

To Solving the Heat Equation with Fractional Load

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Abstract—In the paper, a boundary value problem for a fractionally loaded heat equations is considered in the first quadrant. The questions of the existence and uniqueness of the solution are investigated in the class of continuous functions. The loaded term has the form of the Caputo fractional derivative with respect to the spatial variable, and, the order of the derivative in the loaded term is less than the order of the differential part. The study is based on reducing the boundary value problem to a Volterra integral equation of the second kind. The kernel of the obtained integral equation contains a special function, namely, the generalized hypergeometric series. It is shown that the existence and uniqueness of solutions to the integral equation depends both on the order of the fractional derivative in the loaded term of the initial boundary value problem and on the behavior character of the load.

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1. INTRODUCTION

To date, the loaded heat conduction equations have wide practical application. In addition, loaded equations constitute a special class of equations with specific problems. There is also a need to study loaded equations in the investigation of some inverse problems, in the linearization of nonlinear equations, in the study of some optimal control problems, etc. The most general definition of a loaded equation was given by Nakhushiev in [1–3]. In [3], in particular, using numerous examples, Nakhushiev showed the practical and theoretical importance of research on loaded equations. In the papers [4–7] of Jenaliev and his students, the theory of loaded equations was further developed.

In recent years, we have seen an increase in the number of studies of various boundary value problems for loaded equations (see, for examples [8–14]). Also we note, that a large number of works are devoted to the study of equations with fractional integro-differentiation operators (see, for example [8–13, 15–17]). The distinguishing feature of this problem is the presence of fractional integro-differentiation operators in the boundary conditions. It is interesting to study the boundary value problems for the loaded heat equation, when the loaded term is represented in the form of a fractional derivative. The goal of papers [9, 10] is to clarify the character of the fractional load on the solvability issues of the first boundary value problem for the heat equation, when the load moves with a constant velocity. The loaded term is the trace of the fractional order derivative on the manifold $x = t$, namely, the loaded term is represented as a Riemann–Liouville’s fractional derivative. The resulting Volterra singular integral equation has a nonempty spectrum for certain values of the fractional derivative order.

In the papers [18, 19], the loaded term is represented in the form of the Caputo fractional derivative with respect to the time variable and the spatial variable, and the order of the derivative in the loaded

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term is less than the order of the differential part. In work [19], in particular, a boundary value problem is formulated for the fractionally loaded heat equation. The order of the derivative in the loaded term is less than the order of the differential part, but it is more than the order of the fractional derivative in the loaded term.

In contrast to [19], in our paper there is a discontinuity in the order of the derivative on the left. The boundary value problem is reduced to Volterra integral equation with kernel containing special function. There is the solvability investigation of the integral equation depending both on the order of the fractional derivative in the loaded term of the initial boundary value problem and on the behavior character of the load.

This paper is organized as follows. In Section 2, we recall some previously known concepts and results, and we give some of the most useful properties and presentations of some special functions that will be used throughout the paper. In Section 3, we give the statement of boundary value problems for the fractionally loaded heat equation (the loaded term of the equation is represented as a Caputo fractional derivative) in the class of continuous functions. Section 4 is devoted to reducing the posed boundary value problem to a Volterra integral equation of the second kind with a kernel containing special function, namely, a generalized hypergeometric function. In Section 5, we consider the limiting cases for the fractional derivative order of the term with the load in the equation of the boundary value problem and we show a discontinuity in the order of the derivative on the left. In Section 7, we show that conditions for the solvability of the obtained integral equation depend on the order of the fractional derivative in the loaded term in the equation of the boundary value problem and the nature of the load.

2. PRELIMINARY MATERIAL ON FRACTIONAL CALCULUS AND SPECIAL FUNCTIONS

Let us first recall some previously known concepts and results. The first one is the definition of the Riemann–Liouville fractional derivative.

Definition 1 [13]. Let $f(t) \in L_1[a, b]$. Then, the *Riemann–Liouville derivative of the order β* is defined as follows

$${}_r D_{a,t}^\beta f(t) = \frac{1}{\Gamma(n-\beta)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{\beta-n+1}} d\tau, \quad \beta, a \in \mathbb{R}, \quad n-1 < \beta < n. \quad (1)$$

For practical applications, the definition of the Caputo fractional derivative is significant. It is obtained after interchanging differentiation and integration in (1).

Definition 2 [16]. Let $f(t) \in AC^n[a, b]$ (i.e. $f^{(n-1)}(t)$ is an absolutely continuous function). Then, the *Caputo derivative of the order β* is defined as follows

$${}_c D_{a,t}^\beta f(t) = \frac{1}{\Gamma(n-\beta)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\beta-n+1}} d\tau; \quad \beta, a \in \mathbb{R}, \quad n-1 < \beta < n. \quad (2)$$

From formula (1) it follows that

$${}_r D_{a,t}^0 f(t) = f(t), \quad {}_r D_{a,t}^n f(t) = f^{(n)}(t), \quad n \in \mathbb{N}. \quad (3)$$

The derivatives determined by formulas (1) and (2) are related by the relation [20]

$${}_c D_{a,t}^\beta f(t) = {}_r D_{a,t}^\beta \left[f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} (t-a)^k \right]. \quad (4)$$

We also give definitions and some properties of special functions that are needed throughout the work. Error function and complementary error function have the form

$$\operatorname{erf}z = \frac{2}{\sqrt{\pi}} \int_0^z \exp(-\zeta^2) d\zeta, \quad \operatorname{erfc}z = \frac{2}{\sqrt{\pi}} \int_z^\infty \exp(-\zeta^2) d\zeta = 1 - \operatorname{erf}z.$$

A generalized hypergeometric series is defined by the formula [21, p. 136]

$${}_pF_q(a_1, a_2, \dots, a_p; b_1, b_2, \dots, b_q; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \dots (a_p)_k}{(b_1)_k (b_2)_k \dots (b_q)_k} \frac{z^k}{k!}, \quad (5)$$

where $(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}$ is the Pochhammer symbol.

If $p \leq q$, the singular points of (5) are at $z = 0$ and $z = \infty$; $z = 0$ is a regular singularity, $z = \infty$ is an irregular singularity [21, p. 137]. Then series (5) converges for all finite values z .

Generalized hypergeometric function (5) arises, for example, when calculating the integral by formula 3.478 (3) [22, p. 356]:

$$\begin{aligned} & \int_0^u \xi^{\nu-1} (u-\xi)^{\mu-1} \exp(\beta\xi^n) d\xi \\ &= B(\mu; \nu) u^{\mu+\nu-1} {}_nF_n\left(\frac{\nu}{n}, \frac{\nu+1}{n}, \dots, \frac{\nu+n-1}{n}; \frac{\mu+\nu}{n}, \frac{\mu+\nu+1}{n}, \dots, \frac{\mu+\nu+n-1}{n}; \beta u^n\right), \\ & \operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0, \quad n = 2, 3, \dots \end{aligned} \quad (6)$$

Here and everywhere else $\Gamma(z)$ and $B(\mu; \nu)$ are Euler integrals.

We study boundary value problems for the loaded heat equation, when the loaded term is represented in the form of a fractional derivative. The considered problem is reduced to an integral equation by inverting the differential part. It's known [23, p. 57] that in the domain $Q = \{(x, t) | x > 0, t > 0\}$ the solution to the boundary value problem of heat conduction

$$u_t = a^2 u_{xx} + F(x, t), \quad u|_{t=0} = f(x), \quad u|_{x=0} = g(x),$$

is described by the formula

$$u(x, t) = \int_0^{\infty} G(x, \xi, t) f(\xi) d\xi + \int_0^t H(x, t-\tau) g(\tau) d\tau + \int_0^t \int_0^{\infty} G(x, \xi, t-\tau) F(\xi, \tau) d\xi d\tau, \quad (7)$$

where

$$\begin{aligned} G(x, \xi, t) &= \frac{1}{2\sqrt{\pi at}} \left\{ \exp\left(-\frac{(x-\xi)^2}{4at}\right) - \exp\left(-\frac{(x+\xi)^2}{4at}\right) \right\}, \\ H(x, t) &= \frac{1}{2\sqrt{\pi at}^{3/2}} \exp\left(-\frac{x^2}{4at}\right). \end{aligned}$$

The Green function $G(x, \xi, t-\tau)$ satisfies the relation

$$\int_0^{\infty} G(x, \xi, t-\tau) d\xi = \operatorname{erf}\left(\frac{x}{2\sqrt{t-\tau}}\right). \quad (8)$$

3. STATEMENT OF THE PROBLEM

We consider the problem in a domain $Q = \{(x, t) : x > 0, t > 0\}$

$$u_t - u_{xx} + \lambda \left\{ D_{0,x}^{\beta} u(x, t) \right\} \Big|_{x=\alpha(t)} = f(x, t), \quad (9)$$

$$u(x, 0) = 0, \quad u(0, t) = 0, \quad (10)$$

where λ is a complex parameter, $D_{0,x}^{\beta} u(x, t)$ is Caputo derivative (2) of an order β , $1 < \beta < 2$, $\alpha(t)$ is a continuous increasing function, $\alpha(0) = 0$ or $\alpha(t)$ is a positive constant.

The problem is studied in the class of functions

$$u(x, t) \in AC^2(0, +\infty) \cap C^1(t \in [0, T]).$$

4. REDUCING THE PROBLEM TO A VOLTERRA INTEGRAL EQUATION OF THE SECOND KIND

Lemma 1. *Boundary value problem (9), (10) is equivalently reduced to a Volterra integral equation of the second kind with a kernel that contains a generalized hypergeometric function of the form (5) for $p = q = 2$.*

Proof. We invert the differential part of problem (9), (10) by formula (7)

$$u(x, t) = -\lambda \int_0^t \int_0^\infty G(x, \xi, t - \tau) \left\{ \frac{1}{\Gamma(2 - \beta)} \int_0^x \frac{u_{\xi\xi}(\xi, \tau)}{(x - \xi)^{\beta-1}} d\xi \right\} \Big|_{\xi=\alpha(\tau)} d\xi d\tau + \int_0^t \int_0^\infty G(x, \xi, t - \tau) f(\xi, \tau) d\xi d\tau.$$

Taking into account relation (8) and introducing the notation

$$f_1(x, t) = \int_0^t \int_0^\infty G(x, \xi, t - \tau) f(\xi, \tau) d\xi d\tau,$$

we get the following representation of the solution to problem (9), (10):

$$u(x, t) = -\lambda \int_0^t \operatorname{erf}\left(\frac{x}{2\sqrt{t-\tau}}\right) \mu(\tau) d\tau + f_1(x, t), \tag{11}$$

where

$$\mu(t) = \left\{ D_{0,x}^\beta u(x, t) \right\} \Big|_{x=\alpha(t)} = \frac{1}{\Gamma(2 - \beta)} \int_0^x \frac{u_{\xi\xi}(\xi, t)}{(x - \xi)^{\beta-1}} d\xi \Big|_{x=\alpha(t)}. \tag{12}$$

From representation (11) we take the derivative of the order β with respect to the variables x on both sides and put $x = \alpha(t)$. On the left side, we get the function $\mu(t)$. We also introduce the notation according to formula (11)

$$f_2(t) = \frac{1}{\Gamma(2 - \beta)} \int_0^x \frac{\frac{\partial^2 f_1(\xi, \tau)}{\partial \xi^2}}{(x - \xi)^{\beta-1}} d\xi \Big|_{x=\alpha(t)}. \tag{13}$$

Then taking into account notation (12), equality (11) can be rewritten in the form

$$\mu(t) = -\lambda \left\{ D_{0,x}^\beta \left[\int_0^t \operatorname{erf}\left(\frac{x}{2\sqrt{t-\tau}}\right) \mu(\tau) d\tau \right] \right\} \Big|_{x=\alpha(t)} + f_2(t). \tag{14}$$

We calculate the fractional derivative changing the variable x to ξ :

$$\begin{aligned} & D_{0,x}^\beta \left[\int_0^t \operatorname{erf}\left(\frac{x}{2\sqrt{t-\tau}}\right) \mu(\tau) d\tau \right] \\ &= \frac{1}{\Gamma(2 - \beta)} \int_0^x \frac{1}{(x - \xi)^{\beta-1}} \frac{\partial^2}{\partial \xi^2} \left[\int_0^t \operatorname{erf}\left(\frac{\xi}{2\sqrt{t-\tau}}\right) \mu(\tau) d\tau \right] d\xi \\ &= \frac{1}{\Gamma(2 - \beta)} \int_0^x \frac{1}{(x - \xi)^{\beta-1}} \frac{\partial}{\partial \xi} \left[\int_0^t \mu(\tau) \frac{\partial}{\partial \xi} \left(\frac{2}{\sqrt{\pi}} \int_0^{\frac{\xi}{2\sqrt{t-\tau}}} e^{-\eta^2} d\eta \right) d\tau \right] d\xi \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{\Gamma(2-\beta)} \int_0^x \frac{1}{(x-\xi)^{\beta-1}} \frac{\partial}{\partial \xi} \left[\int_0^t \mu(\tau) \frac{2}{\sqrt{\pi}} \frac{1}{2\sqrt{t-\tau}} \exp\left(-\frac{\xi^2}{4(t-\tau)}\right) d\tau \right] d\xi \\
 &= \frac{1}{\sqrt{\pi}\Gamma(2-\beta)} \int_0^x \frac{1}{(x-\xi)^{\beta-1}} \left(\int_0^t \frac{\mu(\tau)}{\sqrt{t-\tau}} \left(-\frac{\xi}{2(t-\tau)}\right) \cdot \exp\left(-\frac{\xi^2}{4(t-\tau)}\right) d\tau \right) d\xi \\
 &= \frac{-1}{2\sqrt{\pi}\Gamma(2-\beta)} \int_0^x \frac{\xi}{(x-\xi)^{\beta-1}} \left(\int_0^t \frac{\mu(\tau)}{(t-\tau)^{3/2}} \exp\left(-\frac{\xi^2}{4(t-\tau)}\right) d\tau \right) d\xi \\
 &= \frac{-1}{2\sqrt{\pi}\Gamma(2-\beta)} \int_0^t \frac{\mu(\tau)}{(t-\tau)^{3/2}} \left[\int_0^x \frac{\xi}{(x-\xi)^{\beta-1}} \exp\left(-\frac{\xi^2}{4(t-\tau)}\right) d\xi \right] d\tau.
 \end{aligned}$$

We have changed the order of integration. So, we get

$$\begin{aligned}
 &D_{0,x}^\beta \left[\int_0^t \operatorname{erf}\left(\frac{x}{2\sqrt{t-\tau}}\right) \mu(\tau) d\tau \right] \\
 &= -\frac{1}{2\sqrt{\pi}\Gamma(2-\beta)} \int_0^t \frac{\mu(\tau)}{(t-\tau)^{3/2}} \int_0^x \frac{\xi}{(x-\xi)^{\beta-1}} \exp\left(-\frac{\xi^2}{4(t-\tau)}\right) d\xi d\tau. \tag{15}
 \end{aligned}$$

From formula (6), when $n = 2, u = x, \nu = 2, \mu = 2 - \beta$, we obtain that formula (15) takes the form

$$\begin{aligned}
 &D_{0,x}^\beta \left[\int_0^t \operatorname{erf}\left(\frac{x}{2\sqrt{t-\tau}}\right) \mu(\tau) d\tau \right] \\
 &= -\frac{1}{2\sqrt{\pi}\Gamma(2-\beta)} \int_0^t \frac{\mu(\tau)}{(t-\tau)^{3/2}} B(2-\beta; 2) x^{3-\beta} {}_2F_2\left(1, \frac{3}{2}; \frac{4-\beta}{2}, \frac{5-\beta}{2}; -\frac{x^2}{4(t-\tau)}\right) d\tau, \tag{16}
 \end{aligned}$$

where

$$B(2-\beta; 2) = \frac{\Gamma(2-\beta)}{\Gamma(4-\beta)}.$$

For $x = \alpha(t)$, derivative (16) takes the form

$$\begin{aligned}
 &D_{0,x}^\beta \left[\int_0^t \operatorname{erf}\left(\frac{x}{2\sqrt{t-\tau}}\right) \mu(\tau) d\tau \right] \Big|_{x=\alpha(t)} \\
 &= -\frac{(\alpha(t))^{3-\beta}}{2\sqrt{\pi}\Gamma(4-\beta)} \int_0^t \frac{\mu(\tau)}{(t-\tau)^{3/2}} {}_2F_2\left(1, \frac{3}{2}; \frac{4-\beta}{2}, \frac{5-\beta}{2}; -\frac{\alpha^2(t)}{4(t-\tau)}\right) d\tau. \tag{17}
 \end{aligned}$$

In view of (17), equation (14) can be rewritten in the form

$$\mu(t) - \lambda \int_0^t K_\beta(t, \tau) \mu(\tau) d\tau = f_2(t), \tag{18}$$

where

$$K_\beta(t, \tau) = \frac{(\alpha(t))^{3-\beta}}{2\sqrt{\pi}\Gamma(4-\beta)(t-\tau)^{3/2}} {}_2F_2\left(1, \frac{3}{2}; \frac{4-\beta}{2}, \frac{5-\beta}{2}; -\frac{\alpha^2(t)}{4(t-\tau)}\right). \tag{19}$$

Here, by virtue of the formula (5), we have

$${}_2F_2(a_1, a_2; b_1, b_2; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k}{(b_1)_k (b_2)_k} \frac{z^k}{k!}.$$

5. STUDY ON CONTINUITY OF THE FRACTIONAL DERIVATIVE ORDER IN THE INTERVAL OF ITS CHANGING

Lemma 2. For the boundary value problem (9), (10) breaks the continuity of the left boundary interval of changing the order β of the derivative in the loaded term of equation (9).

Proof. We consider the limiting cases for the fractional derivative order of the term with the load in equation (9).

1. $\beta = 1$. Then from formulas (3), (4) and condition (10) we have $D_{0,x}^1 u(x, t) = u_x(x, t)$. Problem (9), (10) will take the form

$$u_t - u_{xx} + \lambda u(x, t) \Big|_{x=\alpha(t)} = f(x, t), \tag{20}$$

$$u(x, 0) = 0, \quad u(0, t) = 0. \tag{21}$$

We denote: $\mu(t) = u_x(x, t) \Big|_{x=\alpha(t)}$. Then we write down solution to problem (20), (21) inverting its differential part by formula (7)

$$u(x, t) = -\lambda \int_0^t \int_0^\infty \mu(\tau) G(x, \xi, t - \tau) d\xi d\tau + f_1(x, t), \tag{22}$$

where

$$f_1(x, t) = \int_0^t \int_0^\infty G(x, \xi, t - \tau) f(\xi, \tau) d\xi d\tau.$$

Taking into account the ratio (8), equality (22) takes the form

$$u(x, t) = -\lambda \int_0^t \mu(\tau) \operatorname{erf}\left(\frac{x}{2\sqrt{t-\tau}}\right) d\tau + f_1(x, t). \tag{23}$$

We differentiate equality (23) with respect to x , taking into account

$$\frac{\partial}{\partial x} \left(\operatorname{erf}\left(\frac{x}{2\sqrt{t-\tau}}\right) \right) = \frac{1}{\sqrt{\pi}\sqrt{t-\tau}} \exp\left(-\frac{x^2}{4(t-\tau)}\right).$$

Further, substituting $x = \alpha(t)$ and taking into account the notation $\mu(t) = u_x(x, t) \Big|_{x=\alpha(t)}$, we obtain Volterra integral equation of the second kind

$$\mu(t) + \lambda \int_0^t K_1(t, \tau) \mu(\tau) d\tau = f_2(t), \tag{24}$$

where

$$K_1(t, \tau) = \frac{1}{\sqrt{\pi}\sqrt{t-\tau}} \exp\left(-\frac{\alpha^2(t)}{4(t-\tau)}\right), \quad f_2(t) = f_1(\alpha(t); t).$$

On the other hand, in equality (15) we take the limit when $\beta \rightarrow 1 + 0$:

$$\lim_{\beta \rightarrow 1+0} D_{0,x}^\beta \left[\int_0^t \operatorname{erf}\left(\frac{x}{2\sqrt{t-\tau}}\right) \mu(t) d\tau \right]$$

$$= \lim_{\beta \rightarrow 1+0} \left\{ \frac{1}{2\sqrt{\pi}\Gamma(2-\beta)} \int_0^t \frac{\mu(\tau)}{(t-\tau)^{3/2}} \left[\int_0^x \frac{\xi}{(x-\xi)^{\beta-1}} \exp\left(-\frac{\xi^2}{4(t-\tau)}\right) d\xi \right] d\tau \right\}.$$

The function under the limit sign is definite and continuous at $\beta = 1$; therefore, it is possible to pass to the limit under the integral sign

$$\begin{aligned} \lim_{\beta \rightarrow 1+0} D_{0,x}^\beta \left[\int_0^t \operatorname{erf}\left(\frac{x}{2\sqrt{t-\tau}}\right) \mu(\tau) d\tau \right] &= -\frac{1}{2\sqrt{\pi}} \int_0^t \frac{\mu(\tau)}{(t-\tau)^{3/2}} \int_0^x \xi \exp\left(-\frac{\xi^2}{4(t-\tau)}\right) d\xi d\tau \\ &= \left\| z = \frac{\xi^2}{4(t-\tau)} \right\| = -\frac{1}{\sqrt{\pi}} \int_0^t \frac{\mu(\tau)}{(t-\tau)^{1/2}} \int_0^{\frac{x^2}{4(t-\tau)}} e^{-z} dz d\tau \\ &= \frac{1}{\sqrt{\pi}} \int_0^t \frac{\mu(\tau)}{\sqrt{t-\tau}} \left(1 - \exp\left(-\frac{x^2}{4(t-\tau)}\right) \right) d\tau. \end{aligned}$$

For $x = \alpha(t)$ we get

$$\lim_{\beta \rightarrow 1+0} D_{0,x}^\beta \left[\int_0^t \operatorname{erf}\left(-\frac{x}{2\sqrt{t-\tau}}\right) \mu(\tau) d\tau \right] \Big|_{x=\alpha(t)} = \int_0^t \frac{\mu(\tau)}{\sqrt{\pi}\sqrt{t-\tau}} \left(1 - \exp\left(-\frac{\alpha^2(t)}{4(t-\tau)}\right) \right) d\tau.$$

Then equation (18) when $\beta \rightarrow 1 + 0$ and $x = \alpha(t)$ takes the form

$$\mu(t) - \lambda \int_0^t K_1(t, \tau) \mu(\tau) d\tau = f_2(t), \tag{25}$$

where

$$K_1(t, \tau) = \frac{1}{\sqrt{\pi}\sqrt{t-\tau}} \exp\left(1 - \frac{\alpha^2(t)}{4(t-\tau)}\right), \quad f_2(t) = f_1(\alpha(t); t).$$

Remark 1. Equation (25) can be obtained by taking the limit of kernel (19) when $\beta \rightarrow 1 + 0$:

$$K_1(t, \tau) = \lim_{\beta \rightarrow 1+0} K_\beta(t, \tau) = \frac{(\alpha(t))^2}{2\sqrt{\pi}\Gamma(3)(t-\tau)^{3/2}} {}_2F_2\left(1, \frac{3}{2}; \frac{3}{2}, 2; -\frac{\alpha^2(t)}{4(t-\tau)}\right).$$

As

$$\begin{aligned} {}_2F_2\left(1, \frac{3}{2}; \frac{3}{2}, 2; -\frac{\alpha^2(t)}{4(t-\tau)}\right) &= \sum_{k=0}^{\infty} \frac{(1)_k}{(2)_k} \cdot \frac{z^k}{k!} \Big|_{z=-\frac{\alpha^2(t)}{4(t-\tau)}} = \sum_{k=0}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k+2)} \cdot \frac{z^k}{k!} \Big|_{z=-\frac{\alpha^2(t)}{4(t-\tau)}} \\ &= \sum_{k=0}^{\infty} \frac{z^k}{(k+1)!} \Big|_{z=-\frac{\alpha^2(t)}{4(t-\tau)}} = \frac{1}{z} \sum_{n=1}^{\infty} \frac{z^n}{n!} \Big|_{z=-\frac{\alpha^2(t)}{4(t-\tau)}} = \frac{1}{z} (e^z - 1) \Big|_{z=-\frac{\alpha^2(t)}{4(t-\tau)}}, \end{aligned}$$

then

$$\lim_{\beta \rightarrow 1+0} K_\beta(t, \tau) = \frac{1}{\sqrt{\pi}\sqrt{t-\tau}} \left(1 - \exp\left(-\frac{\alpha^2(t)}{4(t-\tau)}\right) \right).$$

When $\beta \rightarrow 1 + 0$ equation (18) takes the form

$$\mu(t) - \lambda \int_0^t K_1(\beta, \tau) \mu(\tau) d\tau = f_2(t),$$

where

$$K_1(\phi, \tau) = \frac{1}{\sqrt{\pi}\sqrt{t-\tau}} \exp\left(1 - \frac{\alpha^2(t)}{4(t-\tau)}\right).$$

Taking into account the equation (24), we conclude that equation (18) does not coincide with equation (25) for $\beta = 1$.

II. $\beta = 2$. From formulas (3), (4) and condition (10) we have $D_{0,x}^2 u(x, t) = u_{xx}(x, t)$. Problem (9), (10) will take the form

$$u_t - u_{xx} + \lambda\mu(t) = f(x, t), \tag{26}$$

$$u(x, 0) = 0, \quad u(0, t) = 0, \tag{27}$$

where

$$\mu(t) = u_{xx}(x, t)|_{x=\alpha(t)}. \tag{28}$$

We write down solution to problem (26), (27) inverting its differential part by formula (7):

$$u(x, t) = -\lambda \int_0^t \int_0^{+\infty} \mu(\tau) G(x, \xi, t - \tau) d\xi d\tau + \int_0^t \int_0^{+\infty} f(\xi, \tau) G(x, \xi, t - \tau) d\xi d\tau.$$

In view of relation (8), the last equality can be rewritten in the form

$$u(x, t) = -\lambda \int_0^t \mu(\tau) \operatorname{erf}\left(\frac{x}{2\sqrt{t-\tau}}\right) d\tau + f_1(x, t), \tag{29}$$

where

$$f_1(x, t) = \int_0^t \int_0^{+\infty} f(\xi, \tau) G(x, \xi, t - \tau) d\xi d\tau.$$

We take the Caputo derivative of expression (29) (it is twice differentiable with respect to x) and put $x = \alpha(t)$. Then, taking into account designation (28) and the following equality

$$\begin{aligned} \frac{\partial^2}{\partial x^2} \left(\operatorname{erf}\left(\frac{x}{2\sqrt{t-\tau}}\right) \right) &= \frac{\partial}{\partial x} \left(\frac{1}{\sqrt{\pi(t-\tau)}} \exp\left(-\frac{x^2}{4(t-\tau)}\right) \right) \\ &= \frac{x}{2\sqrt{\pi}(t-\tau)^{3/2}} \exp\left(-\frac{x^2}{4(t-\tau)}\right), \end{aligned}$$

we get

$$\mu(t) - \lambda \int_0^t \frac{\alpha(t)}{2\sqrt{\pi}(t-\tau)^{3/2}} \exp\left(-\frac{\alpha^2(t)}{4(t-\tau)}\right) \mu(\tau) d\tau + f_2(t), \tag{30}$$

where

$$\begin{aligned} f_2(t) &= D_{0,x}^2 f_1(x, t) = \int_0^x \frac{\partial^3 f_1(\xi, t)}{\partial \xi^3} d\xi, \\ K_2(t, \tau) &= \frac{\alpha(t)}{2\sqrt{\pi}(t-\tau)^{3/2}} \exp\left(-\frac{\alpha^2(t)}{4(t-\tau)}\right). \end{aligned} \tag{31}$$

On the other hand, from equality (19) we have

$$\lim_{\beta \rightarrow 2-0} K_\beta(t, \tau) = \frac{\alpha(t)}{2\sqrt{\pi}\Gamma(2)(t-\tau)^{3/2}} {}_2F_2\left(1, \frac{3}{2}; 1, \frac{3}{2}; -\frac{\alpha^2(t)}{4(t-\tau)}\right).$$

As

$${}_2F_2 \left(1, \frac{3}{2}; 1, \frac{3}{2}; -\frac{\alpha^2(t)}{4(t-\tau)} \right) = \sum_{k=0}^{\infty} \frac{(1)_k \cdot \left(\frac{3}{2}\right)_k}{(1)_k \left(\frac{3}{2}\right)_k} \frac{z^k}{k!} \Bigg|_{z=-\frac{\alpha^2(t)}{4(t-\tau)}} = \exp \left(-\frac{\alpha^2(t)}{4(t-\tau)} \right),$$

then

$$\lim_{\beta \rightarrow 2-0} K_\beta(t, \tau) = \frac{\alpha(t)}{2\sqrt{\pi}(t-\tau)} \exp \left(-\frac{\alpha^2(t)}{4(t-\tau)} \right).$$

The result coincides with expression (31) that is the kernel for equation (30).

Remark 2. We cannot directly take the limit $\beta \rightarrow 2 - 0$ from expression (15), as at $\beta \rightarrow 1 + 0$, since passing to the limit, we obtain an uncertainty of the form $(0 \cdot \infty)$. Indeed, according to formula (6) the integral

$$\begin{aligned} \int_0^x \frac{\xi}{x-\xi} \exp \left(-\frac{\xi^2}{4(t-\tau)} \right) d\xi &= \lim_{z \rightarrow 0} B(z, 2) x {}_2F_2 \left(1, \frac{3}{2}; 1, \frac{3}{2}; -\frac{x^2}{4(t-\tau)} \right) \\ &= \lim_{z \rightarrow 0} B(z, 2) x \exp \left(-\frac{x^2}{4(t-\tau)} \right) \end{aligned}$$

diverges. Therefore, equality (15) requires further transformation.

So, the kernel of equation (18) has the form

$$K_\beta(t, \tau) = \begin{cases} \frac{1}{\sqrt{\pi}(t-\tau)} \exp \left(-\frac{\alpha^2(t)}{4(t-\tau)} \right), & \text{if } \beta = 1; \\ \frac{(\alpha(t))^{3-\beta}}{2\sqrt{\pi}\Gamma(4-\beta)(t-\tau)^{3/2}} {}_2F_2 \left(1, \frac{3}{2}; \frac{4-\beta}{2}, \frac{5-\beta}{2}; -\frac{\alpha^2(t)}{4(t-\tau)} \right), & \text{if } 1 < \beta \leq 2 \end{cases}$$

and $K_\beta(t, \tau)$ suffers a discontinuity at the point $\beta = 1$, as a function of β , and the jump of the function $K_\beta(t, \tau)$ is equal to $\frac{1}{\sqrt{\pi}\sqrt{t-\tau}}$ at the point $\beta = 1$. Here we take into account the formula [19, formula (22)].

Lemma 2 is completely proved.

6. CONDITIONS FOR THE SOLVABILITY OF THE INTEGRAL EQUATION. MAIN RESULT

To establish the main result of the paper we investigate the kernel $K_\beta(t, \tau)$ of integral equation (18), which has singularities at $\tau = t$ and $t = 0$.

Direct investigation of kernel $K_\beta(t, \tau)$ in (19) is difficult, since the kernel of this integral equation contains a generalized hypergeometric series. Therefore, we find

$$\lim_{t \rightarrow 0+0} \int_0^t K_\beta(t, \tau) d\tau.$$

Theorem. *Integral equation (18) with kernel (19) for $1 \leq \beta \leq 2$ and with $\alpha(t) \sim t^\omega$ in the neighborhood of $t = 0$ is uniquely solvable in the class of continuous functions for any continuous right-hand side $f_2(t)$ defined by formula (13), if $\omega \geq \frac{1}{2}$ and $1 < \beta < 2$ or $0 \leq \omega < \frac{1}{2}$ and $\beta = 2$ or $\omega \geq 0$ and $\beta = 1$.*

Proof. We use the representation of kernel (19) on formula (15) when $x = \alpha(t)$ and $1 < \beta < 2$:

$$K_\beta(t, \tau) = \frac{1}{2\sqrt{\pi}\Gamma(2-\beta)(t-\tau)^{3/2}} \int_0^{\alpha(t)} \frac{\xi}{(\alpha(t)-\xi)^{\beta-1}} \exp \left(-\frac{\xi^2}{4(t-\tau)} \right) d\xi.$$

Then

$$\begin{aligned} \int_0^t K_\beta(t, \tau) d\tau &= \frac{1}{2\sqrt{\pi}\Gamma(2-\beta)} \int_0^t \frac{1}{(t-\tau)^{3/2}} \left[\int_0^{\alpha(t)} \frac{\xi}{(\alpha(t)-\xi)^{\beta-1}} \exp\left(-\frac{\xi^2}{4(t-\tau)}\right) d\xi \right] d\tau \\ &= \frac{1}{2\sqrt{\pi}\Gamma(2-\beta)} \int_0^{\alpha(t)} \frac{\xi}{(\alpha(t)-\xi)^{\beta-1}} \left[\int_0^t \frac{1}{(t-\tau)^{3/2}} \exp\left(-\frac{\xi^2}{4(t-\tau)}\right) d\tau \right] d\xi. \end{aligned}$$

To calculate the inner integral, we introduce a replacement $z = \frac{\xi}{2\sqrt{t-\tau}}$ and then we apply formula 2.8.2 from [26, p. 92]

$$\begin{aligned} \int_0^t K_\beta(t, \tau) d\tau &= \frac{1}{2\sqrt{\pi}\Gamma(2-\beta)} \int_0^{\alpha(t)} \frac{4}{(\alpha(t)-\xi)^{\beta-1}} \int_{\frac{\xi}{2\sqrt{t}}}^{+\infty} e^{-z^2} dz d\xi \\ &= \frac{1}{\Gamma(2-\beta)} \int_0^{\alpha(t)} \frac{1}{(\alpha(t)-\xi)^{\beta-1}} \operatorname{erfc}\left(\frac{\xi}{2\sqrt{t}}\right) d\xi \\ &= \frac{1}{\Gamma(2-\beta)} \left[-\frac{2(\alpha(t))^{3-\beta}}{\sqrt{\pi}} \frac{1}{2\sqrt{t}} {}_3F_3\left(1, \frac{3}{2}, \frac{1}{2}; \frac{4-\beta}{2}, \frac{5-\beta}{2}, \frac{5}{2}; -\frac{\alpha^2(t)}{4t}\right) \right. \\ &\quad \left. + (\alpha(t))^{2-\beta} B(1; 2-\beta) \right] = -\frac{(\alpha(t))^{3-\beta}}{\sqrt{\pi}\sqrt{t}\Gamma(4-\beta)} {}_2F_2\left(1, \frac{1}{2}; \frac{4-\beta}{2}, \frac{5-\beta}{2}; -\frac{\alpha^2(t)}{4t}\right) + \frac{(\alpha(t))^{2-\beta}}{\Gamma(3-\beta)}. \end{aligned}$$

So, we have

$$\int_0^t K_\beta(t, \tau) d\tau = -\frac{(\alpha(t))^{3-\beta}}{\sqrt{\pi}\sqrt{t}\Gamma(4-\beta)} {}_2F_2\left(1, \frac{1}{2}; \frac{4-\beta}{2}, \frac{5-\beta}{2}; -\frac{\alpha^2(t)}{4t}\right) + \frac{(\alpha(t))^{2-\beta}}{\Gamma(3-\beta)}. \tag{32}$$

We use also the representation of expression (32) through the series

$$\int_0^t K_\beta(t, \tau) d\tau = -\frac{1}{\sqrt{\pi}\Gamma(4-\beta)} \sum_{k=0}^{\infty} \frac{(1)_k \left(\frac{1}{2}\right)_k (-1)^k}{\left(\frac{4-\beta}{2}\right)_k \left(\frac{5-\beta}{2}\right)_k 4^k} (\alpha(t))^{2k+3-\beta} t^{-k-\frac{1}{2}} + \frac{(\alpha(t))^{2-\beta}}{\Gamma(3-\beta)}. \tag{33}$$

Taking $\alpha(t) \sim t^\omega$ at $t \rightarrow 0$, from expression (33) for $1 < \beta < 2$ we obtain

$$\lim_{t \rightarrow 0} \int_0^t K_\beta(t, \tau) d\tau = \lim_{t \rightarrow 0} \left[-\frac{1}{\sqrt{\pi}\Gamma(4-\beta)} \sum_{k=0}^{\infty} \frac{(1)_k \left(\frac{1}{2}\right)_k (-1)^k}{\left(\frac{4-\beta}{2}\right)_k \left(\frac{5-\beta}{2}\right)_k 4^k} t^{2k\omega+3\omega-\beta\omega-k-\frac{1}{2}} + \frac{t^{(2-\beta)\omega}}{\Gamma(3-\beta)} \right].$$

Let $2k\omega + 3\omega - \beta\omega - k - \frac{1}{2} > 0$. Then $(2\omega - 1)k + 3\omega - \beta\omega - \frac{1}{2} > 0$. If $\omega > \frac{1}{2}$, then for $1 < \beta < 2$ we get $3\omega - \beta\omega - \frac{1}{2} > 0$ and $(2\omega - 1)k > 0$ for all $k = 0, 1, 2, \dots$

So, we have

$$\lim_{t \rightarrow 0} \int_0^t K_\beta(t, \tau) d\tau = \begin{cases} 0, & \text{if } \omega \geq \frac{1}{2}; \\ \infty, & \text{if } \omega < \frac{1}{2}. \end{cases}$$

Taking into account that $\alpha(t) \sim \sqrt{t}$ as $t \rightarrow 0$, from (33) for $1 < \beta < 2$ we have

$$\lim_{t \rightarrow 0} K_\beta(t, \tau) = \lim_{t \rightarrow 0} \left[\frac{t^{1-\frac{\beta}{2}}}{\Gamma(3-\beta)} - \frac{t^{1-\frac{\beta}{2}}}{\sqrt{\pi}\Gamma(4-\beta)} \sum_{k=0}^{\infty} \frac{(1)_k (\frac{1}{2})_k (-1)^k}{\left(\frac{4-\beta}{2}\right)_k \left(\frac{5-\beta}{2}\right)_k 4^k} \right] = 0.$$

The series in the second term of the last expression is ${}_2F_2\left(1, \frac{1}{2}; \frac{4-\beta}{2}, \frac{5-\beta}{2}; -\frac{1}{4}\right)$ and converge.

From (33) it follows that for the condition

$$\lim_{t \rightarrow 0} \int_0^t K_\beta(t, \tau) d\tau = 0,$$

when $1 < \beta < 2$, the following inequality must be true

$$2k\omega + 3\omega - \beta\omega - k - \frac{1}{2} > 0.$$

We suppose that $\alpha(t)$ is a continuous increasing function and $\alpha(0) = 0$ in the domain Q or $\alpha(t)$ is a positive constant. Then $\omega \geq 0$. Therefore, the case $\omega < 0$ is not considered.

From the last three inequalities we have $\omega > \frac{2k+1}{4(k+1)}$. So, as $k \rightarrow +\infty$ we get again $\omega > \frac{1}{2}$. Therefore, in this case we have

$$\lim_{t \rightarrow 0} \int_0^t K_\beta(t, \tau) d\tau = 0, \tag{34}$$

for $\alpha(t) \sim t^\omega$ as $t \rightarrow 0, \omega \geq \frac{1}{2}, 1 < \beta < 2$.

Consider the limit values β . It was shown above by (25) that

$$K_1(t, \tau) = \frac{1}{\sqrt{\pi}\sqrt{t-\tau}} \exp\left\{-\frac{\alpha^2(t)}{4(t-\tau)}\right\}.$$

Then for $\beta = 1$ using formula [22, formula 3.461], we obtain

$$\int_0^t K_1(t, \tau) d\tau = \alpha(t) \operatorname{erfc}\left(\frac{\alpha(t)}{2\sqrt{t}}\right) - \frac{2\sqrt{t}}{\sqrt{\pi}} \exp\left(-\frac{\alpha^2(t)}{4t}\right).$$

From here, when $\alpha(t) \sim t^\omega (\omega \geq 0)$ as $t \rightarrow 0$, we get

$$\lim_{t \rightarrow 0} \int_0^t K_1(t, \tau) d\tau = \lim_{t \rightarrow 0} \left(t^\omega \operatorname{erfc}\left(\frac{t^{\omega-\frac{1}{2}}}{2}\right) - \frac{2\sqrt{t}}{\sqrt{\pi}} \exp\left(-\frac{t^{2\omega-1}}{4}\right) \right) = 0. \tag{35}$$

We consider the case $\beta = 2$. It was shown above by (31) that

$$K_2(t, \tau) = \frac{\alpha(t)}{2\sqrt{\pi}(t-\tau)^{3/2}} \exp\left(-\frac{\alpha^2(t)}{4(t-\tau)}\right).$$

Therefore, we have

$$\int_0^t K_2(t, \tau) d\tau = \operatorname{erfc}\left(\frac{\alpha(t)}{2\sqrt{t}}\right).$$

Hence, when $\alpha(t) \sim t^\omega$ ($\omega \geq 0$) as $t \rightarrow 0$, we get

$$\lim_{t \rightarrow 0} \int_0^t K_2(t, \tau) d\tau = \lim_{t \rightarrow 0} \operatorname{erfc} \left(\frac{t^{\omega - \frac{1}{2}}}{2} \right) = \begin{cases} 0, & \text{if } 0 \leq \omega < \frac{1}{2}; \\ \operatorname{erfc} \left(\frac{1}{2} \right), & \text{if } \omega = \frac{1}{2}; \\ 1, & \text{if } \omega > \frac{1}{2}. \end{cases} \quad (36)$$

Summarizing results (34)–(36), we get the main result. The theorem is completely proved.

7. CONCLUSION

Under the conditions of the theorem, kernel (19) of integral equation (18) has a weak singularity. Therefore, the method of successive approximations can be used to find a unique solution to the equation (18) in the class of continuous functions. And the corresponding boundary value problems are well-posed in natural classes of functions, i.e. loaded term is a weak perturbation.

In other cases of the parameter values β and ω integral equation (18) is not solvable by the method of successive approximations. It can be shown that the corresponding homogeneous equation for some values of the parameter λ will have nonzero solutions. If the uniqueness of the solution to the first boundary value problem is violated, then in this case the load can be interpreted as a strong perturbation. So, the existence and uniqueness of solutions to the integral equation depends on the order of the fractional derivative in the loaded term.

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