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Method of constructing general solution of the second order linear ordinary differential equations with variable coefficients

In this article new method of constructing general solution of the second order linear ordinary differential equations with variable coefficients is presented. The general solutions of Airy equation, of the second order linear ordinary differential equations with variable coefficients and coefficient e^x are constructed by this method. Constructed in explicit form general solutions can be used for solving of the Cauchy problem and of the two point's boundary value problems for ordinary differential equations with variable coefficients arising in solving various applied problems of science.

Key words: the second order, linear ordinary differential equation, variable and singular coefficients, general solution.

1 Introduction

Let $-\infty < x_1 < x_2 < \infty$. By $S[x_1, x_2]$ we denote the class of measurable, essentially bounded functions in $[x_1, x_2]$. The norm of an element from $S[x_1, x_2]$ is defined by the formula

$$|f|_0 = \sup_{x\in[x_1,x_2]} vrai|f(x)| = \lim_{p\to\infty}\|f\|_{L_p[x_1,x_2]}.$$

We consider the equation

$$\frac{d^2y}{dx^2} - a(x)y = f(x) \tag{1}$$

in interval $[x_1, x_2]$, where $a(x), f(x) \in S[x_1, x_2]$.

The solution of the equation (1) from class

$$W_\infty^2[x_1,x_2] \cap C^1[x_1,x_2] \tag{2}$$

is sought.

Here $W_\infty^2[x_1,x_2]$ is a class of functions $y(x)$, such that $\frac{d^2y}{dx^2} \in S[x_1,x_2]$.

If $a(x), f(x) \in C[x_1,x_2]$, then general solutions that are found in this article belong to the class $C^2[x_1,x_2]$.

2 Construction of general solution to equation (1)

Let $x_0 \in [x_1, x_2]$. By integrating two times the equation (1) we get

$$y(x) = (By)(x) + g(x) + c_1(x - x_0) + c_2; \tag{3}$$

$$(By)(x) = \int_{x_0}^{x}\int_{x_0}^{\tau} a(t)y(t)dtd\tau,\ g(x) = \int_{x_0}^{x}\int_{x_0}^{\tau} f(t)dtd\tau,$$

where c_1, c_2 are arbitrary real numbers.

Applying the operator $(B\cdot)(x)$ to both sides of the equation (3), we have

$$(By)(x) = (B^2y)(x) + (Bg)(x) + c_1a_1(x) + c_2b_1(x), \tag{4}$$

where

$$(B^2y)(x) = (B(By)(x))(x),\ \ a_1(x) = \int_{x_0}^{x}\int_{x_0}^{\tau}(t - x_0)a(t)dtd\tau,\ \ b_1(x) = \int_{x_0}^{x}\int_{x_0}^{\tau} a(t)dtd\tau.$$

From (3) and (4) it follows

$$y(x) = (B^2y)(x) + (Bg)(x) + g(x) + c_1((x - x_0) + a_1(x)) + c_2(1 + b_1(x)). \tag{5}$$

In the following we use functions and operators

$$(B^ny)(x) = (B(B^{n-1}y)(x))(x),\ \ (n = \overline{2,\infty}),\ (B^1y)(x) = (By)(x);$$

$$a_k(x) = \int_{x_0}^{x}\int_{x_0}^{\tau} a(t)a_{k-1}(t)dtd\tau,\ \ b_k(x) = \int_{x_0}^{x}\int_{x_0}^{\tau} a(t)b_{k-1}(t)dtd\tau,\ \ (k = \overline{2,\infty}).$$

Applying the operator $(B\cdot)(x)$ to both sides of the equation (5) again, we get

$$(By)(x) = (B^3y)(x) + (B^2g)(x) + (Bg)(x) + c_1(a_1(x) + a_2(x)) + c_2(b_1(x) + b_2(x)). \tag{6}$$

From (3) and (6) it follows

$$y(x) = (B^3y)(x) + (B^2g)(x) + (Bg)(x) + g(x) + c_1(x - x_0 + a_1(x) + a_2(x)) + c_2(1 + b_1(x) + b_2(x)).$$

Continuing this procedure n times, we obtain the following integral representation for solutions of the equation (1):

$$y(x) = (B^ny)(x) + c_1\left(x - x_0 + \sum_{k=1}^{n-1} a_k(x)\right) + c_2\left(1 + \sum_{k=1}^{n-1} b_k(x)\right) + \sum_{k=0}^{n-1}(B^kg)(x), \tag{7}$$

where $(B^0g)(x) = g(x)$.

Let $y(x) \in C[x_1, x_2]$. Then the following inequalities are easily obtained

$$|(B^ny)(x)| \le |y|_1 \cdot \frac{(|a|_0 \cdot x^2)^n}{n!} \cdot \frac{1}{(n+1)\cdots 2n}; \tag{8}$$

$$|a_k(x)| \le |a|_0^k \cdot \frac{x^{2k+1}}{(2k+1)!},\ |b_k(x)| \le |a|_0^k \cdot \frac{x^{2k}}{(2k)!}, \tag{9}$$

where $|f|_0 = \sup\limits_{x\in[x_1,x_2]} vrai|f(x)|$, $|f|_1 = \max\limits_{x\in[x_1,x_2]}|f(x)|$.

Passing to the limit with $n \to \infty$ in the representation (7) and using of inequalities (8), (9), we get

$$y(x) = c_1I_1(x) + c_2I_2(x) + F(x); \tag{10}$$

where $I_1(x) = x - x_0 + \sum\limits_{k=1}^{\infty} a_k(x)$, $I_2(x) = 1 + \sum\limits_{k=1}^{\infty} b_k(x)$, $F(x) = \sum\limits_{k=0}^{\infty}(B^kg)(x)$.

Using the estimates (9) and (8), we receive

$$|I_1(x)| \le |x - x_0| + \frac{1}{\sqrt{|a|_0}} sh\left(\sqrt{|a|_0}|x - x_0|\right),\ |I_2(x)| \le ch\left(\sqrt{|a|_0}|x - x_0|\right);$$

$$\left|F(x)\right| \le \left|f\right|_1 + \frac{1}{\sqrt{|a|_0}} ch\left(\sqrt{|a|_0}\left|x - x_0\right|\right).$$

From the form of functions $I_1(x),\ I_2(x)$ and $F(x)$ it follows

$$I_1^{'}(x) = 1 + \int_{x_0}^{x} a(t)I_1(t)dt,\ I_2^{'}(x) = \int_{x_0}^{x} a(t)I_2(t)dt,\ F^{'}(x) = \int_{x_0}^{x} f(t)dt + \int_{x_0}^{x} a(t)F(t)dt;$$

$$I_1^{''}(x) = a(x)I_1(x),\ I_2^{''}(x) = a(x)I_2(x),\ \ F^{''}(x) = f(x) + a(x)F(x); \quad (11)$$

$$I_1(x_0) = F(x_0) = F^{'}(x_0) = I_2^{'}(x_0) = I_1^{''}(x_0) = 0,\ I_2(x_0) = I_1^{'}(x_0) = 1; \quad (12)$$

$$I_2^{''}(x_0) = -a(x_0),\ F^{''}(x_0) = f(x_0).$$

From formulas (11) it follows that the functions $I_1(x),\ I_2(x)$ are particular solutions from the class (2) of the homogeneous equation $\frac{d^2y}{dx^2} - a(x)y = 0$ and the function $F(x)$ is a particular solution from the class (2) of the inhomogeneous equation (1).

From (12) it follows that the Wronskian $W(x)$ of the functions $I_1(x),\ I_2(x)$ is not equal to zero in $x = x_0$:

$$W(x_0) = \begin{vmatrix} I_1(x_0) & I_2(x_0) \\ I_1^{'}(x_0) & I_2^{'}(x_0) \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1.$$

Furthermore, $W^{'}(x) = I_1(x)\cdot I_2^{''}(x) - I_2(x)\cdot I_1^{''}(x) = a(x)\cdot\left(I_1(x)\cdot I_2(x) - I_1(x)\cdot I_2(x)\right) = 0$ and thereby $W(x) = const$. Consequently, $W(x) = -1$. Therefore, the functions $I_1(x)$ and $I_2(x)$ are linearly independent in the interval $\left[x_1, x_2\right]$ and the general solution for the equation (1) is determined by the formula (10) and belongs to the class (2). Summarizing we have proved the following theorem.

Theorem 1. The general solution of the equation (1) from the class (2) is given by formula (10).

Remark. If $a(x), f(x) \in C\left[x_1, x_2\right]$, then the general solution of equation (1) is given by formula (10) and belongs to the class $C^2\left[x_1, x_2\right]$.

3 Construction of the general solution to Airy equation

We consider the Airy equation

$$y^{''} - xy = 0. \quad (13)$$

As known the equation (13) is used in astronomy $\left[1,2\right]$.

Let $0 < x_2 < \infty$. The solution of equation (13) in the class

$$C^2\left[0, x_2\right] \quad (14)$$

is sought. To construct the general solution of the equation (13) we use the formula (10), where $x_0 = 0$, $F(x) \equiv 0,\ \ a(x) = x$. Then we obtain the general solution of the equation (13) in the form

$$y = c_1 I_1(x) + c_2 I_2(x), \quad (15)$$

where $c_1,\ c_2$ are arbitrary real numbers,

$$I_1(x) = x + \sum_{k=1}^{\infty} a_k(x);$$

$$I_2(x) = 1 + \sum_{k=1}^{\infty} b_k(x);$$

$$a_1(x) = \int_0^x \int_0^\tau t^2 dt d\tau;$$

$$a_k(x) = \int_0^x \int_0^\tau t a_{k-1}(t) dt d\tau;$$

$$b_1(x)=\int_0^x\int_0^\tau t\,dtd\tau;$$

$$b_k(x)=\int_0^x\int_0^\tau t b_{k-1}dtd\tau,\ \left(k=\overline{2,\infty}\right).$$

Using these formulas, we find concrete value functions $a_k(x)$ and $b_k(x)$, $\left(k=\overline{1,\infty}\right)$, and thereby the functions $I_1(x)$ and $I_2(x)$

$$a_k(x)=\frac{x^{3k+1}}{3\cdot4\cdot6\cdot7\cdot9\cdot10\cdot\cdots\cdot3k(3k+1)},\ b_k(x)=\frac{x^{3k}}{2\cdot3\cdot5\cdot6\cdot\cdots\cdot(3k-1)\cdot3k},\ \left(k=\overline{1,\infty}\right). \quad (16)$$

The formulas (16) can be proved by mathematical induction.

From formulas (16) and the form of the functions $I_1(x)$ and $I_2(x)$ it follows

$$I_1(x)=x\left(1+\frac{1}{3}\sum_{k=1}^{\infty}\frac{x^{3k}}{k!\prod_{n=1}^{k}(3n+1)}\right),\quad I_2(x)=1+\frac{1}{3}\sum_{k=1}^{\infty}\frac{x^{3k}}{k!\prod_{n=1}^{k}(3n-1)}.$$

For the obtained expressions of the functions $I_1(x)$ and $I_2(x)$ the following estimates can be got easily

$$\left|I_1(x)\right|\le\frac{x}{3}\left(2+\exp x^3\right),\ \left|I_2(x)\right|\le\frac{1}{3}\left(2+\exp x^3\right).$$

Using these inequalities as in section 2, it can be shown that functions $I_1(x)$ and $I_2(x)$ are particular solutions of the equation (13) in the class (14) and form a linearly independent system.

Consequently, the function

$$y=c_1x\left(1+\frac{1}{3}\sum_{k=1}^{\infty}\frac{x^{3k}}{k!\prod_{n=1}^{k}(3n+1)}\right)+c_2\left(1+\frac{1}{3}\sum_{k=1}^{\infty}\frac{x^{3k}}{k!\prod_{n=1}^{k}(3n-1)}\right)$$

is the general solution of the equation (13) in the class (14), where c_1, c_2 are arbitrary real numbers.

4 The general solution of the second order ordinary differential equations with singular point

We consider the equation

$$\frac{d^2y}{dx^2}-\frac{\beta}{x^\alpha}y=0, \quad (17)$$

where $0<\alpha<1$, $\beta>0$ are given parameters.

Let $-\infty<x_1<0<x_2<\infty$. Solution of the equation (17) is sought in the class (2). To construct the general solution of the equation (17) we use the formula (10), where $F(x)\equiv0$, $a(x)=\frac{\beta}{x^\alpha}$. Then we obtain the general solution of the equation (17) in the form $y=c_1I_1(x)+c_2I_2(x)$, where c_1, c_2 are arbitrary real numbers.

$$I_1(x)=x+\sum_{k=1}^{\infty}a_k(x),\ I_2(x)=1+\sum_{k=1}^{\infty}b_k(x);$$

$$a_1(x)=\int_0^x\int_0^\tau t\frac{\beta}{t^\alpha}dtd\tau,\ a_k(x)=\int_0^x\int_0^\tau\frac{\beta}{t^\alpha}a_{k-1}(t)dtd\tau,\ b_1(x)=\int_0^x\int_0^\tau\frac{\beta}{t^\alpha}dtd\tau;$$

$$b_k(x)=\int_0^x\int_0^\tau\frac{\beta}{t^\alpha}b_{k-1}dtd\tau,\ \left(k=\overline{2,\infty}\right).$$

By these formulas we find concrete value functions $a_k(x)$ and $b_k(x)$, $\left(k=\overline{1,\infty}\right)$ and thereby functions $I_1(x)$ and $I_2(x)$:

$$a_k(x)=\frac{\beta^k x^{k(2-\alpha)+1}}{k!(2-\alpha)^k\prod\limits_{n=1}^{k}(n(2-\alpha)+1)},\quad b_k(x)=\frac{\beta^k x^{k(2-\alpha)}}{k!(2-\alpha)^k\prod\limits_{n=1}^{k}(n(2-\alpha)-1)},\quad \left(k=\overline{1,\infty}\right);$$

$$I_1(x)=x+\sum_{k=1}^{\infty}\frac{\beta^k x^{k(2-\alpha)+1}}{k!(2-\alpha)^k\prod\limits_{n=1}^{k}(n(2-\alpha)+1)},\quad I_2(x)=1+\sum_{k=1}^{\infty}\frac{\beta^k x^{k(2-\alpha)}}{k!(2-\alpha)^k\prod\limits_{n=1}^{k}(n(2-\alpha)-1)}.$$

Since the function $a(x)=\frac{\beta}{x^{\alpha}}$ is not continuous in the interval $[x_1,x_2]$, within which there is a point $x=0$, then the proof of the uniform convergence of the series available for the functions $I_1(x)$ and $I_2(x)$ is not suitable for their corresponding estimates of the second paragraph. Therefore, we obtained the following inequalities

$$\left|I_1(x)\right|<x_2\cdot\exp\left(\frac{\beta x_2^{2-\alpha}}{(2-\alpha)^2}\right),\quad \left|I_1(x)\right|<\frac{1}{1-\alpha}\left((2-\alpha)\exp\left(\frac{\beta x_2^{2-\alpha}}{(2-\alpha)^2}\right)-1\right).$$

Using these inequalities as in sections 2 and 3, it can be shown that the functions $I_1(x)$ and $I_2(x)$ are linearly independent and are particular solutions of the equation (17) from the class (2). Consequently, the function

$$y(x)=c_1\left(x+\sum_{k=1}^{\infty}\frac{\beta^k x^{k(2-\alpha)+1}}{k!(2-\alpha)^k\prod\limits_{n=1}^{k}(n(2-\alpha)+1)}\right)+c_2\left(1+\sum_{k=1}^{\infty}\frac{\beta^k x^{k(2-\alpha)}}{k!(2-\alpha)^k\prod\limits_{n=1}^{k}(n(2-\alpha)-1)}\right),$$

is the general solution of the equation (17) from the class (2), where c_1, c_2 are arbitrary real numbers.

5 The general solution of the ordinary second order differential equation with coefficient e^x

Let $0<x_2<\infty$. We consider the equation

$$\frac{d^2y}{dx^2}-e^x y=0 \tag{18}$$

in the interval $[0,x_2]$.

Using formula (10) as described in paragraphs 2–4, we get the general solution of the equation (18) from the class $C^2[0,x_2]$ in the form $y=c_1I_1(x)+c_2I_2(x)$, where c_1, c_2 are arbitrary real numbers,

$$I_1(x)=2(x+1)+2(x-1)e^x+\sum_{k=2}^{\infty}\frac{1}{(k!)^2}\left(x+2\sum_{n=1}^{k}\frac{1}{n}+e^{kx}\left(x-2\sum_{n=1}^{k}\frac{1}{n}\right)\right)+$$
$$+\sum_{k=3}^{\infty}\sum_{n=1}^{k-1}\frac{e^{nx}}{\left(n!(k-n)!\right)^2}\left(x+2\sum_{m=1}^{k-n}\frac{1}{m}-2\sum_{m=1}^{n}\frac{1}{m}\right);$$

$$I_2(x)=-x-(x-2)e^x-\sum_{k=2}^{\infty}\frac{1}{(k-1)!k!}\left(x-\frac{1}{k}+2\sum_{n=1}^{k}\frac{1}{n}-\frac{e^{kx}}{k}\right)-$$
$$-\sum_{k=3}^{\infty}\sum_{n=1}^{k-1}\frac{e^{nx}}{(n!)^2(k-n)!(k-n-1)!}\left(x-\frac{1}{k-n}-2\sum_{m=1}^{n}\frac{1}{m}+2\sum_{m=1}^{k-n}\frac{1}{m}\right).$$

The program Matlab is used for calculating of the last formulas. Namely, to find the concrete values of the functions $a_k(x)$ and $b_k(x)$ we compiled the program Matlab:

```
function integ(n)
tic
   syms x y z;
   A =int(int(exp(z*x)*x,0,y),y,0,x);
   fprintf ('A%d = %s\n',1, char(A));
   B = exp(x)-x-1;
   fprintf ('B%d = %s\n\n',1, char(B));
   k = 2;
   while k<=n,
      A = int (int(exp(z*x)*A,x,0,y),y,0,x);
      fprintf ('A%d = %s\n',k, char(A));
      B = int (int(exp(x)*B,x,0,y),y,0,x);
      fprintf ('B%d = %s\n\n',k, char(B));
      k = k+1;
   end
toc
end
```

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Ә.Түнғатаров, А.А.Сқақов

Айнымалы коэффициенттері бар екінші ретті сызықты жай дифференциалдық теңдеулердің жалпы шешімдерін құрудың бір әдісі

Мақалада айнымалы коэффициенттері бар сызықты жай дифференциалдық теңдеулердің жалпы шешімдерін құрудың жаңа әдісі келтірілген. Осы әдіс арқылы Эйри теңдеуінің, сингулярлы коэффициенті бар сызықты дифференциалды теңдеудің және коэффициенті e^x болып келетін екі ретті жай дифференциалдық теңдеудің жалпы шешімдері құрылған. Ол шешімдер Коши есебін, екі нүктелі шеттік есепті және әр түрлі қолданбалы есептерді шешуде қолданылуы мүмкін.

А.Тунгатаров, А.А.Скаков

Об одном способе построения общих решений линейных дифференциальных уравнений второго порядка с переменными коэффициентами

В статье приведен новый метод построения общих решений линейных обыкновенных дифференциальных уравнений второго порядка с переменными коэффициентами. С помощью этого метода построены общие решения уравнения Эйри, линейных обыкновенных дифференциальных уравнений второго порядка с сингулярным коэффициентом и с коэффициентом e^x. Эти решения могут быть использованы для решения задачи Коши и двух точечных краевых задач для обыкновенных дифференциальных уравнений второго порядка с переменными коэффициентами, возникающих при решении различных прикладных задач естествознания.

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