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Synthesis of uniformly distributed optimal control with nonlinear optimization of oscillatory processes

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In the article the problem of synthesizing uniformly distributed optimal control for nonlinear optimization of oscillatory processes described by integro-differential partial differential equations with the Volterra integral operator was explored. The study was conducted according to the Bellman-Egorov scheme and an algorithm for constructing a uniformly distributed optimal control in the form of a functional from the state of the controlled process was developed. Sufficient conditions for the solvability of the synthesis problem in nonlinear optimization were established.

Keywords: generalized solution, Volterra operator, nonlinear optimization, Bellman functional, Frechet differential, Bellman type equations, synthesis of optimal control.

2020 Mathematics Subject Classification: 49K20.

Introduction

With the advent of studies [1–7], methods of the theory of optimal control with distributed parameter systems began to penetrate into various fields of science and attract the attention of researchers.

However, despite the large flow of research, methods for solving optimal control problems with processes described by integrodifferential partial differential equations [8] have not been sufficiently developed. In particular, the development of methods for solving the synthesis problem is one of the most pressing problems. Research is continuing in this direction, and several papers have been published [9–13]. This article examines the solvability of the problem of synthesis of uniformly distributed optimal control, with nonlinear optimization of oscillatory processes described by integro-differential partial differential equations with the Volterra integral operator. Building on the methodology outlined in [10], we developed synthesis problem-solving method based on the Bellman-Egorov scheme. A Bellman type equation is obtained, which is a non-linear integro-differential equation of a nonstandard form. The structure of its solution is found, which makes it possible to transform Bellman-type equations into a system of two equations, one of which is solved independently of the second. This circumstance significantly simplifies the procedure for constructing a synthesizing control.

The issues of constructing a generalized solution to the boundary value problem of a controlled process with an integral Volterra operator are described in detail and sufficient conditions for unambiguity of the solvability of the synthesis problem are established.

1 *A generalized solution to the boundary value problem of a controlled process*

Let's consider an oscillatory process described by the function $V(t,x)$, which in the domain $Q_T = Q \times (0,T)$ satisfies the integro-differential equation

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$$V_{tt} - AV = \lambda \int_0^t K(t,\tau)V(\tau,x)d\tau + g(t,x)f[u(t)], (t,x) \in Q_T, \qquad (1)$$

along with initial conditions at the boundary of the domain Q_T

$$V(0,x) = \psi(x), \quad V_t(0,x) = \psi_2(x), \quad x \in Q \subset R^n, \qquad (2)$$

and a boundary condition

$$\Gamma V(t,x) \equiv \sum_{i,k=1}^{n} a_{ik}(x)V_{x_k}(t,x)\cos(\sigma, x_i) + a(x)V(t,x) = 0, (t,x) \in \gamma_T = \gamma \times (0,T), \qquad (3)$$

where A is an elliptic operator, Q is a domain in Euclidean space $R^n c$ with a piecewise-smooth boundary γ, σ is the normal vector coming from the point $x\epsilon\gamma$; λ is a parameter; $K(t,\tau)$ is a function defined in the domain $\{0 \le t, \tau \le T\}$ and satisfies the condition

$$\int_0^T \int_0^T K^2(t,\tau)d\tau dt = K_0 < \infty.$$

The functions $\psi_1(x) \in H_1(Q)$, $\psi_2(x) \in H(Q)$, $g(t,x) \in H(Q_T)$, $a_{ik}(x)$, $a(x)$ are considered known; the external source function $f[u(t)] \in H(0,T)$ is nonlinear and monotonic with respect to the functional variable $u(t), t \in [0,T]$; $u(t) \in H(0,T)$ is the control function; H(Y) − Y denotes a Hilbert space of square-integrable functions defined on the set Y; $H_1(Q)$ is the first-order Sobolev space; T is a fixed point in time.

In the context of the problem under consideration, the given functions may be discontinuous, and the existence of a classical solution to the boundary value problem is unlikely. In this regard, following the methodology of reference [9], we will use the following definition of a generalized solution.

Definition 1. A generalized solution of the boundary value problem (1)–(3) is a function $V(t,x) \in H_1(Q_T)$ that satisfies the integral identity

$$\int_Q (V_t(t,x)\Phi(t,x))_{t_1}^{t_2} \quad dx = \int_{t_1}^{t_2} \Bigg\{ \Bigg[\int_Q V_t(t,x)\Phi_t(t,x) \; -$$

$$- \sum_{i,k=1}^{\infty} a_{ik}(x)V_{x_k}(t,x)\Phi_{x_i}(t,x) - c(x)V(t,x)\Phi(t,x) +$$

$$+ \left(\lambda \int_0^t K(t,\tau)V(\tau,x)d\tau + g(t,x)f[u(t)] \right) \Phi(\mathrm{t},x) \Bigg) \Bigg] dx -$$

$$- \int_\gamma a(x)V(T,x)\Phi(t,x) \Bigg\} dt \qquad (4)$$

for all $t \, (0 \le t \le t_2 \le T)$ and for any function $\Phi(t,x) \in H_1(Q_T)$, as well as the initial condition (2) in the weak sense, i.e., as $t_1 \to 0$

$$\int_Q [(V(t,x) - \psi_1(x))] \, \Phi_0(x)dx = 0, \int_Q [V_t(t,x) - \psi_2(x)] \, \Phi_1(x)dx = 0$$

for any functions $\Phi_0(x) \in H(Q)$ and $\Phi_1(x) \in H(Q)$.

We seek the generalized solution of the boundary value problem (1)–(3) in the form

$$V(t,x) = \sum_{n=1}^{\infty} V_n(t) z_n(x), \quad V_n(t) = \int_Q V(t,x) z_n(x) dx, \tag{5}$$

where $z_n(x)$ is a generalized eigenfunction of the boundary value problem of the form [9]

$$\begin{aligned}
&D_n \left[V(t,x), z_j(x)\right] \equiv \\
&\equiv \int_Q \left[\sum_{i,k=1}^{n} a_{ik}(x) V_{x_k}(t,x) z_{jx_i}(x) + c(x) V(t,x) z_{j(x)}\right] dx + \\
&+ \int_\gamma a(x) V(t,x) z_j(x) dx = \lambda_j^2 \int_Q V(t,x) z_j(x) dx, \\
&\Gamma z_j(x) = 0, \ j = 1, 2, 3, \ldots
\end{aligned}$$

and the corresponding eigenvalues satisfy the properties

$$\lambda_j \leq \lambda_{j+1} \leq \cdots \text{ and } \lim_{j \to \infty} \lambda_j = \infty$$

and the functions $z_n(x), n = 1, 2, 3, \ldots$, form a complete orthonormal system of generalized eigenfunctions of the boundary value problem (6) in a closed domain $\bar{Q} = Q \cup \gamma$.

According to the methodology of [9], it can be shown that the Fourier coefficients $V_n(t)$ are determined as the solution of the Cauchy problem

$$\begin{gathered}
V_n''(t) + \lambda_n^2 V_n(t) = \lambda \int_0^t K(t,\tau) V_n(\tau) d\tau + g_n(t) f[u(t)], \forall t \in [t_1, t_2], \\
V_n(t_1) = \int_Q V(t_1, x) z_n(x) dx, \quad V_n'(t_1) = \int_Q V_t(t_1, x) z_n(x) dx, \\
g_n(t) = \int_Q g(t,x) z_n(x) dx, \quad n = 1, 2, 3, \ldots,
\end{gathered}$$

which is obtained from the integral identity (4) with $\Phi(t,x) \equiv z_n(x)$.

We find the solution of this problem using the formula

$$\begin{aligned}
V_n(t) = &\int_Q V(t_1, x) z_n(x) dx \cos \lambda_n t + \frac{1}{\lambda_n} \int_Q V_t(t_1, x) z_n(x) dx \sin \lambda_n t + \\
&+ \frac{1}{\lambda_n} \int_{t_1}^t \sin \lambda_n (t - \tau) \left[\lambda \int_0^\tau K(\tau, y) V_n(y) dy + g_n(\tau) f[u(\tau)]\right] d\tau
\end{aligned}$$

which as $t_1 \to 0$ becomes

$$V_n(t) = \psi_{1n} \cos \lambda_n t + \frac{1}{\lambda_n} \psi_{2n} \sin \lambda_n t + \frac{1}{\lambda_n} \int_0^t \sin \lambda_n (t-\tau) \left[\lambda \int_0^t K(\tau, y) V_n(y) dy + q_n(\tau) f[u(\tau)]\right] d\tau,$$

where

$$\psi_{1n} = \lim_{t_1 \to 0} \int_Q V(t_1, x) z_n(x) dx, \quad \psi_{2n} = \lim_{t_1 \to 0} \int_Q V_t(t_1, x) z_n(x) dx.$$

Using the Liouville approach, this solution can be represented as a linear integral equation of the Volterra 2nd kind of the following form:

$$V_n(t) = \lambda \int_0^t K_n(t,y) V_n(y) dy + q_n(t), \quad n = 1, 2, \ldots, \tag{6}$$

where

$$K_n(t,y) = \int_y^t \frac{1}{\lambda_n} \sin \lambda_n (t-\tau) K(\tau, y) d\tau, \tag{7}$$

$$q_n(t) = \psi_{1n} \cos \lambda_n t + \frac{1}{\lambda_n} \psi_{2n} \sin \lambda_n t + \int_0^t \frac{1}{\lambda_n} \sin \lambda_n (t-\tau) g_n(\tau) f[u(\tau)] d\tau.$$

The solution of equation (6) is found using the formula [14]

$$V_n(t) = \lambda \int_0^t R_n(t,y,\lambda) q_n(y) dy + q_n(t), \tag{8}$$

where the resolvent $R_n(t,y,\lambda)$ is the sum of the Neumann series, i.e.,

$$R_n(t,y,\lambda) = \sum_{i=1}^{\infty} \lambda^{i-1} K_{n,i}(t,y),$$

$$K_{n,i+1}(t,y) = \int_y^t K_n(t,\tau) K_{n,i}(\tau,y) d\tau, i = 1, 2, 3, \ldots.$$

By direct calculations, we establish the estimates

$$|K_{n,i}(t,y)| \le \frac{K_0^i T^{i-1}}{\lambda_n^i} \cdot \frac{(t-y)^i}{i!}, \quad i = 1, 2, 3 \ldots$$

which imply the ratio

$$|R_n(t,y,\lambda)| \le \sum_{i=1}^{\infty} |\lambda|^{i-1} |K_{n,i}(t,y)| \le$$
$$\le \frac{1}{|\lambda| T} \left(\sum_{i=0}^{\infty} \frac{1}{i!} \left[\frac{|\lambda| K_0 T}{\lambda_n} (t-y) \right]^i - 1 \right) \frac{1}{|\lambda| T} \left(e^{\frac{|\lambda| K_0 T}{\lambda_n}(t-y)} - 1 \right),$$

from which it follows that the resolvent $R_n(t,y,\lambda)$ for each $n = 1, 2, 3 \ldots$, for any value of the parameter $\lambda \neq 0$ is a continuous function of the arguments. Note that for the resolvent, there is an estimate

$$\int_0^{]T} |R_n(t,y,\lambda)|^2 dy \le \int_0^T \left[\frac{1}{|\lambda| T^2} \left(e^{\frac{|\lambda| K_0 T}{\lambda_n}(t-y)} - 1 \right) \right]^2 dy \le$$
$$\le \frac{2}{|\lambda|^2 T^2} \int_0^T \left(\int_0^T \left(e^{\frac{2|\lambda| K_0 T}{\lambda_n}(t-y)} + 1 \right) \right) dy \le \tag{9}$$

$$\le \frac{2}{|\lambda|^2 T} \left(\frac{e^{\alpha_n} - 1}{\lambda_n} + 1 \right) \le \frac{2}{|\lambda|^2 T} \left(\frac{e^{\lambda_1} - 1}{\lambda_1} + 1 \right)$$

since

$$\alpha_n = \frac{2|\lambda| K_0 T^2}{\lambda_n} \text{ and } \lim_{\alpha_n \to 0} \frac{e^{\alpha_n} - 1}{\alpha_n} = 1, \text{ as } \lambda_n \to \infty.$$

Next, we substitute the Fourier coefficients $V_n(t)$ found by formula (8) into (5), and we find the formal solution of the boundary value problem (1)–(3) using the formula

$$V(t,x)=\sum_{n=1}^{\infty}\left(\lambda\int_0^t R_n(t,y,\lambda)q_n(y)dy+q_n(t)\right)z_n(x), \tag{10}$$

where the function $q_n(t)$ has the form (7).

Lemma 1.1 The function (10) is an element of the space $H_1(Q_T)$.

Proof. Differentiating (10) by t, we obtain the function

$$V_t(t,x)=\sum_{n=1}^{\infty}\left(\lambda\int_0^t R'_{nt}(t,y,\lambda)q_n(y)dy+\lambda R_n(t,y,\lambda)q_n(t)+q'_n(t)\right)z_n(x). \tag{11}$$

Taking into account (9) and (10)-(11), by direct calculations, the following relations are established:

$$\|V(t,x)\|^2_{H(Q_T)}\leq\frac{6T}{{\lambda_1}^2}\left(1+2\left(\frac{e^{\alpha_1-1}}{\alpha_1}+1\right)\right)\times$$

$$\times\left(\|\psi_1(x)\|^2_{H_1(Q)}+\|\psi_2(x)\|^2_{H_{(Q)}}+\|g(t,x)\|^2_{H(Q_T)}\|f[u(t)]\|^2_{H(0,T)}\right)<\infty;$$

$$\|V_t(t,x)\|^2_{H(Q_T)}\leq 9T\left(1+\frac{\lambda^2K_0^2T^2}{\lambda_1^4}\cdot\frac{e^{\alpha_1-1}}{\alpha_1}\right)\times$$

$$\times\left(\|\psi_1(x)\|^2_{H_1(Q)}+\|\psi_2(x)\|^2_{H_{(Q)}}+\|g(t,x)\|^2_{H(Q_T)}\|f[u(t)]\|^2_{H(0,T)}\right)<\infty.$$

From these relations, the statement of the lemma follows.

Theorem 1.1 Let the given functions and parameters satisfy the conditions of the boundary value problem (1)–(3). Then the boundary value problem (1)–(3), for any value of the parameter λ, has a unique generalized solution of the form (10).

Proof. According to Lemma 1.1, the function of the form (10) belongs to the space $H_1(Q_T)$. By construction, it satisfies the integral identity, and its Fourier coefficients are uniquely determined as the solution to the Cauchy problem. It is also worth noting that due to the monotonicity of the function $f[u(t)]$ with respect to the functional variable, there is a one-to-one correspondence between the elements of the control space $\{u(t)\}=H(0,T)$ and the space of states of the controlled process $\{V(t,x)\}$. If we assume the existence of two generalized solutions, we will arrive at a contradiction.

2 Formulation of the optimal control synthesis problem

Let's consider a nonlinear optimization problem, where the goal is to minimize the integral functional

$$I[u(t)]=\int_Q\|w(T,x)-\xi(x)\|^2dx+\beta\int_0^T p[t,u(t)]dt,\quad \beta>0, \tag{12}$$

over the set of generalized solutions of the boundary value problem (1)-(3). Here, the symbol $\|\cdot\|$ denotes the norm of a vector; $w(t,x)=\{V(t,x),V_t(t,x)\}$ is the vector function describing the state of the controlled process, the vector function $\xi(x)=\{\xi_1(x),\ \xi_2(x)\}\in H^2(Q)=H(Q)\times H(Q)$ is considered known; and the function $p[t,u(t)]\in H(0,T)$ is convex with respect to the functional variable $u(t)\in H(0,T)$.

In the optimal control synthesis problem, the sought control $u^0(t)\in H(0,T)$ is to be found as a function (functional) of the state vector of the controlled process, i.e., in the form $u^0(t)=u[t,w(t,x)]$.

First, let's note one property of the functional (12).

Lemma 1.2 Suppose the function $f[t,u(t)]$ is monotonic and the function $p[t,u(t)]$ is convex with respect to the functional variable $u(t), \forall t \in [0,T]$. Then, the functional $I[u(t)]$ attains its minimum value at a unique element $u^0(t) \in H(0,T)$.

Proof. The monotony condition of the function $f[t,u(t)]$ implies, that each control $[u(t)]$ corresponds to a unique state of the controlled process $w(t,x)$. For example, for the control $u_1(t)+u_2(t)$ the corresponding state of the controlled process is $w_1(t,x)+w_2(t,x)$, leading to the relationship:

$$I\left(\frac{u_1(t)+u_2(t)}{2}\right) = \int_Q \left\| \frac{w_1(T,x)+w_2(T,x)}{2} - \xi(x) \right\|^2 dx + \beta \int_0^T p\left[t, \frac{u_1(t)+u_2(t)}{2}\right] dt. \qquad (13)$$

By analogy with the known methodology [1], direct calculations easily establish the equality

$$I\left[u_1(t)\right] + I\left[u_2(t)\right] = 2\int_Q \left\| \frac{w_1(T,x)+w_2(T,x)}{2} - \xi(x) \right\|^2 dx +$$
$$+ \frac{1}{2}\int_Q \left\| w_1(T,x) - w_2(T,x) \right\|^2 dx + \beta \int_0^T \left(p\left[t,u_1(t)\right] + p_2\left[t,u_2(t)\right]\right) dt,$$

from which, considering the convexity property of the function $p[t,u(t)]$, we obtain the inequality

$$I\left[u_1(t)\right] + I\left[u_2(t)\right] \geq 2\int_Q \left\| \frac{w_1(T,x)+w_2(T,x)}{2} - \xi(x) \right\|^2 dx +$$
$$+ \frac{1}{2}\int_Q \left\| w_1(T,x) - w_2(T,x) \right\|^2 dx + 2\beta \int_0^T p\left(t, \frac{u_1(t)+u_2(t)}{2}\right) dt >$$
$$> 2I\left[\frac{u_1(t)+u_2(t)}{2}\right]. \qquad (14)$$

Suppose that the functional $I[u(t)]$ attains its minimum value $I_{\min}$ for the controls $u_1(t)$ and $u_2(t)$. Then, according to (13)-(14), we obtain the inequality

$$I\left[\frac{u_1(t)+u_2(t)}{2}\right] < I\left[u_1(t)\right] + I\left[u_2(t)\right] = 2I_{\min},$$

which contradicts the optimality of the controls $u_1(t)$ and $u_2(t)$.

3 About the solvability of the synthesis problem

According to (12), the Bellman functional takes the form

$$S[t,w(t,x)] = \min_{\substack{u(\tau)\in U \\ t\leq\tau\leq T}} \left\{ \beta \int_t^T p[\tau,u(\tau)]d\tau + \int_Q \|w(T,x)-\xi(x)\|^2 dx \right\}, \qquad (15)$$

where U is the set of admissible control values $u(t) \in H(0,T)$. According to the Bellman-Egorov scheme, assuming that $S[t,w(t,x)]$ as a function is differentiable by t and as a functional is differentiable by Fresche, the relation (15) is reduced to the form

$$-\frac{\partial S[t,w(t,x)]}{\partial t}\Delta t = \min_{\substack{u(\tau)\in U \\ t\leq\tau\leq t}} \left\{ \beta \int_t^{t+\Delta t} p[\tau,u(\tau)]d\tau + ds[t,w(t,x);\Delta w(t,x)] + \right.$$
$$\left. + o_1(\Delta t) + \delta[t,w(t,x);\Delta w(t,x)] \right\} = \min_{u(\tau)\in U} \left\{ \beta \int_t^{t+\Delta t} p[\tau,u(\tau)]d\tau + \int_Q m^*(t,x)\Delta w(t,x)dx + o_1(\Delta t) + \right.$$
$$\left. + \delta[t,w(t,x);\Delta w(t,x)] \right\}, \qquad (16)$$

where $m(t,x) = \{m_1(t,x), m_2(t,x)\}$ is the gradient of the functional $S[t,w(t,x)]$; $o_1(\Delta t), \delta[t,w(t,x);\Delta w(t,x)$ are infinitesimal quantities; * denotes transposition.

Further, using an identity of the form

$m^*(t,x)\Delta w(t,x) = m_1(t,x)\Delta V(t,x) - \Delta m_2(t,x)V_t(t+\Delta t,x) + \left(V_t(t,x)m_2(t,x)\right)_t^{t+\Delta t}$; and an integral identity

$$\int_Q \left(V_t(t,x)m_2(t,x)\right)_t^{t+\Delta t} dx = \int_t^{t+\Delta t}\left\{\int_Q [V_t(y,x)m_{2t}(y,x) - \right.$$
$$-\sum_{i,k=1}^{k} a_{i,k}(x)V_{x_k}(y,x)m_{2x_i}(y,x) - c(x)V(y,x)m_2(y,x)+$$
$$+\left(\lambda\int_0^y K(y,\tau)V(\tau,x)d\tau + g(y,x)f[y,u(y)]\right)m_2(y,x)\Big]\,dx-$$
$$\left. -\int_\gamma a(x)V(y,x)m_2(y,x)dx\right\}dy,$$

which is derived from (4) with $t_1 = t, t_2 = t+\Delta t, \Phi(t,x) \equiv m_2(t,x)$, relation (16) can be represented as

$$-\frac{\partial S[t,w(t,x)]}{\partial t}\Delta t = \min_{\substack{u(\tau)\in U\\ t\le\tau\le t+\Delta t}}\left\{\beta\int_t^{t+\Delta t} p[\tau,u(\tau)]d\tau + \right.$$
$$+\int_Q \left(m_1(t,x)\Delta V(t,x) - \Delta m_2(t,x)V_t(t+\Delta t,x)\right)+$$
$$+\int_t^{t+\Delta t}\left(\int_Q [\,V_t(y,x)m_{2t}(y,x) - \sum_{i,k=1}^{n} a_{i,k}(x)V_{x_k}(y,x)m_{2x_i}(y,x) - c(x)V(y,x)m_2(y,x)+\right.$$
$$+\left(\lambda\int_0^y K(y,\tau)V(\tau,x)d\tau + g(y,x)f[y,u(y)]\right)m_2(y,x)\Big]\,dx-$$
$$\left.\left.-\int_\gamma a(x)V(y,x)m_2(y,x)dx\right)dy + o_1(\Delta t) + \delta[t,w(t,x);\Delta w(t,x)]\right\}.$$

We divide this equality by $\Delta \mathrm{t}$ and for $\Delta \mathrm{t} \to 0$, after simple calculations, we have equality in the limit

$$-\frac{\partial S[t,w(t,x)]}{\partial t} = \min_{\substack{u(\tau)\in U\\ t\le\tau\le t}}\left\{\beta p[t,u(t)] + \int_Q g(t,x)m_2(t,x)dx f[t,u(t)] + \int_Q [m_1(t,x)V_t(t,x) - \right.$$
$$-\sum_{i,k=1}^{n} a_{i,k}(x)V_{x_k}(t,x)m_{2x_i}(t,x) - c(x)V(t,x)m_2(t,x)+$$
$$+\left(\lambda\int_0^t K(t,\tau)V(\tau,x)d\tau\right)m_2(t,x)\Big]\,dx-$$
$$\left.-\int_\gamma a(x)V(t,x)m_2(t,x)dx\right\}, \tag{17}$$

which we will call the Bellman-type equation. Note that here the equality holds for the variable $t \in (0,\mathrm{T})$ almost everywhere. We will consider this equation together with the condition

$$S[T,w(T,x)] = \int_Q \|w(T,x) - \xi(x)\|^2 dx. \tag{18}$$

Thus, the Bellman functional $S[t, w(t,x)]$ should be found as a solution for the Cauchy-Bellman problem (17)-(18), which is called the Cauchy-Bellman problem.

In the first stage of solving equation (17), we will consider the minimization problem over the control $u(t)$, $\forall t \in [0,T]$, which, depending on the properties of the set U, is solved by different methods.

Let U be an open set. Then, the extremal problem is solved by the classical method, and the first-order optimality condition is given by

$$\beta p_u[t, u(t)] + \int_Q g(t,x) m_2(t,x) dx f_u[t, u(t)] = 0, \tag{19}$$

and the second-order optimality condition is determined by a differential inequality of the form

$$\beta p_{uu}[t, u(t)] + \int_Q g(t,x) m_2(t,x) dx f_{uu}[t, u(t)] > 0,$$

which, with (19) taken into account, can be transformed into the form [10–13]

$$f_u[t, u(t)] \left(\frac{p_u[t, u(t)]}{f_u[t, u(t)]} \right)_u > 0. \tag{20}$$

This inequality is one of the constraints, meaning that the problem of optimal control synthesis in nonlinear optimization of controlled processes is solvable only for those pairs of functions $(f[t, u(t)], p[t, u(t)])$ that satisfy condition (20). When condition (20) is met, according to the implicit function theorem, equation (19) is uniquely solvable for the control $u(t)$. In other words, there exists a unique function $\varphi(\cdot)$, such that

$$u^0(t) = \varphi\left[t, \int_Q g(t,x) m_2(t,x) dx, \beta\right]. \tag{21}$$

Substituting the found $u^0(t)$ into (17), we obtain a simplified version of the Bellman type equation.

$$\begin{aligned} -\frac{\partial S[t, w(t,x)]}{\partial t} &= \beta p\left[t, \int_Q g(t,x) m_2(t,x) dx, \beta\right] + \\ &+ \int_Q g(t,x) m_2(t,x) dx f\left[t, \int_Q g(t,x) m_2(t,x) dx, \beta\right] + \\ &+ \int_Q [m_2(t,x) V_t(t,x) - \\ &- \sum_{i,k=1}^{n} a_{i,k}(x) V_{x_k}(t,x) m_{2x_i}(t,x) - c(x) V(t,x) m_2(t,x) + \\ &+ \lambda \int_0^t K(t,\tau) V(\tau,x) d\tau \Big) m_2(t,x) \Big) \Big] dx - \\ &- \int_\gamma a(x) V(t,x) m_2(t,x) dx. \end{aligned} \tag{22}$$

This equation is a nonlinear integro-differential equation of a complex nature and is not of a standard form. According to the methodology developed by A. Kerimbekov [10–13], we seek the solution to equation (22) in the form

$$S[t, w(t,x)] = S_0[t, w(t,x)] + \lambda S_1(t), \tag{23}$$

where $S_0[t, w(t,x)]$ and $S_1(t)$ are to be determined. In this case, equation (22) splits into two equations, and the functional $S_0[t, w(t,x)]$ is determined as the solution to a problem of the form

$$\begin{aligned}
-\frac{\partial S_0[t,w]}{\partial t} &= \beta p\left[t, \int_Q g(t,x)m_2(t,x)dx, \beta\right] + \\
&+ \int_Q g(t,x)m_2(t,x)dx f\left[t, \int_Q g(t,x)m_2(t,x)dx, \beta\right] + \\
&+ \int_Q \Big[m_1(t,x)V_t(t,x) - \\
&- \sum_{i,k=1}^{n} a_{i,k}(x)V_{x_k}(t,x)m_{2x_i}(t,x) - c(x)V(t,x)m_2(t,x)\Big] dx - \\
&- \int_\gamma a(x)V(t,x)m_2(t,x)dx, \qquad (24)\\
S_0[T, w(T,x)] &= \int_Q \|w(T,x) - \xi(x)\|^2 dx, \qquad (25)
\end{aligned}$$

and the function $S_1(t)$ is determined as the solution to the following problem

$$-\frac{\partial S_1(t)}{\partial t} = \int_Q m_2(t,x)\int_0^t K(t,\tau)V(\tau,x)d\tau dx, \qquad (26)$$

$$S_1(T) = 0. \qquad (27)$$

According to (23), the equality

$$\operatorname{grad} S[t, w(t,x)] = \operatorname{grad} S_0[t, w(t,x)]$$

holds, which implies that in formula (21) the function $m_2(t,x)$ can be determined by solving problem (24)-(25). This circumstance significantly simplifies the procedure of constructing optimal control depending on the state of the controlled process, i.e. the solution of the synthesis problem.

Let $S_0[t, w(t,x)]$ be the solution to problem (24)-(25), and $S_1(T)$ be the solution to problem (26)-(27). Then, according to (23) and (15), the minimum value of the functional (12) is found by the formula

$$\begin{aligned}
I\left[u^0(t)\right] = S[0, w(0,x)] &= S_0\left[0, V(0,x), V_t(0,x)\right] + \lambda S_1(0) = \\
&= S_0\left[0, \psi_1(x), \psi_2(x)\right] + \lambda S_1(0).
\end{aligned}$$

In conclusion, it should be noted that in the general case, methods for solving problem (24)-(25) are not developed. However, in some particular cases, it is possible to find the solution to problem (24)-(25) and, using formula (21), to write down the explicit form of the sought control $u^0(t)$ depending on the state of the controlled process.

Author Contributions

1. A. K.: Development of the solution method for the nonlinear integro-differential equation of the Bellman type and general analysis of erroneous results.
2. Zh. A.: Justification of results on the application of implicit function theory and analysis.
3. A. B.: Justification of results on the application of integral equations theory and analysis.

Conflict of Interest

The authors declare no relevant financial or non-financial competing interests.

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