FROBENIUS GROUPS OF FINITE MORLEY RANK

Bruno Poizat
Camille Jordan Institute, Lyon, France
E-mail: poizat@math.univ-lyon1.fr

Frobenius group: a group F whith a *malnormal* subgroup T, i.e. T is proper, self-normalizing, and disjoint from its conjugates; U(T) denotes the complement of the conjugates of T, augmented with the identity.

Facts when F is finite : card(F) = card(T).card(U(T)) (easy); U(T) is a group (FROBENIUS 1901), which is nilpotent (THOMPSON 1959), i.e. F is the semi-direct product of U(T) by T (F *splits*), and U(T) is the largest normal nilpotent subgroup of F; when T contains an involution i, U(T) is commutative and inverted by i.

Our aim: improve the pp. 203-219 of BOROVIK-NESIN 1994, on Frobenius groups of finite Morley rank.

What to expect when F *has a finite Morley rank ?*

Basic fact: T is definable (B&N p. 206).

RM(F) = RM(T) + RM(U(T)) ? Conflicts with the possible existence of Frobenius *bad groups*, where U(T) = {1}.

F splits? In this case the splitting factor U is definable.

The semi-direct product of U by T is Frobenius iff T acts *freely* on U, i.e. $t.u = 1 \Leftrightarrow t=1 \vee u=1$, otherwise said iff for each $t \neq 1$ the commutator map $(t.u).u^{-1}$ is injective. In this case U = U(T) iff the commutator is surjective.

Injectivity implies surjectivity when U is solvable by finite.

When T *contains an involution :*

- it contains only one (improves B&N p. 207-208)
- two involutions in F are conjugate by a unique involution (they form a *symmetron* I)
- F = T.I = T.iI where i is the involution of T
- if there is a non-trivial point inversed by every involution, U(T) is a commutative group inversed by every involution
- if not, U(T) contains no non trivial subgoup normalized by every involution, and the socle of T° is a simple (non algebraic)

Frobenius group.

When F is pseudo-locally-finite (in particular when F is an algebraic group) :

- U(T) is a nilpotent group
- T is abelian by finite ; in fact T° can be embedded in the multiplicative subgroup of a definable field
- when F is connected, it is solvable and U(T) is its derivate subgroup.

Example : $\begin{bmatrix} t & u & w \\ 0 & 1 & v \\ 0 & 0 & t^{-1} \end{bmatrix}$ where $t \in K^*$ and char(K) = 2,
or $t \in M$, for a bad field (K,M)
with no involutions in M

Back to the general finite RM *case,* F *connected*
(for simplification)

The largest solvable normal subgroup of F is connected and nilpotent, and contained in U(T).

When F has no abelian normal subgroup, its socle is either a finite product of simple groups included in U(T), or a single simple Frob. group; these groups are not algebraic.

If T acts freely on an abelian group U, any infinite abelian subgroup M of T is embeddable in the multiplicative subgroup of a definable field, and any connected solvable subgroup of T is commutative (T° is commutative or contradicts the Algebraicity Conjecture).

Proof of the last sentence

(i) The action on U of the ring R = Z(M) is definable.

(ii) The restriction R_1 of R to the subgroup U_1 generated by the minimal M-submodules of U has no nilpotent element.

(iii) By ABC 2008 p. 44, R_1 is a finite product of fields; therefore U_1 is a product of vector spaces over these fields.

(iv) If N is the normalizer of M, N° preserves this decomposition of U_1, and acts linearly on each of the components; since the action is free, it has no unipotents.

(v) In characteristic p, by POIZAT 2001 no simple group is involved, and N° is commutative.

(vi) In characteristic 0, the intersection of all the Z(M) is infinite, and we finally obtain a vector space over a field K of characteristic 0 on which T° acts linearly and without unipotents; then use Lie-Kolchin-Mal'cev.

References

1. Tuna Altınel, Aleksandr Borovik & Gregory Cherlin, *Simple Groups of Finite Morley rank*, A.M.S., 2008.
2. Aleksandr Borovik & Ali Nesin, *Groups of Finite Morley Rank*, Oxford, Clarendon Press, 1994.
3. G. Frobenius, *Uber auslöbare Gruppen IV*, Berl. Sitz., 1901, P. 1223-1225.
4. Bruno Poizat, *Quelques modestes remarques à propos d'une conséquence inattendue d'un résultat surprenant de Monsieur Frank Olaf Wagner*, J.S.L., 2001, 66, P. 1637-1648
5. Bruno Poizat, *Quelques modestes compléments aux travaux de Messieurs Mark DeBonis, Franz Delahan, David Epstein et Ali Nesin sur les Groupes de Frobenius de rang de Morley fini*, submitted
6. J. G. Thompson, *Finite groups with fixed-point-free automorphisms of prime order*, Proc. Nat. Acad. Sci., 1959, 45, P. 578-581.

FAMILIES OF ELEMENTARY THEORIES AND THEIR BASIC CHARACTERISTICS

Sudoplatov S.V.
Sobolev Institute of Mathematics, Novosibirsk State Technical University, Novosibirsk State University, Novosibirsk, Russia
E-mail: sudoplat@math.nsc.ru

We present a survey of results on families of elementary theories and their basic characteristics including the following main items:

1. Topological, spectral and syntactic characterization of total transcendence for families of theories and their closures both in general for complete and incomplete theories, and for families of theories of abelian groups [1, 2, 3, 4, 5].

2. Description of rank values and their dynamics for various families of theories and their subfamilies [1, 2, 6, 7].

3. Description of the approximability and approximations of theories by various families [8, 9].

4. Characterization and description of generating sets, *P*-closures $\mathrm{Cl}_P(T)$ and *E*-closures $\mathrm{Cl}_E(T)$ for families *T* of theories and their combinations [5, 10, 11].

5. Characterization and description of formulas for families of theories, as well as of their characteristics [3, 12, 13].

6. Description of arities of theories, their dynamics and characteristics under transition to closures [14, 15, 16].

7. Description of countable spectra and Hasse diagrams for various families of theories [17, 18] including linearly, circularly and spherically ordered ones [19, 20, 21].

8. Minimality conditions for ordered structures [22, 23, 24].

As for linear and circular orders we divide the class of spherically ordered structures into dense, discrete and mixed ones. For these cases we introduce minimality conditions similar to ones in [22, 23, 24]. In particular, dense spherical orders produce analogues of (weak) o-minimality, and discrete minimal ones assume possibilities to define finite or cofinite sets only. Moreover, these spherical orders are connected with correspondent linear ones, as well as their automorphism groups are naturally linked.