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ON ONE SOLUTION OF A NONLOCAL BOUNDARY VALUE PROBLEM FOR A NONLINEAR PARTIAL DIFFERENTIAL EQUATION OF THE THIRD ORDER

In this paper, a nonlocal boundary value problem for the Benjamin-Bona-Mahony-Burgers equation is studied in a rectangular domain. By introducing new functions, the nonlocal boundary value problem for a nonlinear third-order partial differential equation is reduced to a boundary value problem for a second-order hyperbolic equation with a mixed derivative and functional relations. Before using the approximate method, the nonlinear problem under consideration is examined for the presence of solutions, it is necessary to clarify where these solutions are located, that is, to find the region of isolation of solutions. The isolation area of the solution in our case is a ball in which there is a unique solution to the problem. Next, an algorithm for finding a solution to a nonlocal boundary value problem is proposed. In terms of the initial data, conditions for the convergence of the algorithms are established, which simultaneously ensure the existence and isolation of a solution to a nonlinear nonlocal boundary value problem. Estimates between the exact and approximate solutions of the problem under consideration are obtained. The results obtained are of a theoretical nature and can be used in the construction of computational algorithms for solving nonlocal boundary value problems for the Benjamin-Bona-Mahony-Burgers equation.
Key words: Benjamin-Bona-Mahony-Burgers equation, differential equations with partial derivatives, algorithm, approximate solution.

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Үшінші ретті дербес туындылы сызықтық емес дифференциалдық теңдеу үшін бейлокал шеттік есебінің бір шешімі жайында

Бұл жұмыста тікбұрышты облыста Бенджамин-Бона-Махони-Бюргерс теңдеуі үшін бейлокал шеттік есебі зерттеледі. Жаңа функциялар енгізе отырып үшінші ретті дербес туындылы сызықтық емес дифференциалдық теңдеу үшін бейлокал шеттік есебі аралас туындылы екінші ретті гиперболалық теңдеу үшін бастапқы-шеттік есепке келтіріледі. Жуық әдісті қолданбас бұрын қарастырылып отырған сызықтық емес есептің шешімінің бар болуын зерттейміз, шешімдердің қайда орналасқанын, яғни шешімнің оқшауланған облысын анықтау қажет. Біздің жағдайымызда оқшауланған облыс - есептің шешімі бар және жалғыз болатын шар болып табылады. Әрі қарай бейлокал шеттік есептің шешімін табу алгоритмі ұсынылады. Бастапқы берілгендер терминінде сызықтық емес бейлокал шеттік есептің шешімінің бар болуы мен оқшауланғанын қамтамасыз ететін алгоритмдердің жинақтылық шарттары алынған. Қарастырылып отырған есептің нақты және жуық шешімі арасындағы бағалаулар табылған. Алынған нәтижелер теориялық сипатқа ие және үшінші ретті дербес туындылы сызықтық емес дифференциалдық теңдеулер үшін бейлокал шеттік есептерді шешуде есептеу алгоритмдерін құру үшін қолданысын таба алады.
Түйін сөздер: Бенджамин-Бона-Махони-Бюргерс теңдеуі, дербес туындылы дифференциалдық теңдеулер, алгоритм, жуық шешімі.

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Об одном решении нелокальной краевой задачи для нелинейного дифференциального уравнения в частных производных третьего порядка

В данной работе в прямоугольной области исследуется нелокальная краевая задача для уравнения Бенджамина-Бона-Махони-Бюргерса. Вводя новые функций нелокальная краевая задача для нелинейного дифференциального уравнения в частных производных третьего порядка сводится к начально-краевой задаче для гипереболического уравнения второго порядка со смешанной производной и функциональным соотношениям. Прежде чем использовать приближенный метод рассматриваемая нелинейная задача исследуется на наличие решений, необходимо уточнить, где эти решения находятся, то есть найти область изоляции решений. Область изоляции решения в нашем случае является шар, в котором решения задачи существует и единственно. Далее, предложен алгоритм нахождения решения нелокальной краевой задачи. В терминах исходных данных установлены условия сходимости алгоритмов, одновременно обеспечивающие существование и изолированность решения нелинейной нелокальной краевой задачи. Получены оценки между точным и приближенным решениями рассматриваемой задачи. Полученные результаты носят теоретический характер и могут быть использованы при построении вычислительных алгоритмов решения нелокальных краевых задач для нелинейных дифференциальных уравнений в частных производных третьего порядка.

Ключевые слова: уравнение Бенджамина-Бона-Махони-Бюргерса, дифференциальные уравнения в частных производных, алгоритм, приближенное решение.

1 Introduction

The article considers a nonlocal boundary value problem for a third-order nonlinear partial differential equation or the Benjamin-Bona-Mahony-Burgers equation. Various types of the BBMB equation were studied in [1]-[14]. Despite the presence of a large number of works devoted to the study of solving problems for the BBMB equation, interest in them has not waned to this day. This is due to the fact that the Benjamin-Bona-Mahony-Burgers equations represent an interesting and important object of study, combining wave theory, mathematical physics and practical applications. Previously, the authors used the parameterization method [15]-[17] to study more general boundary value problems for a system of linear [18]-[19] and nonlinear third-order pseudoparabolic equations. Constructive algorithms for finding approximate solutions to the problems under study were proposed and necessary and sufficient conditions for the existence of a solution were established. Subsequently, a nonlocal boundary value problem for the Benjamin-Bona-Mahony equation was studied [20]. Due to the fact that the problems under consideration are nonlinear, direct use of previously obtained results is not always possible. In this article, the Benjamin-Bona-Mahony-Burgers equation with general nonlocal boundary conditions that differ from the conditions specified in [20] is investigated. An algorithm for finding an approximate solution is proposed and conditions for the solvability of the problem under study are obtained.

2 Statement of the initial boundary problem

A nonlocal boundary value problem for the nonlinear Benjamin-Bona-Mahony-Burgers equation is considered.

$$\frac{\partial^3 w}{\partial x^2 \partial y} = \frac{\partial w}{\partial y} - \alpha \frac{\partial^2 w}{\partial x^2} + \beta \frac{\partial w}{\partial x} + w \frac{\partial w}{\partial x}, \quad (x,y) \in \Omega = [0,X] \times [0,Y], \tag{1}$$

$$w(x,0) = \varphi(x), \quad x \in [0,X], \tag{2}$$

$$w(0,y) = \gamma(y) w(X,y) + \psi(y), \quad y \in [0,Y], \tag{3}$$

$$\frac{\partial w(0,y)}{\partial x} = \theta(y), \quad y \in [0,Y], \tag{4}$$

here α, β-const, function $\varphi(x)$ continuously differentiable on $[0,X]$, functions $\psi(y), \theta(y), \gamma(y)$ are continuously differentiable on $[0,Y]$, $\gamma(y) \neq 1$.

Let $C(\Omega, R)$ be the set of functions $w : \Omega \to R$. continuous on Ω

Function $w(x,y) \in C(\Omega,R)$ having partial derivatives $\frac{\partial w(x,y)}{\partial x} \in C(\Omega,R)$, $\frac{\partial w(x,y)}{\partial y} \in C(\Omega,R)$, $\frac{\partial^2 w(x,y)}{\partial x \partial y} \in C(\Omega,R)$, $\frac{\partial^2 w(x,y)}{\partial x^2} \in C(\Omega,R)$, $\frac{\partial^3 w(x,y)}{\partial x^2 \partial y} \in C(\Omega,R)$ is called a solution to problem (1)-(4) if it satisfies equation (1), for all $(x,y) \in \Omega$, and boundary conditions (2)-(4).

To find solutions to problem (1)-(4) we introduce new functions

$$w(0,y) = \mu(y), \quad z(x,y) = w(x,y) - \mu(y).$$

Then problem (1)-(4) can be written in the form (5)-(9)

$$\frac{\partial^3 z(x,y)}{\partial x^2 \partial y} = \frac{\partial z(x,y)}{\partial y} + \mu'(y) - \alpha \frac{\partial^2 z(x,y)}{\partial x^2} + \beta \frac{\partial z(x,y)}{\partial x} + [z(x,y) + \mu(y)] \frac{\partial z(x,y)}{\partial x}, \tag{5}$$

$$z(0,y) = 0, \quad y \in [0,Y], \tag{6}$$

$$z(x,0) + \lambda(0) = \varphi(x), \quad \mu(0) = \varphi(0), \tag{7}$$

$$\mu(y) = \frac{\gamma(y) z(X,y) + \psi(y)}{1 - \gamma(y)}, y \in [0,Y], \tag{8}$$

$$\frac{\partial z(0,y)}{\partial x} = \theta(y), \quad y \in [0,Y]. \tag{9}$$

3 Methods and materials

Differentiating equation (8) with respect to the variable y we obtain equation (10).

$$\mu'(y) = \frac{\gamma'(y)}{[1-\gamma(y)]^2} z(X,y) + \frac{\gamma(y)}{1-\gamma(y)} \frac{\partial z(X,y)}{\partial y} + \frac{\psi'(y)}{\left[1-\gamma(y)\right]^2}. \tag{10}$$

By reintroducing the new function $v(x,y)=\frac{\partial z(x,y)}{\partial x}$, problem (5)-(9) and relation (10) can be written in the following form

$$\frac{\partial^2 v(x,y)}{\partial x\partial y}=\int\limits_0^x\frac{\partial v(\xi,y)}{\partial y}d\xi+\mu'(y)-\alpha\frac{\partial v(x,y)}{\partial x}+\beta v(x,y)+\Big[\int\limits_0^x v(\xi,y)d\xi+\mu(y)\Big]v(x,y), \quad (11)$$

$$v(x,0)=\varphi'(x),\quad x\in[0,X], \quad (12)$$

$$\mu(y)=\frac{\gamma(y)}{1-\gamma(y)}\int\limits_0^X v(x,y)dx+\frac{\psi(y)}{1-\gamma(y)}, \quad (13)$$

$$\mu'(y)=\frac{\gamma'(y)}{[1-\gamma(y)]^2}\int\limits_0^X v(x,y)dx+\frac{\gamma(y)}{1-\gamma(y)}\int\limits_0^X\frac{\partial v(x,y)}{\partial(y)}dx+\frac{\psi'(y)}{[1-\gamma(y)]^2}, \quad (14)$$

$$v(0,y)=\theta(y),\quad y\in[0,Y], \quad (15)$$

where $z(x,y)=\int\limits_0^x v(\xi,y)d\xi$. In (11), integrating over the variable x and taking into account conditions (15), we obtain

$$\frac{\partial v(x,y)}{\partial y}=\theta'(y)-\alpha v(x,y)+\alpha\theta(y)+\mu'(y)x+\int\limits_0^x\Big(\int\limits_0^\xi\frac{\partial v(\xi_1,y)}{\partial y}d\xi_1+$$
$$+\Big[\int\limits_0^\xi v(\xi_1,y)d\xi_1+\mu(y)\Big]v(\xi,y)+\beta v(\xi,y)\Big)d\xi. \quad (16)$$

After repeated integration over the variable y and application of condition (12), we obtain the following:

$$v(x,y)=\varphi'(x)+\int\limits_0^y\Big(\theta'(\eta)-\alpha v(x,\eta)+\alpha\theta(\eta)+\mu'(\eta)x+\int\limits_0^x\int\limits_0^\xi\frac{\partial v(\xi_1,\eta)}{\partial\eta}d\xi_1 d\xi+$$
$$+\int\limits_0^x\Big[\int\limits_0^\xi v(\xi_1,\eta)d\xi_1+\mu(\eta)\Big]v(\xi,\eta)d\xi+\beta\int\limits_0^x v(\xi,\eta)d\xi\Big)d\eta. \quad (17)$$

Assuming that $v(x,y)=\varphi'(x)$, from equations (13) and (14) we determine

$$\mu^{(0)}(y)=\frac{\gamma(y)}{1-\gamma(y)}[\varphi(X)-\varphi(0)]+\frac{\psi(y)}{1-\gamma(y)},$$

$$\mu'^{(0)}(y)=\frac{\gamma'(y)}{[1-\gamma(y)]^2}\Big[\varphi(X)-\varphi(0)\Big]+\frac{\psi'(y)}{[1-\gamma(y)]^2}.$$

Using equation (16) under the condition $\mu(y)=\mu^{(0)}(y)$, we can find

$$\frac{\partial v^{(0)}(x,y)}{\partial y}=\theta'(y)-\alpha\varphi'(x)+\alpha\theta(y)+$$

$$+{\mu'}^{(0)}(y)x+\int\limits_0^x\Big[\int\limits_0^\xi\varphi'(\xi_1)d\xi_1+\mu^{(0)}(y)\Big]\varphi'(\xi)d\xi+\beta\int\limits_0^x\varphi'(\xi)d\xi=$$

$$=\theta'(y)-\alpha\varphi'(x)+\alpha\theta(y)+{\mu'}^{(0)}(y)x+\int\limits_0^x\Big[\varphi(\xi)-\varphi(0)+\mu^{(0)}(y)\Big]\varphi'(\xi)d\xi+\beta\int\limits_0^x\varphi'(\xi)d\xi.$$

Next, from equation (17) it follows

$$v^{(0)}(x,y)=\varphi'(x)+\int\limits_0^y\Big(\theta'(\eta)-\alpha\varphi'(x)+\alpha\theta(\eta)+{\mu'}^{(0)}(y)x+$$

$$+\int\limits_0^x\Big[\varphi(\xi)-\varphi(0)+\mu^{(0)}(\eta)\Big]\varphi'(\xi)d\xi+\beta\int\limits_0^x\varphi'(\xi)d\xi\Big)d\eta.$$

Taking the found functions $\mu^{(0)}(y)$ and $v^{(0)}(x,y)$, the numbers $r_1>0$ and $r_2>0$, we construct the following sets:

$$S(\mu^{(0)}(y),r_1)=\Big\{\mu(y)\in C([0,Y],R):\|\mu(y)-\mu^{(0)}(y)\|<r_1\Big\},$$

$$S(v^{(0)}(x,y),r_2)=\Big\{v(x,y)\in C(\Omega,R):\|v(x,y)-v^{(0)}(x,y)\|<r_2,(x,y)\in\Omega\Big\},$$

$$G^0(r_1,r_2)=\Big\{(x,y,w,v):(x,y)\in\Omega,\quad\Big\|w(x,y)-\int\limits_0^x v^{(0)}(\xi,y)d\xi-\mu^{(0)}(y)\Big\|<$$

$$<r_1+r_2,\quad\|v(x,y)-v^{(0)}(x,y)\|<r_2\Big\}.$$

Let $U(l_1,l_2,x,y)$ denote the collection of quadruples $\Big(\mu^{(0)}(y),v^{(0)}(x,y),r_1,r_2\Big)$, for which the function $f(x,y,w,v)$ in $G^0(r_1,r_2)$ has continuous partial derivatives $f'_w(x,y,w,v)$, $f'_v(x,y,w,v)$ and $\|f'_w(x,y,w,v)\|\le l_1,\quad\|f'_v(x,y,w,v)\|\le l_2,\quad l_{1,2}-const$. Taking the pair $\{\lambda^{(0)}(y),v^{(0)}(x,y)\}$ as the initial approximation of problem (11)-(15), we construct successive approximations using the algorithm

1. Assuming $v(x,y)=v^{(k-1)}(x,y)$, from (13) and (14) we determine ${\mu'}^{(k)}(y)$ и $\mu^{(k)}(y)$.
2. Using equation (16), $\mu(y)=\mu^{(k)}(y)$, we find $\frac{\partial v^{(k)}(x,y)}{\partial y}$.
3. Then, using equation (17) we find $v^{(k)}(x,y)$.

As a result, we get the system $\Big\{{\mu'}^{(k)}(y),\mu_r^{(k)}(x),\ \frac{\partial v^{(k)}(x,y)}{\partial y},v_r^{(k)}(x,t)\Big\},k=1,2,....$

The following statement ensures the feasibility and convergence of the proposed algorithm, as well as the solvability of problem (11)-(15).

Theorem 1. Let the following conditions be satisfied

a) the function $\varphi(x)$ is continuously differentiable on $[0, X]$,

b) the functions $\gamma(y), \psi(y), \theta(y)$ are continuously differentiable on $[0, Y]$, $\gamma(y) \neq 1$,

c) $q = \alpha Y + \frac{X^2 Y \gamma}{1-\gamma} + \frac{X^2 \gamma}{1-\gamma} + \frac{X^2}{2} + \frac{X^2 Y l_2}{2} + X^2 Y \frac{l_2 \gamma}{1-\gamma} + (l_1 + \beta) XY < 1$,

d) $\frac{\sigma}{1-q} \frac{\gamma}{1-\gamma} XY^2 < r_1, \quad \frac{qY\sigma}{1-q} < r_2$,

where $\gamma = \max\limits_{y \in [0,Y]} \|\gamma(y)\|$, $\psi = \max\limits_{y \in [0,Y]} \|\psi(y)\|$, $\theta = \max\limits_{y \in [0,Y]} \|\theta(y)\|, \alpha, \beta - const$,

$$\sigma = \theta' + \alpha \max_{x \in [0,X]} \|\varphi'(x)\| + \alpha\theta + X\left(\frac{\gamma'}{[1-\gamma]^2}\Big[\varphi(X) - \varphi(0)\Big] + \frac{\psi'}{[1-\gamma]^2}\right) +$$

$$+ X \max_{x \in [0,X]} \|\varphi(x)\| \max_{x \in [0,X]} \|\varphi'(x)\| + X\left(\frac{\gamma}{1-\gamma}[\varphi(X) - \varphi(0)] + \frac{\psi}{1-\gamma}\right) \max_{x \in [0,X]} \|\varphi'(x)\| +$$

$$+ \beta \int\limits_0^X \varphi'(x) dx,$$

then the sequence of functions $\{\mu_r^{(k)}(x), v_r^{(k)}(x,t)\}, k = 1, 2, ...$, determined by the algorithm is contained in $S(\mu^{(0)}(y), r_1) \times S(v^{(0)}(x,y), r_2)$, converges to $\{\mu^*(x), v^*(x,t)\}$ - solving problem (11)-(15). Moreover, any solution to problem (11)-(15) in $S(\mu^{(0)}(y), r_1) \times S(v^{(0)}(x,y), r_2)$ isolated and fair estimates:

a) $\|\mu^*(y) - \lambda^{(k)}(y)\| \leq \frac{\gamma}{1-\gamma} \frac{XY^2}{2} \sum\limits_{i=k}^{\infty} q^i \sigma$,

b) $\|v^*(x,y) - v^{(k)}(x,y)\| \leq Y \sum\limits_{i=k+1}^{\infty} q^i \sigma$.

Proof. For $k = 0$ the following inequalities hold

$$\|\mu^{(0)}(y)\| \leq \frac{\gamma}{1-\gamma}[\varphi(X) - \varphi(0)] + \frac{\psi}{1-\gamma},$$

$$\left\|\mu'^{(0)}(y)\right\| \leq \frac{\gamma'}{[1-\gamma]^2}\Big[\varphi(X) - \varphi(0)\Big] + \frac{\psi'}{[1-\gamma]^2},$$

$$\left\|\frac{\partial v^{(0)}(x,y)}{\partial y}\right\| \leq \theta' + \alpha \max_{x \in [0,X]} \|\varphi'(x)\| + \alpha\theta + X\left(\frac{\gamma'}{[1-\gamma]^2}\Big[\varphi(X) - \varphi(0)\Big] + \frac{\psi'}{[1-\gamma]^2}\right) +$$

$$+ X \max_{x \in [0,X]} \|\varphi(x)\| \max_{x \in [0,X]} \|\varphi'(x)\| + X\left(\frac{\gamma}{1-\gamma}[\varphi(X) - \varphi(0)] + \frac{\psi}{1-\gamma}\right) \max_{x \in [0,X]} \|\varphi'(x)\| +$$

$$+ \beta \int\limits_0^X \varphi'(x) dx = \sigma,$$

$$\|v^{(0)}(x,y)-\varphi'(x)\| \le \int\limits_0^y \left\|\frac{\partial v^{(0)}(x,\eta)}{\partial \eta}\right\| d\eta.$$

For $k=1$, when $v(x,y)=v^{(0)}(x,y)$, the following inequalities follow

$$\|\mu^{(1)}(y)-\mu^{(0)}(y)\| \le \frac{\gamma'}{1-\gamma}\int\limits_0^X \|v^{(0)}(x,\eta)-\varphi'(x)\|dx < \frac{\gamma}{1-\gamma}XY\sigma < r_1,$$

$$\left\|\mu'^{(1)}(y)-\mu'^{(0)}(y)\right\| \le \frac{\gamma'}{[1-\gamma]^2}\int\limits_0^X \|v^{(0)}(x,y)-\varphi'(x)\|dx + \frac{\gamma}{1-\gamma}\int\limits_0^X \left\|\frac{\partial v^{(0)}(x,y)}{\partial y}\right\| dx,$$

$$\left\|\frac{\partial v^{(1)}(x,y)}{\partial y}-\frac{\partial v^{(0)}(x,y)}{\partial y}\right\| \le$$

$$\le \alpha \int\limits_0^y \left\|\frac{\partial v^{(0)}(x,\eta)}{\partial \eta}\right\| d\eta + \frac{\gamma' x}{[1-\gamma]^2}\int\limits_0^y \left\|\frac{\partial v^{(0)}(x,\eta)}{\partial \eta}\right\| d\eta +$$

$$+\frac{\gamma x}{1-\gamma}\int\limits_0^X \left\|\frac{\partial v^{(0)}(x,y)}{\partial y}\right\| dx + \int\limits_0^x \int\limits_0^\xi \left\|\frac{\partial v^{(0)}(\xi_1,y)}{\partial \eta}\right\| d\xi_1 d\xi +$$

$$+l_2 \int\limits_0^x \int\limits_0^\xi \int\limits_0^y \left\|\frac{\partial v^{(0)}(\xi_1,\eta)}{\partial \eta}\right\| d\eta d\xi_1 d\xi + \frac{\gamma l_2 x}{1-\gamma}\int\limits_0^y \int\limits_0^X \left\|\frac{\partial v^{(0)}(x,\eta)}{\partial \eta}\right\| dx d\eta +$$

$$+l_1 \int\limits_0^x \int\limits_0^y \left\|\frac{\partial v^{(0)}(\xi,\eta)}{\partial \eta}\right\| d\eta d\xi + \beta \int\limits_0^x \int\limits_0^y \left\|\frac{\partial v^{(0)}(\xi,\eta)}{\partial \eta}\right\| d\eta d\xi \le$$

$$\le \left(\alpha Y + \frac{X^2 Y\gamma}{1-\gamma} + \frac{X^2\gamma}{1-\gamma} + \frac{X^2}{2} + \frac{X^2 Y l_2}{2} + X^2 Y \frac{l_2\gamma}{1-\gamma} + (l_1+\beta)XY\right) \max_{(x,y)\in\Omega} \left\|\frac{\partial v^{(0)}(x,y)}{\partial y}\right\| \le$$

$$\le q \max_{(x,y)\in\Omega} \left\|\frac{\partial v^{(0)}(x,y)}{\partial y}\right\| \le q\sigma.$$

$$\|v^{(1)}(x,y)-v^{(0)}(x,y)\| \le \int\limits_0^y \left\|\frac{\partial v^{(1)}(x,\eta)}{\partial \eta}-\frac{\partial v^{(0)}(x,\eta)}{\partial \eta}\right\| d\eta \le \int\limits_0^y q\sigma < r_2.$$

For $k=2$ the following estimates hold:

$$\|\mu^{(2)}(y)-\mu^{(1)}(y)\| \le \frac{\alpha}{1-\alpha}\int\limits_0^X \|v^{(1)}(x,y)-v^{(0)}(x,y)\|dx \le$$

$$\leq \frac{\gamma}{1-\gamma}\int\limits_0^y\int\limits_0^X q\sigma dxd\eta \leq \frac{\gamma}{1-\gamma}\frac{XY}{2}q\sigma,$$

$$\left\|\frac{\partial v^{(2)}(x,y)}{\partial y}-\frac{\partial v^{(1)}(x,y)}{\partial y}\right\| \leq q\max_{(x,y)\in\Omega}\left\|\frac{\partial v^{(1)}(x,y)}{\partial y}-\frac{\partial v^{(0)}(x,y)}{\partial y}\right\| \leq q^2\max_{(x,y)\in\Omega}\left\|\frac{\partial v^{(0)}(x,y)}{\partial y}\right\| \leq q^2\sigma,$$

$$\|v^{(2)}(x,y)-v^{(1)}(x,y)\| \leq \int\limits_0^y\left\|\frac{\partial v^{(2)}(x,\eta)}{\partial\eta}-\frac{\partial v^{(1)}(x,\eta)}{\partial\eta}\right\|d\eta \leq Yq^2\sigma.$$

$$\|\mu^{(2)}(y)-\mu^{(0)}(y)\| \leq \frac{\gamma}{1-\gamma}\frac{XY}{2}q\sigma+\frac{\gamma}{1-\gamma}\frac{XY}{2}\sigma \leq \frac{\gamma}{1-\gamma}\frac{XY}{2}(1+q)\sigma < r_1,$$

$$\left\|\frac{\partial v^{(2)}(x,y)}{\partial y}-\frac{\partial v^{(0)}(x,y)}{\partial y}\right\| \leq (1+q)\max_{(x,y)\in\Omega}\left\|\frac{\partial v^{(1)}(x,y)}{\partial y}-\frac{\partial v^{(0)}(x,y)}{\partial y}\right\| \leq$$

$$\leq (q+q^2)\max_{(x,y)\in\Omega}\left\|\frac{\partial v^{(0)}(x,y)}{\partial y}\right\| \leq (q+q^2)\sigma.$$

$$\|v^{(2)}(x,y)-v^{(0)}(x,y)| \leq Y(q^2+q)\sigma < r_2.$$

At the $k+1-$th step of the algorithm, for $v(x,y)=v^{(k)}(x,y)$, the following estimates hold

$$\|\mu^{(k+1)}(y)-\mu^{(k)}(y)\| \leq \frac{\gamma}{1-\gamma}\int\limits_0^y\int\limits_0^X \|v^{(k)}(x,\eta)-v^{(k-1)}(x,\eta)\|dxd\eta, \tag{18}$$

$$\left\|{\mu'}^{(k+1)}(y)-{\mu'}^{(k)}(y)\right\| \leq \frac{\gamma}{1-\gamma}\int\limits_0^X \|v^{(k)}(x,y)-v^{(k-1)}(x,y)\|dx, \tag{19}$$

$$\left\|\frac{\partial v^{(k+1)}(x,y)}{\partial y}-\frac{\partial v^{(k)}(x,y)}{\partial y}\right\| \leq q\max_{(x,y)\in\Omega}\left\|\frac{\partial v^{(k)}(x,y)}{\partial y}-\frac{\partial v^{(k-1)}(x,y)}{\partial y}\right\|, \tag{20}$$

$$\|v^{(k+1)}(x,y)-v^{(k)}(x,y)\| \leq \int\limits_0^y\left\|\frac{\partial v^{(k+1)}(x,\eta)}{\partial\eta}-\frac{\partial v^{(k)}(x,\eta)}{\partial\eta}\right\|d\eta. \tag{21}$$

$$\|\mu^{(k+1)}(y)-\mu^{(0)}(y)\| \leq$$

$$\leq \frac{\gamma}{1-\gamma}\int\limits_0^y\int\limits_0^X \|v^{(k)}(x,\eta)-\varphi'(x)\|dxd\eta \leq \frac{\gamma}{1-\gamma}\frac{XY}{2}\sum_{i=0}^{k}q^i\sigma < r_1,$$

$$\|v^{(k+1)}(x,y)-v^{(0)}(x,y)\| \leq Y\sum_{i=1}^{k+1}q^i\sigma < r_2.$$

Consequently, from inequalities (18)-(21) and $q < 1$ it follows that the sequence $\{\mu^{(k)}(y), v^{(k)}(x,y)\}$ at $k \to \infty$, converges to $\{\mu^*(y), v^*(x,y)\}-$ solution of problem (10)-(14) in $S(\mu^{(0)}(y), r_1) \times S(v^{(0)}(x,y), r_2)$.

Let's establish inequalities

$$\|\mu^{(k+p)}(y) - \mu^{(k)}(y)\| \leq \frac{\gamma}{1-\gamma}\frac{XY}{2}\sum_{i=k}^{k+p-1} q^i\sigma, \tag{22}$$

$$\|v^{(k+p)}(x,y) - v^{(k)}(x,y)\| \leq Y\sum_{i=k+1}^{k+p} q^i\sigma. \tag{23}$$

for $p \to \infty$ we obtain estimates a), b) of Theorem 1.

Let's prove uniqueness. The uniqueness of the solution of problem (1)-(4) is proved similarly to the proof of Theorem 1 from [18]. Theorem 1 is proved.

The function $w^{(k)}(x,y), k = 1,2,3...$ is determined from the equality

$$w^{(k)}(x,y) = \mu^{(k)}(x,y) + \int_0^x v^{(k)}(\xi,y)d\xi, (x,y) \in \Omega.$$

Let $S_1(w^{(0)}(x,y), r_1 + r_2 x)$ denote the set of continuously differentiable with respect to x, y functions $w : \Omega \to R$, satisfying the inequality $\|w(x,y) - w^{(0)}(x,y)\| < r_1 + r_2 x$.

In view of the equivalence of problems (1)-(4) and (11)-(15), Theorem 1 implies.

Theorem 2. If the conditions of Theorem 1 are satisfied, then the sequence of functions $w^{(k)}(x,y), k = 1,2,...,$ is contained in $S(w^{(0)}(x,y), r_1 + r_2)$ converges to the unique solution $w^*(x,y)$ of problem (1)-(4) in $S(w^{(0)}(x,y), r_1 + r_2)$ and the inequality

$$\|w^*(x,y) - w^{(k)}(x,y)\| \leq \frac{\gamma}{1-\gamma}\frac{X^2Y^2}{2}\sum_{i=k+1}^{\infty} q^i\sigma + Y\sum_{i=k}^{\infty} q^i\sigma.$$

4 Conclusion

Thus, the third-order nonlinear Benjamin-Bona-Mahony-Burgers equation with nonlocal conditions has been studied. The Benjamin-Bona-Mahony-Burgers equation describes the propagation of small amplitude waves in a nonlinear dispersive medium when simulating unidirectional plane waves. In this paper, an algorithm for searching for an approximate solution to the problem under consideration is proposed and the conditions for the convergence of the proposed algorithm are determined. An estimate between the exact and approximate solution of the nonlinear problem is obtained.

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