DOI 10.31489/2020M3/105-119

MSC 65M06,65M12, 35G10

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On the stable difference scheme for the time delay telegraph equation

The stable difference scheme for the approximate solution of the initial boundary value problem for the telegraph equation with time delay in a Hilbert space is presented. The main theorem on stability of the difference scheme is established. In applications, stability estimates for the solution of difference schemes for the two type of the time delay telegraph equations are obtained. As a test problem, one-dimensional delay telegraph equation with nonlocal boundary conditions is considered. Numerical results are provided.

Keywords: difference schemes, delay telegraph equations, stability.

Introduction

Time delays appear in a diversity of science and engineering, such as biology, physics, chemistry, dynamical processes. The delay term can cause oscillatory instabilities and chaos. However, to find more realistic solutions to the problems encountered in life, the delay term should be taken into consideration in mathematical modeling. Many scientists have worked to solve such problems (see [1-10]).

Telegraph equation is mostly interested in physical systems. Many physicists, engineers and mathematicians have studied on telegraph equation without time delay (see [11-18]) paranthesis is missed. Operator theory is used in [19] for the investigation of stability of the initial value problem for the telegraph equation in a Hilbert space. Ashyralyev, Agirseven and Turk in [20] studied the stability of the initial value problem for the telegraph differential equation with time delay

$$\begin{cases} \frac{d^2u(t)}{dt^2} + \alpha\frac{du(t)}{dt} + Au(t) = aAu([t]),\ t > 0, \\ u(0) = \varphi, \quad u'(0) = \psi \end{cases} \tag{1}$$

in a Hilbert space H with a self-adjoint positive definite operator A, $A \geq \delta I$, φ and ψ are elements of $D(A)$ and $[t]$ denotes the greatest-integer function, here $\delta > \frac{\alpha^2}{4}$ and $0 < a < 1$.

In the present paper, the first order of accuracy stable two-step difference scheme

$$\begin{cases} \frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2} + \alpha\frac{u_{k+1}-u_k}{\tau} + Au_{k+1} = aAu_{\left[\frac{k-mN}{N+1}\right]N+mN}, \\ N\tau = 1,\ (m-1)N+1 \leq k \leq mN-1,\ m = 1,2,..., \\ u_0 = \varphi, \quad \left((1+\alpha\tau)I + \tau^2A\right)\frac{u_1-u_0}{\tau} = \psi, \\ \left((1+\alpha\tau)I + \tau^2A\right)\frac{u_{mN+1}-u_{mN}}{\tau} = \frac{u_{mN}-u_{mN-1}}{\tau}, \quad m = 1,2,... \end{cases} \tag{2}$$

for the solution of the problem (1) is constructed. The main theorem on stability estimates for the solution of difference problem (2) is established. In applications, stability estimates for the solution of the difference scheme for the two type of the time delay telegraph equations are established. As a test problem, an initial-boundary value problem for one-dimensional delay telegraph equations with nonlocal boundary conditions is considered. Numerical results are given.

The stability of difference scheme (2)

Throughout this paper, the operator B is defined by the formula

$$B = A - \frac{\alpha^2}{4} I.$$

It is easy to show that for $\delta > \frac{\alpha^2}{4}$, the operator B is a self-adjoint positive definite operator in a Hilbert space H with $B \geq (\delta - \frac{\alpha^2}{4})I$. Operator functions R and $\widetilde{R}$ are given by formulas

$$Ru = \left(\left(1 + \frac{\alpha\tau}{2}\right) I - i\tau B^{1/2}\right)^{-1} u, \quad \widetilde{R}u = \left(\left(1 + \frac{\alpha\tau}{2}\right) I + i\tau B^{1/2}\right)^{-1} u.$$

Lemma 1. The following estimates hold:

$$\|B^{-1/2}\|_{H\to H} \leq \frac{1}{\sqrt{\delta - \frac{\alpha^2}{4}}}, \tag{3}$$

$$\|R\|_{H\to H} \leq 1, \quad \|\tau B^{1/2} R\|_{H\to H} \leq 1, \quad \| \widetilde{R}\|_{H\to H} \leq 1, \|\tau B^{1/2} \ \widetilde{R}\|_{H\to H} \leq 1, \tag{4}$$

$$\left\|\tau B^{1/2}((1+\alpha\tau)I + \tau^2 A)^{-1}\right\|_{H\to H} \leq 1. \tag{5}$$

The proof of Lemma 1 is based on the spectral representation of the self-adjoint positive definite operator B in Hilbert space H (see [21]).

Theorem 1. For the solution of difference problem (2), the following estimates hold:

$$\max_{1\leq k\leq N} \|u_k\|_H \leq b \|\varphi\|_H + \left\|B^{-1/2}\psi\right\|_H, \tag{6}$$

$$\max_{1\leq k\leq N} \left\|B^{-1/2}\frac{u_k - u_{k-1}}{\tau}\right\|_H \leq c \|\varphi\|_H + d \left\|B^{-1/2}\psi\right\|_H, \tag{7}$$

$$\max_{mN+1\leq k\leq (m+1)N} \|u_k\|_H \leq b \max_{(m-1)N\leq k\leq mN} \|u_k\|_H + \max_{(m-1)N+1\leq k\leq mN} \left\|B^{-1/2}\frac{u_k - u_{k-1}}{\tau}\right\|_H, \ m = 1, 2, ..., \tag{8}$$

$$\max_{mN+1\leq k\leq (m+1)N} \left\|B^{-1/2}\frac{u_k - u_{k-1}}{\tau}\right\|_H \leq c \max_{(m-1)N\leq k\leq mN} \|u_k\|_H + d \max_{(m-1)N+1\leq k\leq mN} \left\|B^{-1/2}\frac{u_k - u_{k-1}}{\tau}\right\|_H, \ m = 1, 2, ..., \tag{9}$$

where

$$b = |a| + |1-a|d, \quad c = |1-a|\frac{\delta}{\delta - \frac{\alpha^2}{4}}, \quad d = 1 + \frac{\frac{\alpha}{2}}{\sqrt{\delta - \frac{\alpha^2}{4}}}.$$

Proof. Difference problem (2) can be rewritten as the equivalent initial value problem for the second order difference equations with operator coefficients

$$\begin{cases} \left((1+\alpha\tau)\, I+\tau^2 A\right) u_{k+1}-(2+\alpha\tau)\, u_k+u_{k-1}=a\tau^2 A u_{\left[\frac{k-mN}{N+1}\right]N+mN}, \\ N\tau=1,\ (m-1)N+1\le k\le mN-1,\ m=1,2,... \\ u_0=\varphi, \quad u_1=\varphi+\tau((1+\alpha\tau)I+\tau^2A)^{-1}\psi, \\ u_{mN+1}=u_{mN}+R\widetilde{R}(u_{mN}-u_{mN-1}),\ m=1,2,... \end{cases}$$

Let $1\le k\le N$. It is clear that

$$u_1=\varphi+\tau B^{1/2}R\widetilde{R}B^{-1/2}\psi$$

and

$$B^{-1/2}\frac{u_1-u_0}{\tau}=((1+\alpha\tau)I+\tau^2A)^{-1}B^{-1/2}\psi=R\widetilde{R}B^{-1/2}\psi.$$

Then, using the triangle inequality and estimate (5), we get

$$\|u_1|_H\le\|\varphi\|_H+\|B^{-1/2}\psi\|_H$$

and

$$\|B^{-1/2}\frac{u_1-u_0}{\tau}\|_H\le\|\varphi\|_H+\|B^{-1/2}\psi\|_H.$$

Therefore, they follow the estimates (6) and (7) for $k=1$. Now, we prove estimates (6) and (7) for $2\le k\le N$. We have that (see [21])

$$\begin{aligned} u_k&=R\widetilde{R}\left(\widetilde{R}-R\right)^{-1}\left(R^{k-1}-\widetilde{R}^{k-1}\right)u_0+\left(\widetilde{R}-R\right)^{-1}\left(\widetilde{R}^k-R^k\right)u_1 \\ &+\sum_{j=1}^{k-1}R\widetilde{R}\left(\widetilde{R}-R\right)^{-1}\left(\widetilde{R}^{k-j}-R^{k-j}\right)a\tau^2Au_{\left[\frac{j-mN}{N+1}\right]N+mN}. \end{aligned} \tag{10}$$

Using the formula (10) and the following identities

$$\left(I-\widetilde{R}\right)(I-R)=\tau^2AR\widetilde{R},\ \left(\widetilde{R}-R\right)^{-1}=\left(-2i\tau B^{1/2}\right)^{-1}R^{-1}\widetilde{R}^{-1},$$

we get

$$\begin{aligned} u_k=&\left\{a+(1-a)\,\frac{i}{2}\left(B^{-1/2}\left(-\frac{\alpha}{2}I-iB^{1/2}\right)R^{k-1}\right.\right. \\ &\left.\left.-B^{-1/2}\left(-\frac{\alpha}{2}I+iB^{1/2}\right)\widetilde{R}^{k-1}\right)\right\}\varphi+\frac{i}{2}\left(\widetilde{R}^k-R^k\right)B^{-1/2}\psi \end{aligned} \tag{11}$$

and

$$\begin{aligned} B^{-1/2}\frac{u_{k+1}-u_k}{\tau}=&\left\{(1-a)\,\frac{i}{2}B^{-1/2}\left(B^{-1/2}\left(-\frac{\alpha}{2}I-iB^{1/2}\right)\left(-\frac{\alpha}{2}I+iB^{1/2}\right)R^k\right.\right. \\ &\left.\left.-B^{-1/2}\left(-\frac{\alpha}{2}I+iB^{1/2}\right)\left(-\frac{\alpha}{2}I-iB^{1/2}\right)\widetilde{R}^k\right)\right\}\varphi \\ &+\frac{i}{2}B^{-1/2}\left\{\left(-\frac{\alpha}{2}I-iB^{1/2}\right)\widetilde{R}^{k+1}-\left(-\frac{\alpha}{2}I+iB^{1/2}\right)R^{k+1}\right\}B^{-1/2}\psi. \end{aligned} \tag{12}$$

Applying the formulas (11) and (12), using the triangle inequality and the estimates (3) and (4), we obtain

$$\|u_k\|_H \leq \left(|a| + |1-a|\left(1+\frac{\alpha}{2}\frac{1}{\sqrt{\delta^2-\frac{\alpha^2}{4}}}\right)\right)\|\varphi\|_H + \left\|B^{-1/2}\psi\right\|_H \tag{13}$$

and

$$\left\|B^{-1/2}\frac{u_{k+1}-u_k}{\tau}\right\|_H \leq |1-a|\frac{\delta}{\delta-\frac{\alpha^2}{4}}\|\varphi\|_H + \left(1+\frac{\alpha}{2}\frac{1}{\sqrt{\delta^2-\frac{\alpha^2}{4}}}\right)\left\|B^{-1/2}\psi\right\|_H. \tag{14}$$

From (13) and (14), they follow the estimates (6) and (7) for $2 \leq k \leq N$.

Now, let $mN+1 \leq k \leq (m+1)N$ for $m = 1,2,3....$ It is clear that

$$u_{mN+1} = u_{mN} + \tau B^{1/2}R\widetilde{R}\left(B^{-1/2}\frac{u_{mN}-u_{mN-1}}{\tau}\right) \tag{15}$$

and

$$B^{-1/2}\frac{u_{mN+1}-u_{mN}}{\tau} = R\widetilde{R}B^{-1/2}\frac{u_{mN}-u_{mN-1}}{\tau}. \tag{16}$$

Applying formulas (15), (16) and using triangle inequality and estimates (3) and (4), we get

$$\|u_{mN+1}\|_H \leq \|u_{mN}\|_H + \left\|B^{-1/2}\frac{u_{mN}-u_{mN-1}}{\tau}\right\|_H \tag{17}$$

and

$$\left\|B^{-1/2}\frac{u_{mN+1}-u_{mN}}{\tau}\right\|_H \leq \|u_{mN}\|_H + \left\|B^{-1/2}\frac{u_{mN}-u_{mN-1}}{\tau}\right\|_H. \tag{18}$$

So, from these estimates they follow the estimates (8) and (9) for $k = mN$, respectively. Now, we will prove estimates (6) and (7) for $mN+2 \leq k \leq (m+1)N,\ m = 1,2,....$ We have that (see [21])

$$u_k = R\widetilde{R}\left(\widetilde{R}-R\right)^{-1}\left(R^{k-mN-1}-\widetilde{R}^{k-mN-1}\right)u_{mN} + \left(\widetilde{R}-R\right)^{-1}\left(\widetilde{R}^{k-mN}-R^{k-mN}\right)u_{mN+1}$$
$$+\sum_{j=mN+1}^{k-1} R\widetilde{R}\left(\widetilde{R}-R\right)^{-1}\left(\widetilde{R}^{k-j}-R^{k-j}\right)a\tau^2 Au_{\left[\frac{j-mN}{N+1}\right]N+mN} \tag{19}$$

for the solution of the difference problem (2). Using formula (19), we get

$$u_k = \left[a + (1-a)\frac{i}{2}B^{-1/2}\left[\left(\frac{-\alpha}{2}I - iB^{1/2}\right)R^{k-mN-1} - \left(\frac{-\alpha}{2}I + iB^{1/2}\right)\widetilde{R}^{k-mN-1}\right]\right]u_{mN}$$
$$+\frac{i}{2}B^{-1/2}\left(R\widetilde{R}\right)^{-1}\left[\widetilde{R}^{k-mN}-R^{k-mN}\right]\left(\frac{u_{mN+1}-u_{mN}}{\tau}\right). \tag{20}$$

Applying the formula (20) and using triangle inequality, we get

$$\|u_k\|_H \leq b\|u_{mN}\|_H + \left\|B^{-1/2}\left(\frac{u_{mN+1}-u_{mN}}{\tau}\right)\right\|_H. \tag{21}$$

From (21) it follows the estimate (8). Using (20), we obtain

$$B^{-1/2}\frac{u_{k+1}-u_k}{\tau}=\left[(1-a)\frac{i}{2}\left[B^{-1}AR^{k-mN}-B^{-1}A\widetilde{R}^{k-mN}\right]\right]u_{mN}$$
$$+\frac{i}{2}\left[\left(-\frac{\alpha}{2}B^{-1/2}-i\right)R^{-1}\widetilde{R}^{k-mN}-\left(-\frac{\alpha}{2}B^{-1/2}+i\right)\widetilde{R}^{-1}R^{k-mN}\right]B^{-1/2}\left(\frac{u_{mN+1}-u_{mN}}{\tau}\right). \quad (22)$$

Now, applying (22) and using triangle inequality, we get

$$\left\|B^{-1/2}\frac{u_{k+1}-u_k}{\tau}\right\|_H\le|1-a|\frac{\delta}{\delta-\frac{\alpha^2}{4}}\|u_{mN}\|_H+\left(1+\frac{\frac{\alpha}{2}}{\sqrt{\delta-\frac{\alpha^2}{4}}}\right)\left\|B^{-1/2}\left(\frac{u_{mN+1}-u_{mN}}{\tau}\right)\right\|_H. \quad (23)$$

From (23) it follows the estimate (9). Therefore, the proof of Theorem 1 is completed. □

By applying operator $B^{1/2}$, in the same manner of proof of Theorem 1, we can obtain the following stability results.

Theorem 2. For the solution of difference problem (2), the following estimates hold:

$$\max_{1\le k\le N}\left\|B^{1/2}u_k\right\|_H\le b\left\|B^{1/2}\varphi\right\|_H+\|\psi\|_H, \quad (24)$$

$$\max_{1\le k\le N}\left\|\frac{u_k-u_{k-1}}{\tau}\right\|_H\le c\left\|B^{1/2}\varphi\right\|_H+d\|\psi\|_H, \quad (25)$$

$$\max_{mN+1\le k\le(m+1)N}\left\|B^{1/2}u_k\right\|_H\le b\max_{(m-1)N\le k\le mN}\left\|B^{1/2}u_k\right\|_H$$
$$+\max_{(m-1)N+1\le k\le mN}\left\|\frac{u_k-u_{k-1}}{\tau}\right\|_H,\ m=1,2,..., \quad (26)$$

$$\max_{mN+1\le k\le(m+1)N}\left\|\frac{u_k-u_{k-1}}{\tau}\right\|_H\le c\max_{(m-1)N\le k\le mN}\left\|B^{1/2}u_k\right\|_H$$
$$+d\max_{(m-1)N+1\le k\le mN}\left\|\frac{u_k-u_{k-1}}{\tau}\right\|_H,\ m=1,2,.... \quad (27)$$

Applications

Now, we consider the applications of abstract Theorem 1 and Theorem 2.

As first application, we consider the initial value problem for the delay telegraph equations with nonlocal boundary conditions

$$\begin{cases} u_{tt}(t,x)+\alpha u_t(t,x)-(a(x)u_x(t,x))_x+\delta u(t,x) \\ =a\left(-(a(x)u_x([t],x)))_x+\delta u([t],x)\right),\ 0<t<\infty,\ 0<x<l, \\ u(0,x)=\varphi(x),\ u_t(0,x)=\psi(x),\ 0\le x\le l, \\ u(t,0)=u(t,l),\ u_x(t,0)=u_x(t,l),\ 0\le t<\infty. \end{cases} \quad (28)$$

Problem (28) has a unique smooth solution $u(t,x)$ for smooth funtions $a(x)\ge a_0>0$, $(x\in(0,l))$, $a(l)=a(0)$, $\delta>0$, $\varphi(x)$, $\psi(x)$, $(x\in[0,l])$ and $0<a<1$. This allows us to reduce the problem (28) to the initial value problem (1) in a Hilbert space $H = L_2[0,l]$ with a self-adjoint positive definite operator A^x defined by the formula (28).

The discretization of problem (28) is carried out in two steps. In the first step, we define the grid space

$$[0,l]_h = \{x = x_n : x_n = nh,\ 0 \le n \le M,\ Mh = l\}.$$

We introduce the Hilbert spaces $L_{2h} = L_2([0,l]_h)$ and $W_{2h}^1 = W_2^1([0,l]_h)$ of the grid functions $\varphi^h(x) = \{\varphi_n\}_0^M$ defined on $[0,l]_h$, equipped with the norms

$$\left\|\varphi^h\right\|_{L_{2h}} = \left(\sum_{x\in[0,l]_h} \left|\varphi^h(x)\right|^2 h\right)^{1/2},$$

$$\left\|\varphi^h\right\|_{W_{2h}^1} = \left\|\varphi^h\right\|_{L_{2h}} + \left(\sum_{x\in[0,l]_h} \left|\varphi^h_{x,j}(x)\right|^2 h\right)^{1/2},$$

respectively. To the differential operator A^x defined by (28), we assign the difference operator A_h^x by the formula

$$A_h^x\varphi^h(x) = \{-(a(x)\varphi_{\bar{x}})_{x,n} + \delta\varphi_n\}_1^{M-1} \tag{29}$$

acting in the space of grid functions $\varphi^h(x) = \{\varphi_n\}_0^M$ satisfying the conditions $\varphi_0 = \varphi_M$, $\varphi_1 - \varphi_0 = \varphi_M - \varphi_{M-1}$. It is well-known that A_h^x is a self-adjoint positive definite operator in L_{2h}. With the help of A_h^x, we reach the initial value problem

$$\begin{cases} \frac{d^2u^h(t,x)}{dt^2} + \alpha\frac{du^h(t,x)}{dt} + A_h^x u^h(t,x) = aA_h^x u^h([t],x), \\ 0 < t < \infty,\ x \in [0,l]_h, \\ u^h(0,x) = \varphi^h(x),\ u_t^h(0,x) = \psi^h(x),\ x \in [0,l]_h. \end{cases} \tag{30}$$

In the second step, we replace (30) with the difference scheme (2) and we get

$$\begin{cases} \frac{u_{k+1}^h(x) - 2u_k^h(x) + u_{k-1}^h(x)}{\tau^2} + \alpha\frac{u_{k+1}^h(x) - u_k^h(x)}{\tau} + A_h^x u_{k+1}^h(x) = aA_h^x u^h_{\left[\frac{k-mN}{N+1}\right]N+mN}(x), \\ t_k = k\tau,\quad x \in [0,l]_h,\ N\tau = 1,\ (m-1)N + 1 \le k \le mN - 1,\quad m = 1,2,..., \\ u_0^h(x) = \varphi^h(x),\ \left((1+\alpha\tau)I_h + \tau^2A_h^x\right)\frac{u_1^h(x) - u_0^h(x)}{\tau} = \psi^h(x),\ x \in [0,l]_h, \\ \left((1+\alpha\tau)I_h + \tau^2A_h^x\right)\frac{u_{mN+1}^h(x) - u_{mN}^h(x)}{\tau} = \frac{u_{mN}^h(x) - u_{mN-1}^h(x)}{\tau}, m = 1,2,...,\ x \in [0,l]_h. \end{cases} \tag{31}$$

Theorem 3. Suppose that $\delta > \frac{\alpha^2}{4}$. Then, for the solution $\left\{u_k^h(x)\right\}_0^N$ of problem (31) the following stability estimates hold:

$$\max_{1\le k\le N} \|u_k^h\|_{L_{2h}} \le M_1\left\{\|\varphi^h\|_{L_{2h}} + \|\psi^h\|_{L_{2h}}\right\},$$

$$\max_{1\le k\le N} \|u_k^h\|_{W_{2h}^1} + \max_{1\le k\le N}\left\|\frac{u_k^h - u_{k-1}^h}{\tau}\right\|_{L_{2h}} \le M_2\left\{\|\varphi^h\|_{W_{2h}^1} + \|\psi^h\|_{L_{2h}}\right\},$$

$$\max_{mN+1\le k\le (m+1)N}\left\|u_k^h\right\|_{L_{2h}}$$
$$\le M_3\left\{\max_{(m-1)N\le k\le mN}\left\|u_k^h\right\|_{L_{2h}}+\max_{(m-1)N+1\le k\le mN}\left\|\frac{u_k^h-u_{k-1}^h}{\tau}\right\|_{L_{2h}}\right\},\ m=1,2,...,$$

$$\max_{mN+1\le k\le (m+1)N}\left\|u_k^h\right\|_{W_{2h}^1}+\max_{mN+1\le k\le (m+1)N}\left\|\frac{u_k^h-u_{k-1}^h}{\tau}\right\|_{L_{2h}}$$
$$\le M_4\left\{\max_{(m-1)N\le k\le mN}\left\|u_k^h\right\|_{W_{2h}^1}+\max_{(m-1)N+1\le k\le mN}\left\|\frac{u_k^h-u_{k-1}^h}{\tau}\right\|_{L_{2h}}\right\},\ m=1,2,...,$$

where M_1, M_2, M_3 and M_4 do not depend on $\varphi^h(x)$ or $\psi^h(x)$.

Proof. Difference scheme (31) can be written in abstract form

$$\begin{cases}\frac{u_{k+1}^h-2u_k^h+u_{k-1}^h}{\tau^2}+\alpha\frac{u_{k+1}^h-u_k^h}{\tau}+A_h^x u_{k+1}^h=aA_h^x u^h_{\left[\frac{k-mN}{N+1}\right]N+mN},\\ t_k=k\tau,\quad N\tau=1,\ (m-1)N+1\le k\le mN-1,\quad m=1,2,...,\\ u_0^h=\varphi^h,\ \left((1+\alpha\tau)I_h+\tau^2A_h^x\right)\frac{u_1^h-u_0^h}{\tau}=\psi^h,\\ \left((1+\alpha\tau)I_h+\tau^2A_h^x\right)\frac{u_{mN+1}^h-u_{mN}^h}{\tau}=\frac{u_{mN}^h-u_{mN-1}^h}{\tau},\ m=1,2,...\end{cases}$$

in a Hilbert space L_{2h} with self-adjoint positive definite operator $A_h=A_h^x$ by formula (29). Here, $u_k^h=u_k^h(x)$ is unknown abstract mesh function defined on $[0,l]_h$ with the values in $H=L_{2h}$. Therefore, estimates of Theorem 3 follow from estimates (6), (7), (8) and (9), respectively. □

For second application of abstract Theorem 1 and Theorem 2, let $\Omega\subset R^n$ be an open bounded domain with smooth boundary S, $\overline{\Omega}=\Omega\cup S$. In $[0,\infty)\times\Omega$, we consider the initial-boundary value problem for the delay telegraph equations

$$\begin{cases}u_{tt}(t,x)+\alpha u_t(t,x)-\sum\limits_{r=1}^{n}(a_r(x)u_{x_r}(t,x))_{x_r}=a\left(-\sum\limits_{r=1}^{n}(a_r(x)u_{x_r}([t],x))_{x_r}\right),\\ x=(x_1,...,x_n)\in\Omega,\ 0<t<\infty,\\ u(0,x)=\varphi(x),\ \frac{\partial u(0,x)}{\partial t}=\psi(x),\ x\in\overline{\Omega},\\ u(t,x)=0,\ x\in S,\ 0\le t<\infty,\end{cases}\tag{32}$$

where $a_r(x)$, $(x\in\Omega)$, $\varphi(x)$, $\psi(x)$, $(x\in\overline{\Omega})$ are given smooth functions and $a_r(x)>0$ and $0<a<1$.

We introduce the Hilbert space $L_2(\overline{\Omega})$, the space of all integrable functions defined on $\overline{\Omega}$, equipped with the norm

$$\|f\|_{L_2(\overline{\Omega})}=\left\{\int\cdots\int_{x\in\overline{\Omega}}|f(x)|^2\,dx_1...dx_n\right\}^{\frac{1}{2}}.$$

The discretization of problem (32) is carried out in two steps. In the first step, we define the grid space

$$\overline{\Omega}_h = \left\{x = x_r = (h_1 j_1, \cdots, h_n j_n)\,,\ j = (j_1, \cdots, j_n)\,,\ 0 \le j_r \le N_r,\ N_r h_r = 1,\ r = 1, \cdots, n\right\},$$
$$\Omega_h = \overline{\Omega}_h \cap \Omega,\ S_h = \overline{\Omega}_h \cap S.$$

We introduce the Hilbert spaces $L_{2h} = L_2(\overline{\Omega}_h)$ and $W_{2h}^1 = W_2^1(\overline{\Omega}_h)$ of the grid functions $\varphi^h(x) = \{\varphi(h_1 r_1, ..., h_n r_n)\}$ defined on $\overline{\Omega}_h$, equipped with the norms

$$\left\|\varphi^h\right\|_{L_{2h}} = \left(\sum_{x \in \overline{\Omega}_h} \left|\varphi^h(x)\right|^2 h_1 \cdots h_n\right)^{1/2},$$

$$\left\|\varphi^h\right\|_{W_{2h}^1} = \left\|\varphi^h\right\|_{L_{2h}} + \left(\sum_{x \in \overline{\Omega}_h} \sum_{r=1}^{m} \left|\varphi^h_{x_r, j_r}(x)\right|^2 h_1 \cdots h_n\right)^{1/2},$$

respectively. To the differential operator A^x defined by (32), we assign the difference operator A_h^x by the formula

$$A_h^x u^h = -\sum_{r=1}^{n} \left(a_r(x) u^h_{x_r}\right)_{x_r, j_r},$$

where A_h^x is known as self-adjoint positive definite operator in L_{2h}, acting in the space of grid functions $u^h(x)$ satisfying the conditions $u^h(x) = 0$ for all $x \in S_h$. With the help of the difference operator A_h^x, we arrive at the following initial value problem

$$\begin{cases} \frac{d^2 u^h(t,x)}{dt^2} + \alpha \frac{du^h(t,x)}{dt} + A_h^x u^h(t,x) = a A_h^x u^h([t]\,, x), \\ 0 < t < \infty,\ x \in \Omega_h, \\ u^h(0,x) = \varphi^h(x), u_t^h(0,x) = \psi^h(x),\ x \in \overline{\Omega}_h \end{cases} \tag{33}$$

for an infinite system of ordinary differential equations.

In the second step, we replace (33) with the difference scheme (2) and we get

$$\begin{cases} \frac{u^h_{k+1}(x) - 2u^h_k(x) + u^h_{k-1}(x)}{\tau^2} + \alpha \frac{u^h_{k+1}(x) - u^h_k(x)}{\tau} + A_h^x u^h_{k+1}(x) = a A_h^x u^h_{\left[\frac{k-mN}{N+1}\right]N + mN}(x), \\ t_k = k\tau,\quad x \in \Omega_h,\ N\tau = 1,\ (m-1)N + 1 \le k \le mN - 1,\quad m = 1, 2, ..., \\ u_0^h(x) = \varphi^h(x),\ \left((1 + \alpha\tau)\, I_h + \tau^2 A_h^x\right) \frac{u_1^h(x) - u_0^h(x)}{\tau} = \psi^h(x),\ x \in \overline{\Omega}_h, \\ \left((1 + \alpha\tau)\, I_h + \tau^2 A_h^x\right) \frac{u^h_{mN+1}(x) - u^h_{mN}(x)}{\tau} = \frac{u^h_{mN}(x) - u^h_{mN-1}(x)}{\tau},\ x \in \overline{\Omega}_h,\ m = 1, 2, ... \end{cases} \tag{34}$$

Theorem 4. Suppose that $\delta > \frac{\alpha^2}{4}$. Then, for the solution $\left\{u_k^h(x)\right\}_0^N$ of problem (34) the following stability estimates hold:

$$\max_{1 \le k \le N} \|u_k^h\|_{L_{2h}} \le M_5 \left\{\|\varphi^h\|_{L_{2h}} + \|\psi^h\|_{L_{2h}}\right\},$$

$$\max_{1\le k\le N}\|u_k^h\|_{W_{2h}^1}+\max_{1\le k\le N}\left\|\frac{u_k^h-u_{k-1}^h}{\tau}\right\|_{L_{2h}}\le M_6\left\{\|\varphi^h\|_{W_{2h}^1}+\|\psi^h\|_{L_{2h}}\right\},$$

$$\max_{mN+1\le k\le (m+1)N}\|u_k^h\|_{L_{2h}}$$

$$\le M_7\left\{\max_{(m-1)N\le k\le mN}\|u_k^h\|_{L_{2h}}+\max_{(m-1)N+1\le k\le mN}\left\|\frac{u_k^h-u_{k-1}^h}{\tau}\right\|_{L_{2h}}\right\},\ m=1,2,...,$$

$$\max_{mN+1\le k\le (m+1)N}\|u_k^h\|_{W_{2h}^1}+\max_{mN+1\le k\le (m+1)N}\left\|\frac{u_k^h-u_{k-1}^h}{\tau}\right\|_{L_{2h}}$$

$$\le M_8\left\{\max_{(m-1)N\le k\le mN}\|u_k^h\|_{W_{2h}^1}+\max_{(m-1)N+1\le k\le mN}\left\|\frac{u_k^h-u_{k-1}^h}{\tau}\right\|_{L_{2h}}\right\},\ m=1,2,...,$$

where M_5, M_6, M_7 and M_8 do not depend on $\varphi^h(x)$ or $\psi^h(x)$.

Proof. Difference scheme (34) can be written in abstract form (2) in a Hilbert space $L_{2h}=L_2(\overline{\Omega}_h)$ with self-adjoint positive definite operator $A_h=A_h^x$ by formula (33). Here, $u_k^h=u_k^h(x)$ is unknown abstract mesh function defined on $\overline{\Omega}_h$ with the values in $H=L_{2h}$. Therefore, estimates of Theorem 4 follow from estimates (6), (7), (8) and (9) and the following theorem on the coercivity inequality for the solution of the elliptic difference problem in [22]. □

Theorem 5. For the solutions of the elliptic difference problem

$$A_h^x u^h(x)=\omega^h(x),\ x\in\Omega_h,\ u^h(x)=0,\ x\in S_h,$$

the following coercivity inequality holds:

$$\sum_{r=1}^{n}\left\|u^h{}_{x_r\bar{x_r}}\right\|_{L_{2h}}\le M_9||\omega^h||_{L_{2h}},$$

where M_9 does not depend on h and ω^h.

Numerical results

When the analytical methods do not work properly, the numerical methods for obtaining approximate solutions of telegraph differential equations play an important role in applied mathematics. In this section the first order of accuracy difference scheme for the solution of the initial boundary value problem for one dimensional telegraph differential equation with nonlocal boundary conditions is presented.

We consider the initial-boundary value problem

$$\begin{cases} u_{tt}(t,x)+2u_t(t,x)-u_{xx}(t,x)+u(t,x)=0.001\left(-u_{xx}([t],x)+u([t],x)\right), \\ 0<t<\infty,\ 0<x<\pi, \\ u(t,x)=e^{-t}\sin(2x),\ -1\le t\le 0,\ 0\le x\le\pi, \\ u(t,0)=u(t,\pi),\ u_x(t,0)=u_x(t,\pi),\ 0\le t<\infty \end{cases} \tag{35}$$

for the delay telegraph differential equation with nonlocal conditions.

By using step by step method and Fourier series method, it can be shown that the exact solution of the problem (35) is

$$u(t,x) = T_n(t)\sin(2x),\ n-1 \le t \le n,\ n = 1,2,...,$$

where

$$T_1(t) = \frac{999}{1000}e^{-t}\cos(2t) - \frac{1}{2000}e^{-t}\sin(2t) + \frac{1}{1000},$$

$$T_{n+1}(t) = T_n(n)e^{-t}\cos(2t) + \frac{T_n(n) + T_n'(n)}{2}e^{-t}\sin(2t)$$
$$+\frac{T_n(n)}{2000}\left(2 - 2e^{-(t-n)}\cos(2(t-n)) - e^{-(t-n)}\sin(2(t-n))\right),\ n = 1,2....$$

Using first order of accuracy difference scheme for the approximate solutions of problem (35), we get the following system of equations

$$\begin{cases} \frac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2} + 2\frac{u_n^{k+1}-u_n^k}{\tau} - \frac{u_{n+1}^{k+1}-2u_n^{k+1}+u_{n-1}^{k+1}}{h^2} + u_n^{k+1} \\ = 0.001\left(-\frac{u_{n+1}^{\left[\frac{k-mN}{N+1}\right]N+mN}-2u_n^{\left[\frac{k-mN}{N+1}\right]N+mN}+u_{n-1}^{\left[\frac{k-mN}{N+1}\right]N+mN}}{h^2} + u_n^{\left[\frac{k-mN}{N+1}\right]N+mN}\right), \\ t_k = k\tau,\ N\tau = 1,\ mN+1 \le k \le (m+1)N-1,\ m = 0,1,2,..., \\ x_n = nh,\ Mh = \pi,\ 1 \le n \le M-1, \\ u_n^0 = \sin(2nh),\ (1+2\tau)\frac{u_n^1-u_n^0}{\tau} + \tau\left(-\frac{u_{n+1}^1-2u_n^1+u_{n-1}^1}{h^2} + u_n^1\right) \\ +\tau\left(\frac{u_{n+1}^0-2u_n^0+u_{n-1}^0}{h^2} - u_n^0\right) = -\sin(2nh),\ 0 \le n \le M, \\ (1+2\tau)\frac{u_n^{mN+1}-u_n^{mN}}{\tau} + \tau\left(-\frac{u_{n+1}^{mN+1}-2u_n^{mN+1}+u_{n-1}^{mN+1}}{h^2} + u_n^{mN+1}\right) \\ +\tau\left(\frac{u_{n+1}^{mN}-2u_n^{mN}+u_{n-1}^{mN}}{h^2} - u_n^{mN}\right) = \frac{u_n^{mN}-u_n^{mN-1}}{\tau},\ 0 \le n \le M,\quad m = 1,2,..., \\ u_0^k = u_M^k,\quad u_1^k - u_0^k = u_M^k - u_{M-1}^k,\ mN \le k \le (m+1)N,\ m = 0,1,2,.... \end{cases} \tag{36}$$

We can rewrite system (36) in the matrix form

$$\begin{cases} CU^{k+1} + DU^k + EU^{k-1} = \varphi\left(U^{\left[\frac{k-mN}{mN+1}\right]N+mN}\right),\ k = 1,2,3,... \\ U^0 = \begin{bmatrix} 0 \\ \sin(2h) \\ \cdot \\ \cdot \\ \cdot \\ \sin(2(M-1)h) \\ 0 \end{bmatrix}_{(M+1)\times 1},\ U^1 = F^{-1}G\begin{bmatrix} 0 \\ \sin(2h) \\ \cdot \\ \cdot \\ \cdot \\ \sin(2(M-1)h) \\ 0 \end{bmatrix}_{(M+1)\times 1}, \\ U^{mN+1} = F^{-1}HU^{mN} - F^{-1}U^{mN-1},\ m = 1,2,..., \end{cases} \tag{37}$$

where C, D, E, F, G and H are $(M+1)\times(M+1)$ matrices, $\varphi\left(U^{\left[\frac{k-mN}{mN+1}\right]N+mN}\right)$ and U^r, $r=k,k\pm1$ are $(M+1)\times 1$ column vectors defined by

$$C=\begin{bmatrix} 1 & 0 & 0 & 0 & . & 0 & 0 & 0 & -1\\ a & b & a & 0 & . & 0 & 0 & 0 & 0\\ 0 & a & b & a & . & 0 & 0 & 0 & 0\\ . & . & . & . & . & . & . & . & .\\ 0 & 0 & 0 & 0 & . & a & b & a & 0\\ 0 & 0 & 0 & 0 & . & 0 & a & b & a\\ 1 & -1 & 0 & 0 & . & 0 & 0 & -1 & 1 \end{bmatrix}_{(M+1)\times(M+1)},$$

$$D=\begin{bmatrix} 0 & 0 & 0 & . & 0 & 0 & 0 & 0\\ 0 & c & 0 & . & 0 & 0 & 0 & 0\\ 0 & 0 & c & . & 0 & 0 & 0 & 0\\ . & . & . & . & . & . & . & .\\ 0 & 0 & 0 & . & 0 & c & 0 & 0\\ 0 & 0 & 0 & . & 0 & 0 & c & 0\\ 0 & 0 & 0 & . & 0 & 0 & 0 & 0 \end{bmatrix}_{(M+1)\times(M+1)},$$

$$E=\begin{bmatrix} 0 & 0 & 0 & . & 0 & 0 & 0 & 0\\ 0 & d & 0 & . & 0 & 0 & 0 & 0\\ 0 & 0 & d & . & 0 & 0 & 0 & 0\\ . & . & . & . & . & . & . & .\\ 0 & 0 & 0 & . & 0 & d & 0 & 0\\ 0 & 0 & 0 & . & 0 & 0 & d & 0\\ 0 & 0 & 0 & . & 0 & 0 & 0 & 0 \end{bmatrix}_{(M+1)\times(M+1)},$$

$$F=\begin{bmatrix} 1 & 0 & 0 & 0 & . & 0 & 0 & 0 & -1\\ e & p & e & 0 & . & 0 & 0 & 0 & 0\\ 0 & e & p & e & . & 0 & 0 & 0 & 0\\ . & . & . & . & . & . & . & . & .\\ 0 & 0 & 0 & 0 & . & e & p & e & 0\\ 0 & 0 & 0 & 0 & . & 0 & e & p & e\\ 1 & -1 & 0 & 0 & . & 0 & 0 & -1 & 1 \end{bmatrix}_{(M+1)\times(M+1)},$$

$$G=\begin{bmatrix} 1 & 0 & 0 & 0 & . & 0 & 0 & 0 & -1\\ e & s & e & 0 & . & 0 & 0 & 0 & 0\\ 0 & e & s & e & . & 0 & 0 & 0 & 0\\ . & . & . & . & . & . & . & . & .\\ 0 & 0 & 0 & 0 & . & e & s & e & 0\\ 0 & 0 & 0 & 0 & . & 0 & e & s & e\\ 1 & -1 & 0 & 0 & . & 0 & 0 & -1 & 1 \end{bmatrix}_{(M+1)\times(M+1)},$$

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & . & 0 & 0 & 0 & -1 \\ e & g & e & 0 & . & 0 & 0 & 0 & 0 \\ 0 & e & g & e & . & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & e & g & e & 0 \\ 0 & 0 & 0 & 0 & . & 0 & e & g & e \\ 1 & -1 & 0 & 0 & . & 0 & 0 & -1 & 1 \end{bmatrix}_{(M+1)\times(M+1)},$$

$$\varphi\left(U^{\left[\frac{k-mN}{mN+1}\right]N+mN}\right) = \begin{bmatrix} 0 \\ \varphi_1^k \\ . \\ . \\ . \\ \varphi_{M-1}^k \\ 0 \end{bmatrix}_{(M+1)\times 1}, \quad U^r = \begin{bmatrix} U_0^r \\ U_1^r \\ . \\ . \\ . \\ U_{M-1}^r \\ U_M^r \end{bmatrix}_{(M+1)\times 1}, \text{ for } r = k, k \mp 1,$$

where

$$\varphi_n^k = 0.001\left(-\frac{u_{n+1}^{\left[\frac{k-mN}{N+1}\right]N+mN} - 2u_n^{\left[\frac{k-mN}{N+1}\right]N+mN} + u_{n-1}^{\left[\frac{k-mN}{N+1}\right]N+mN}}{h^2} + u_n^{\left[\frac{k-mN}{N+1}\right]N+mN}\right) \text{ for } k = 1, 2, ...,$$

$m = 0, 1, 2, ..., 1 \leq n \leq M - 1$.

Here, we denote $a = -1/h^2$, $b = 1/\tau^2 + 2/\tau + 2/h^2 + 1$, $c = -2/\tau^2 - 2/\tau$, $d = 1/\tau^2$, $e = -\tau^2/h^2$, $p = 1 + 2\tau + \tau^2 + 2\tau^2/h^2$, $s = 1 + \tau + \tau^2 + 2\tau^2/h^2$ and $g = 2 + 2\tau + \tau^2 + 2\tau^2/h^2$.

Hence, we have a second order of difference equation with matrix coefficients. We find the numerical solutions for different values of N and M and here, u_n^k represents the numerical solutions of the difference scheme at (t_k, x_n). For $N = M = 40$, $N = M = 80$ and $N = M = 160$ in $t \in [0, 1]$, $t \in [1, 2]$ and $t \in [2, 3]$, the errors computed by the following formula are given in Table 1.

$$E_M^N = \max_{\substack{mN+1 \leq k \leq (m+1)N,\ m = 0, 1, ... \\ 0 \leq n \leq M}} \left| u(t_k, x_n) - u_n^k \right|.$$

Table 1

Errors of Difference Scheme (36)

	N=M=40	N=M=80	N=M=160
$t \in [0, 1]$	0.045895	0.023073	0.011568
$t \in [1, 2]$	0.042967	0.021574	0.010810
$t \in [2, 3]$	0.019786	0.010107	0.0051085

As it is seen in Table 1, the errors in the first order of accuracy difference scheme decrease approximately by a factor of 1/2 when the values of M and N are doubled.

Conclusion

In this study, we consider the initial-boundary value problem for telegraph equations with time delay in a Hilbert space. Theorem on stability estimates for the solution of the first order of accuracy difference scheme is established. In practice, stability estimates for the solution of the difference schemes for the two type of the time delay telegraph equations are obtained. As a test problem, one-dimensional delay telegraph equation with nonlocal boundary conditions is considered. Numerical solutions of this problem are provided.

Acknowledgement

We are grateful to the TUBITAK Graduate Scholarship Programme for supporting Koray Turk. The publication has been prepared with the support of the "RUDN University Program 5-100".

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Кідіртпелі телеграф теңдеуі үшін орнықты айырымдық схемасы туралы

Гильберт кеңістігінде кешігулі телеграф теңдеуі үшін бастапқы-шеттік есебінің жуықтау шешімінің орнықты айырымдық схемасы ұсынылған. Айырымдық схемасының орнықтылығы туралы негізгі теоремасы берілген. Қосымшасында уақыт кідіртпесі бар телеграф теңдеуінің екі түрі үшін айырымдық схемасының шешімінің орнықтылық бағамы алынды. Тестілік есебі ретінде, бейлокальді шарттарымен берілген кідіртпелі телеграф бірөлшемді теңдеуі қарастырылды. Сандық есептеулері мақалада көрсетілген.

Кілт сөздер: айырымдық схемасы, кешігулі телеграф теңдеуі, орнықтылық.

А. Ашыралыев, К. Турк, Д. Агирсевен

Об устойчивой разностной схеме для уравнения телеграфа с задержкой

Представлена устойчивая разностная схема для приближенного решения начально-краевой задачи для телеграфного уравнения с запаздыванием в гильбертовом пространстве. Установлена основная теорема об устойчивости разностной схемы. В приложениях получены оценки устойчивости решения разностных схем для двух типов телеграфных уравнений с временной задержкой. В качестве тестовой задачи рассмотрено одномерное уравнение задержки телеграфа с нелокальными условиями. Численные результаты приведены в статье.

Ключевые слова: разностные схемы, уравнения телеграфа с запаздыванием, устойчивость.

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