-\infty}^{\infty} \left\|\varphi_j'(l_{t,j}+\lambda I)^{-1}\varphi_j f\right\|^2_{L_2(\Delta_j)} \le$$

$$\le 9 \sum_{j=-\infty}^{\infty} \left\|\varphi_j'(l_{t,j}+\lambda I)^{-1}\varphi_j f\right\|^2_{L_2(R)} \le 9\cdot c \sum_{j=-\infty}^{\infty} \left\|(l_{t,j}+\lambda I)^{-1}\right\|^2_{L_2(R)\to L_2(R)} \cdot \|\varphi_j f\|^2_{L_2(R)} \le$$

$$\le \frac{9\cdot c}{(\delta_0+\lambda)^2} \cdot \int_{-\infty}^{\infty} (\sum_j \varphi_j^2)|f|^2 dy = \frac{9\cdot c}{(\delta_0+\lambda)^2} \cdot \|f\|^2_{L_2(R)}.$$

Hence

$$\|B_\lambda\|_{L_2(R)\to L_2(R)} \le \frac{9\cdot c}{(\delta_0+\lambda)^2}. \tag{8}$$

From (8) it follows that there exists a number $\lambda_0 > 0$, such that $\lambda \ge \lambda_0$, $\|B_\lambda\|_{L_2(R)\to L_2(R)} < 1$. Lemma 2.4 is proved.

Now consider the initial operator

$$(l_t+\lambda I)z = z'(y) + (-it^3R_2(y) + itR_1(y) + R_0(y))z(y),$$

where $z(y) = u(y) + i\vartheta(y)$, $z(y) \in C_0^\infty(R)$, $-\infty < t < \infty$, $(R = (-\infty, \infty))$.

Lemma 2.5. Let the condition i) be fulfilled. Then the estimate

$$\|(l_t+\lambda I)z\|_2 \ge (\delta_0+\lambda)\,\|z\|_2 \tag{9}$$

holds for all $z \in D(l_t)$.

Proof. The proof follows from the functional $< (l_t+\lambda I)z, z >$, $z \in D(l_t)$.

Lemma 2.6. Let the condition i) be fulfilled. Then there is a number λ_0 such that operator $l_t+\lambda I$ is boundedly invertible for $\lambda \ge \lambda_0$ and the equality

$$(l_t+\lambda I)^{-1} = K_\lambda(I-B_\lambda)^{-1} \tag{10}$$

holds for the inverse operator $(l_t+\lambda I)^{-1}$.

Proof. Using estimates (7), (9) and Lemma 2.4, we obtain the proof of Lemma 2.6.

On the existence of the resolvent. Proof of Theorem 1.

In this subsection, we prove Theorem 1. Firstly, we define the following definition:

Definition 2. The function $u \in L_2(R^2)$ is called a solution of the equation $(L+\lambda I)u = f$ in $L_2(R^2)$ if there exists a sequence $\{u_n\}_{n=1}^{\infty} \subset C_0^{\infty}(R^2)$ such that

$$\|u_n - u\|_2 \to 0, \ \ \|(L+\lambda I)u - f\|_2 \to 0, \ \ as \ \ n \to \infty.$$

Consider the equation

$$(L+\lambda I)u = \frac{\partial u}{\partial y} + R_2(y)\frac{\partial^3 u}{\partial x^3} + R_1(y)\frac{\partial u}{\partial x} + R_0(y)u + \lambda u = f \in C_0^{\infty}(R^2) \tag{11}$$

Applying the Fourier transform to the equation (11) with respect to the variable x, we obtain

$$(l_t+\lambda I)u = -\widetilde{u}'(t,y) + (-it^3R_2(y) + itR_1(y) + R_0(y))\widetilde{u} = \widetilde{f}(t,y), \tag{12}$$

where $\widetilde{u}(t,y)$, $\widetilde{f}(t,y)$ are the Fourier transform of functions $u(x,y)$ and $f(x,y)$ with respect to the variable x. Further, we denote the Fourier transform by $F_{x\to t}$ and the Fourier inverse formula by $F^{-1}_{t\to x}$.

Hence, the problem of solving of the equation (11) turns into the problem of solving of the equation (12). Therefore, according to Lemma 2.6, we have

$$\widetilde{u} = (l_t+\lambda I)\widetilde{f} = K_\lambda(I - B\lambda)^{-1}\widetilde{f}.$$

By using the inverse operator $F^{-1}_{t\to x}$, we find

$$u(x,y) = F^{-1}_{t\to x}\widetilde{u} = F^{-1}_{t\to x}(l_\tau+\lambda I)^{-1}\widetilde{f}. \tag{13}$$

The set $C_0^{\infty}(R^2)$ is dense in $L_2(R^2)$. From here and passing to the limit, through the boundedness and continuity of the Fourier transform, we obtain a proof for any $f(x,y) \in L_2(R^2)$. The uniqueness follows from Lemma 2.1. Theorem 1 is proved.

On the separability of the operator. Proof of theorem 2

To prove separability, first, we give the following lemmas.

Lemma 2.7. Let $z(y) \in D(l_{t,j}+\lambda I)$ and $z(y) = u(y) + i\vartheta(y)$, then $it^3R_2(y)z(y) \in L_2(R)$ if and only if $t^3R_2(y)u(y) \in L_2(R)$ and $t^3R_2(y)\vartheta(y) \in L_2(R)$.

Proof. The proof of Lemma 2.7 is obvious.

Remark. This Lemma is also true for $it^3R_{1,j}(y)z(y)$.

Using this lemma, we consider the operator

$$(l_{t,j}+\lambda I)u = u'(y) + (-it^3R_{2,j}(y) + itR_{1,j}(y) + R_0(y) + \lambda)u(y)$$

on the set of infinitely differentiable, finite and real-valued functions.

Lemma 2.8. Let the condition i) be fulfilled. Then the estimates:

$$\|(l_{t,j}+\lambda I)u(y)\|_2 \geq R_0(y_j)\,\|u\|_2\,, \ n = 0, \pm1, \pm2..., \ where \ R_0(y_j) = \min_{y\in\overline{\Delta}_j} R_{0,j}(y); \tag{14}$$

$$\|(l_{t,j}+\lambda I)u(y)\|_2 \geq |t|R_1(\overline{y_j})\,\|u\|_2\,, \ n = 0, \pm1, \pm2..., \ where \ R_1(\overline{y_j}) = \min_{y\in\overline{\Delta}_j} R_{1,j}(y); \tag{15}$$

$$\|(l_{t,j}+\lambda I)u(y)\|_2 \geq |t|^3R_2(\overline{\overline{y_j}})\,\|u\|_2 \ \ n = 0, \pm1, \pm2..., \ where \ R_2(\overline{\overline{y_j}}) = \min_{y\in\overline{\Delta}_j} |R_{2,j}(y)| \tag{16}$$

hold for any $u \in D(l_{t,j}+\lambda I)$.

Proof. Let $u(y) \in C_0^{\infty}(R)$. It is easy to verify that $\int_{-\infty}^{\infty} u'(y)u(u)dy = 0$ and reproducing the computations used in the proof of Lemma 2.1, we have

$$|<(l_{t,j}+\lambda I)u, u>| = |\int_{-\infty}^{\infty}(-it^3R_{2,j}(y) + itR_{1,j}(y) + R_{0,j}(y) + \lambda)u^2dy|. \tag{17}$$

From (17) we obtain

$$|<(l_{t,j}+\lambda I)u,u>|\geq|\int_{-\infty}^{\infty}(R_{0,j}(y)+\lambda)|u|^2dy|\geq\min_{y\in\overline{\Delta}_j}R_0(y)\left\|u\right\|_2^2. \quad (18)$$

Using the Cauchy-Bunyakovsky inequality, from (18) we obtain

$$\left\|l_{t,j}+\lambda I)u\right\|_2\geq R_0(y_j)\left\|u\right\|_2, \quad (19)$$

where $R_0(y_j)=\min\limits_{y\in\overline{\Delta}_j}R_0(y)$.

From (19) we obtain the proof of inequality (14) of Lemma 2.8. Inequalities (15) and (16) can be proved in the same way as inequality (14). Lemma 2.8 is proved.

Lemma 2.9. Let the condition i) be fulfilled and $\lambda\geq\lambda_0$, $\alpha=0,1,2,3$, $p(y)$ is a continuous function defined on R. Then the estimate

$$\left\|p(y)|t|^{\alpha}(l_t+\lambda I)^{-1}\right\|^2_{L_2(R)\to L_2(R)}\leq c(\lambda)\sup_{j\in Z}\left\|p(y)|t|^{\alpha}\varphi_j(l_{t,j}+\lambda I)^{-1}\right\|^2_{L_2(\Delta_j)\to L_2(\Delta_j)} \quad (20)$$

holds, where $-\infty<t<\infty$

Proof. Let $f\in C_0^{\infty}(R)$. From the representation (10), considering the properties of the functions φ_j $(j\in Z)$ we have

$$\left\|p(y)|t|^{\alpha}(l_t+\lambda I)^{-1}f\right\|^2_{L_2(R)}=\left\|p(y)|t|^{\alpha}K_{\lambda}(I-B_{\lambda})^{-1}f\right\|^2_{L_2(R)}=$$

$$=\int\limits_{-\infty}^{\infty}|p(y)|t|^{\alpha}\sum_{\{j\}}\varphi_j(l_{t,j}+\lambda I)^{-1}\varphi_j(I-B_{\lambda})^{-1}f|^2dy.$$

From the construction it follows that on the interval$\Delta_j(j\in Z)$, only functions φ_{j-1}, φ_j, φ_{j+1} are nonzero, therefore

$$\left\|p(y)|t|^{\alpha}(l_t+\lambda I)^{-1}f\right\|^2_{L_2(R)}\leq\sum_{j=-\infty}^{\infty}\int\limits_{\Delta_j}|p(y)|t|^{\alpha}\sum_{j-1}^{j+1}\varphi_j(l_{t,j}+\lambda I)^{-1}\varphi_j(I-B_{\lambda})^{-1}f|^2dy\leq$$

$$\leq 9\sup_{j\in Z}\left\|p(y)|t|^{\alpha}\varphi_j(l_{t,j}+\lambda I)^{-1}\right\|^2_{L_2(\Delta_j)}\cdot\int\limits_{-\infty}^{\infty}(\sum_j\varphi_j^2)|(I-B_{\lambda})^{-1}f|^2dy. \quad (21)$$

As is known $(\sum\limits_j\varphi_j^2)=1$, then from (21) we obtain

$$\left\|p(y)|t|^{\alpha}(l_t+\lambda I)^{-1}f\right\|^2_{L_2(R)}\leq 9\sup_{j\in Z}\left\|p(y)|t|^{\alpha}\varphi_j(l_{t,j}+\lambda I)^{-1}\right\|^2_{L_2(\Delta_j)}\cdot\int\limits_{-\infty}^{\infty}|(I-B_{\lambda})^{-1}f|^2dy\leq$$

$$\leq 9\sup_{j\in Z}\left\|p(y)|t|^{\alpha}\varphi_j(l_{t,j}+\lambda I)^{-1}\right\|^2_{L_2(\Delta_j)}\cdot\left\|(I-B_{\lambda})^{-1}\right\|^2_{2\to2}\cdot\left\|f\right\|_2^2. \quad (22)$$

From Lemma 2.4 it follows that $\left\|I-B_{\lambda}\right\|^2_{2\to2}<c(\lambda)$. From this and (22), we obtain

$$\left\|p(y)|t|^{\alpha}](l_{t,j}+\lambda I)^{-1}\right\|_{L_2(R)\to L_2(R)}\leq 9\cdot c(\lambda)\sup_{j\in Z}\left\|p(y)|t|^{\alpha}\varphi_j(l_{t,j}+\lambda I)^{-1}\right\|^2_{L_2(\Delta_j)}.$$

Lemma 2.9 is proved.

Lemma 2.10. Let the conditions i)-ii) be fulfilled. Then the estimates

$$\left\|R_0(y)(l_t+\lambda I)^{-1}\right\|_{L_2(R)\to L_2(R)}\leq C_0<\infty; \quad (23)$$

$$\left\|R_1(y)|t|(l_t+\lambda I)^{-1}\right\|_{L_2(R)\to L_2(R)}\leq C_1<\infty; \quad (24)$$

$$\left\| R_2(y)|t|^3(l_t+\lambda I)^{-1}\right\|_{L_2(R)\to L_2(R)} \le C_2 < \infty, \tag{25}$$

hold, where C_0, C_1, C_1 are independent of t $(n = 0, \pm1, \pm2...)$.

Proof. (10) shows that the operator $R_0(l_t + \lambda I)$ is bounded if $\sup\limits_{j\in Z}\left\| R_0(y)\varphi_j(l_{t,j}+\lambda I)^{-1}\right\|_{L_2(\Delta_j)}$ is bounded. Therefore, we will estimate the last expression

$$\left\| R_0(y)\varphi_j(l_t+\lambda I)^{-1}\right\|^2_{L_2(R)\to L_2(R)} \le C(\lambda)\sup_{j\in Z}\left\| R_0(y)\varphi_j(l_{t,j}+\lambda I)^{-1}\right\|^2_{L_2(\Delta_j)} \le$$

$$\le C(\lambda)\sup_{j\in Z}\max_{y\in\Delta_j}|R_0(y)\varphi_j|^2\left\|(l_{t,j}+\lambda I)^{-1}\right\|^2_{L_2(\Delta_j)} \le C(\lambda)\sup_{j\in Z}\max_{y\in\Delta_j}R_0^2(y)\left\|(l_{t,j}+\lambda I)^{-1}\right\|^2_{L_2(\Delta_j)}.$$

From the last inequality and taking into account inequality (14) and the condition ii), we obtain

$$\left\| R_0(y)(l_t+\lambda I)^{-1}\right\|^2_{L_2(R)\to L_2(R)} \le C(\lambda)\sup_{|y-t|\le 1}\frac{R_0^2(y)}{R_0^2(t)} < C(\lambda)\cdot\mu_0^2 \le C_0^2 < \infty.$$

This estimate proves inequality (23) of Lemma 2.10.

Now we prove inequality (24). By virtue of estimate (20), we have

$$\left\| R_1(y)|t|(l_t+\lambda I)^{-1}\right\|^2_{L_2(R)\to L_2(R)} \le C(\lambda)\sup_{j\in Z}\left\| R_1(y)|t|\varphi_j(l_{t,j}+\lambda I)^{-1}\right\|^2_{L_2(\Delta_j)\to L_2(\Delta_j)} \le$$

$$\le C(\lambda)\sup_{j\in Z}\max_{y\in\overline{\Delta_j}}R_1^2(y)|t|^2\left\|(l_{t,j}+\lambda I)^{-1}\right\|^2_{L_2(\Delta_j)\to L_2(\Delta_j)}.$$

From the last inequality, Lemma 2.8 and condition ii), we obtain

$$\left\| R_1(y)|t|(l_t+\lambda I)^{-1}\right\|^2_{L_2(R)\to L_2(R)} \le C(\lambda)\cdot\mu_1^2 \le C_1^2 < \infty.$$

The inequality (24) is proved.

The inequality (25) is proved in the same way as the inequality (24). Lemma 2.10 is proved completely.

Proofs of Theorem 2.

According to Theorem 1 and equality (13), we obtain

$$R_0(y)u(x,y) = R_0(y)F^{-1}_{t\to x}(l_t+\lambda I)^{-1}\widetilde{f}(t,y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}R_0(y)(l_t+\lambda I)^{-1}\widetilde{f}(t,y)\cdot e^{itx}dt =$$

$$= F^{-1}_{t\to x}R_0(y)(l_t+\lambda I)^{-1}\widetilde{f}(t,y).$$

Hence, using the unitarity property of the operator $F^{-1}_{t\to x}$, we find

$$\|R_0(y)u(x,y)\|_2^2 = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}R_0(y)(l_t+\lambda I)^{-1}(\widetilde{f}(t,y))^2dy\right)dt =$$

$$= \int_{-\infty}^{\infty}\left\| R_0(y)(l_t+\lambda I)^{-1}\widetilde{f}(t,y)\right\|_2^2 dt \le \int_{-\infty}^{\infty}\left\| R_0(y)(l_t+\lambda I)^{-1}\right\|^2_{2\to 2}\cdot\left\|\widetilde{f}(t,y)\right\|^2 dt.$$

From the last inequality and using Parseval's equality in $L_2(R)$, we obtain

$$\|R_0(y)u(x,y)\|_2^2 \le \sup_{t\in R}\left\| R_0(y)(l_t+\lambda I)^{-1}\right\|^2_{2\to 2}\cdot\int_{-\infty}^{\infty}\left\|\widetilde{f}(t,y)\right\|^2 dt \le$$

$$\le \sup_{t\in R}\left\| R_0(y)(l_t+\lambda I)^{-1}\right\|^2_{2\to 2}\cdot\|f(x,y)\|_2^2\,dt.$$

From the last inequality and estimate (23) it follows that

$$\|R_0(y)u(x,y)\|_2^2 \le C_0^2\,\|f(x,y)\|_2^2\,,$$

i.e.

$$\|R_0(y)u(x,y)\|_2 \le C_0 \|(L+\lambda I)u\|_2^2, \tag{26}$$

where $(L+\lambda I)u = f(x,y)$.

Further, using estimate (24) and repeating the computations and arguments that were used in the proof of (26), we have

$$\left\|R_1(y)\frac{\partial u}{\partial x}\right\|_2 \le C_1 \|(L+\lambda I)u\|_2. \tag{27}$$

Similarly, we have

$$\left\|R_2(y)\frac{\partial^3 u}{\partial x^3}\right\|_2 \le C_2 \|(L+\lambda I)u\|_2. \tag{28}$$

Now from inequalities (26)–(28) we have

$$\left\|\frac{\partial u}{\partial y}\right\|_2 = \left\|(L+\lambda I) - R_2(y)\frac{\partial^3 u}{\partial x^3} - R_1(y)\frac{\partial u}{\partial y} - R_0(y)u - \lambda u\right\|_2 \le$$

$$\le \|(L+\lambda I)u\|_2 + C_2\|(L+\lambda I)u\|_2 + C_1\|(L+\lambda I)u\|_2 + C_0\|(L+\lambda I)u\|_2 +$$
$$+\lambda\|(L+\lambda I)u\|_2 \le C(\lambda)\|(L+\lambda I)u\|_2. \tag{29}$$

From (26)–(29) it follows that

$$\left\|\frac{\partial u}{\partial y}\right\|_2 + \left\|R_2(y)\frac{\partial^3 u}{\partial x^3}\right\|_2 + \left\|R_1(y)\frac{\partial u}{\partial y}\right\|_2 + \|R_0(y)u\|_2 \le C(\lambda)(\|Lu\|_2 + \|u\|_2),$$

where $C > 0$ is constant number independent of $u(x,y)$. Theorem 2 is proved.

References

1 Temam R. Sur un probleme non linearie / R. Temam // Math.Pures. Apple. — 1969. — 48. — №2. — P. 159–172.

2 Лионс Ж.Л. Некоторые методы решения нелинейных краевых задач / Ж.Л. Лионс. — М.: Мир, 1972. — 586 с.

3 Дубинский Ю.А. Об одной абстрактной теореме и ее приложении к краевым задачам для неклассических уравнений / Ю.А. Дубинский // Мат. сб. — 1969. — 79 (121). — №1. — С. 91–117.

4 Юрчук Н.И. О граничных задачах для уравнений, содержащих в главной части оператор вида d^{2m+1}/dt^{2m+1} / Н.И. Юрчук // Дифф. уравн. — 1974. — 10. — № 4. — С. 759–762.

5 Пятков С.Г. Об одном линейном уравнении неклассического типа высокого порядка / С.Г. Пятков // Препринт им. СО АН СССР. — 1981. — 24 с.

6 Bona J.Y. A nonhomogeneous boundary– value problem for the Korteweg-de Vries equation posed on a finite domain / J.Y. Bona, Shu Ming Sun, Bing–Yu Zhang // Comm. Partial Differential Equations. — 2003. — 28. — № 7–8. — P. 1391–1436.

7 Тихонов А.Н. Уравнения математической физики / А.Н. Тихонов, А.А. Самарский. — М.: Наука, 1972. — 735 с.

8 Everitt W.N. Some properties of the domains of certain differential operators / W.N. Everitt, M. Giertz // Proceedings of the London Mathematical Society. — 1971 — 3. — № 23. — P. 301–324.

9 Отелбаев М. Коэрцитивные оценки и теоремы разделимости для эллиптических уравнений в R^n / М. Отелбаев // Тр. МИАН. — 1933. — 161. — С. 195–217.

10 Muratbekov M. On the existence of a resolvent and separability for a class of singular hyperbolic type differential operators on an unbounded domain / M. Muratbekov, M. Otelbaev // Eurasian Mathematical Journal. — 2011. — 7. — No. 1. — P. 50–67.

11 Muratbekov M.B. On existence of the resolvent and discreteness of the spectrum of a class of differential operators of hyperbolic type / M.B. Muratbekov, M.M. Muratbekov, A.M. Abylayeva // Electronic Journal of Theory of Differential Equations. — 2013. — 63. — P. 1–10.

12 Muratbekov M.B. Estimates of spectrum for a class of mixed type operators / M.B. Muratbekov, M.M. Muratbekov // Differential Equations. — 2007. — 43. — P. 143–146.

13 Ospanov K.N. Qualitative and approximate characteristics of solutions of Beltrami type systems / K.N. Ospanov // Complex Variables and Elliptic Equations. — 2015. — 60. — № 7. — P. 1005–1014.

14 Ospanov K.N. Discreteness and estimates of spectrum of a first order difference operator / K.N. Ospanov // Eurasian Mathematical Journal. — 2018. — 9. — No. 2. — P. 89–94.

15 Muratbekov M.M. On the existence of the resolvent and separability of a class of the Korteweg-de Vriese type linear singular operators / M.M. Muratbekov, A.O. Suleimbekova // Bulletin of the Karaganda University. — 2021. — 101. — № 1. — P. 87–97.

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Бүкіл жазықтықта берілген үшінші ретті дифференциалдық оператордың бөліктенуі туралы

Мақалада $L_2(R^2)$ кеңістігінде коэффициенттері $R(-\infty, +\infty)$-да үзіліссіз үшінші ретті дифференциалдық оператор қарастырылған. Мұнда бұл коэффициенттер шексіздікте шектеусіз функциялар болуы мүмкін. Сонымен қатар, коэффициенттерге қатысты жоғарыдағы шарттардан бөлек кей шектеу қою арқылы берілген оператордың шектеулі қайтымдылығы дәлелденген және коэрцитивті бағалау алынған, яғни бөліктену анықталған.

Кілт сөздер: резольвента, үшінші ретті дифференциалдық оператор, бөліктену.

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Разделимость дифференциального оператора третьего порядка, заданного на всей плоскости

В статье в пространстве $L_2(R^2)$ изучен дифференциальный оператор третьего порядка с непрерывными коэффициентами в $R(-\infty, +\infty)$. Здесь данные коэффициенты могут быть неограниченными функциями на бесконечности. Автором при некоторых ограничениях на коэффициенты, помимо указанных выше условий, доказана ограниченная обратимость заданного оператора и получена коэрцитивная оценка, т.е. разделимость.

Ключевые слова: резольвента, дифференциальные уравнения третьего порядка, разделимость, неограниченные функции.

References

1 Temam, R. (1969). Sur un probleme non linearie. *J. Math.Pures. Apple, 48*(2), 159–172 [in French].

2 Lyons, Zh.L. (1972). *Nekotorye metody resheniia nelineinykh kraevykh zadach [Some methods for solving nonlinear boundary value problems]*. Moscow: Mir [in Russian].

3 Dubinsky, Yu.A. (1969). Ob odnoi abstraktnoi teoreme i ee prilozhenii k kraevym zadacham dlia neklassicheskikh uravnenii [On an abstract theorem and its applications to boundary value problems for non-classical equations]. *Matematicheskii sbornik — Mathematical collection, 79*(121), 1, 91–117 [in Russian].

4 Yurchuk, N.I. (1974). O granichnykh zadachakh dlia uravnenii, soderzhashchikh v glavnoi chasti operator vida d^{2m+1}/dt^{2m+1} [On boundary value problems for equations containing operators of the form d^{2m+1}/dt^{2m+1} in the main part]. *Differentsialnye uravneniia — Differential equations, 10*(4), 759–762 [in Russian].

5 Pyatkov, S.G. (1981). Ob odnom lineinom uravnenii neklassicheskogo tipa vysokogo poriadka [On a non-classical-type high-order linear equation]. *Preprint IPM SO AN SSSR — Preprint of Institute of Mathematics, Siberian Branch of the USSR Academy of Sciences, 24* [in Russian].

6 Bona, J.Y., Shu Ming Sun, & Bing–Yu Zhang (2003). A nonhomogeneous boundary-value problem for the Korteweg-de Vries equation posed on a finite domain. *Comm. Partial Differential Equations, 28*(7, 8), 1391–1436.

7 Tikhonov, A.N., & Samarskii, A.A. (1972). *Uravneniia matematicheskoi fiziki [Equations of mathematical physics]*. Moscow: Nauka, 735 [in Russian].

8 Everitt, W.N., & Giertz, M. (1971). Some properties of the domains of certain differential operators. *Proceedings of the London Mathematical Society, (3)23*, 301–324

9 Otelbaev, M. (1983). Koertsitivnye otsenki i teoremy razdelimosti dlia ellipticheskikh uravnenii v R^n [Coercive estimates and separability theorems for elliptic equations in R^n]. *Trudy Matematicheskogo instituta imeni V.A. Steklova — Proceedings of the V.A. Steklov Mathematical Institute, 161,* 195–217 [in Russian].

10 Muratbekov, M., & Otelbaev, M. (2016). On the existence of a resolvent and separability for a class of singular hyperbolic type differential operators on an unbounded domain. *Eurasian Mathematical Journal, 7*(1), 50–67.

11 Muratbekov, M.B., Muratbekov, M.M., & Abylayeva, A.M.(2013). On existence of the resolvent and discreteness of the spectrum of a class of differential operators of hyperbolic type. *Electronic Journal of Theory of Differential Equations, 63,* 1–10.

12 Muratbekov, M.B., & Muratbekov, M.M. (2007). Estimates of spectrum for a class of mixed type operators. *Differential Equations, 43*, 143–146.

13 Ospanov, K.N.(2015). Qualitative and approximate characteristics of solutions of Beltrami type systems. *Complex Variables and Elliptic Equations, 60,* 7, 1005–1014.

14 Ospanov, K.N.(2018). Discreteness and estimates of spectrum of a first order difference operator. *Eurasian Mathematical Journal, 9*(2), 89–94.

15 Muratbekov, M.M., & Suleimbekova, A.O. (2021). On the existence of the resolvent and separability of a class of the Korteweg-de Vriese type linear singular operators. *Bulletin of the Karaganda University, 1*(101), 87–97.