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A combined problem with local and nonlocal conditions for a class of mixed-type equations

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This paper investigates the issues of existence and uniqueness of a solution to a combined boundary value problem with local and nonlocal conditions for a specific class of mixed elliptic-hyperbolic-type equations with singular coefficients. A distinctive feature of the considered problem is that on one part of the boundary characteristic, the values of the desired function are specified, while on the other part, nonlocal conditions are imposed. These conditions establish pointwise connections between the values of the sought function on different parts of the boundary characteristics using the Riemann-Liouville fractional differentiation operator. At the same time, a portion of hyperbolic domain's boundary remains free from boundary conditions. The proof of the solution's uniqueness is based on the application of an analogue of A.V. Bitsadze's extremum principle for mixed-type equations with singular coefficients. The existence of the solution is reduced to the analysis of a Tricomi singular integral equations' system with a shift, containing a non-Fredholm operator with isolated first-order singularity in kernel. By applying the Carleman-Vekua regularization method, these equations are reduced to a Wiener-Hopf integral equation, for which it is proved that the index is equal to zero. This, in turn, reduces the problem to a Fredholm integral equation of the second kind, the uniqueness of whose solution ensures the well-posedness of the given problem.

Keywords: mixed-type equation with singular coefficients, nonlocal condition, regularization, systems of singular integral equations, Wiener-Hopf equation, index.

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Introduction

The theory of local and nonlocal boundary value problems for mixed-type equations plays an important role in engineering and nature, particularly in gas dynamics, state processes, the development of oil reservoirs, groundwater filtration, heat and mass transfer in objects with complex structures, electrical oscillations in conductors, fluid flow in a channel surrounded by a porous medium, aerodynamics, and other phenomena.

The development of the theory of degenerate equations of mixed type originates from the fundamental works of G. Darboux, F. Tricomi, E. Holmgren, and S. Gellerstedt, published in 1894, 1923, 1927, and 1938, respectively. The problem for the model equation of mixed type was first formulated and solved by F. Tricomi, and it is now known as the Tricomi problem. After this work, the theory of local and nonlocal problems for mixed-type equations was developed in fundamental studies of E. Holmgren, S. Gellerstedt, A.V. Bitsadze, A.A. Samarskii, V.I. Zhegalov, A.M. Nakhushev, I. Frankl, S.G. Mikhlin, K.I. Babenko, M.M. Smirnov, M. Protter, M.M. Meredov, Sh.A. Alimov, E.I. Moiseev, A.P. Soldatov, M.S. Salakhitdinov, and others.

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Among the works devoted to boundary value problems for mixed-type equations, special attention should be given to the work of A.V. Bitsadze [1], in which a number of important problems in both two-dimensional and spatial cases were studied. These studies have stimulated further research in this direction and have attracted numerous mathematicians to this field.

By the late 1970s, many issues in the theory of boundary value problems for degenerate partial differential equations, including mixed-type equations with smooth coefficients in the considered domain, had acquired a mathematically complete form. Further progress in this field was largely determined by qualitatively new problems for equations with non-smooth coefficients, particularly for mixed-type equations with singular coefficients. Despite the large number of studies on hyperbolic equations and equations of mixed type, problems with combined local and nonlocal conditions for a degenerate mixed-type equation with singular coefficients remain poorly studied.

A nonlocal problem for a mixed-type equation with a singular coefficient in an unbounded domain was studied in the work of M. Ruziev and M. Reissig [2], while the works of Z.G. Feng [3], Z. Feng, and J. Kuang [4] are devoted to the study of boundary value problems for nonlinear mixed-type equations.

This work is devoted to the study of the solvability of a combined problem with local and nonlocal conditions for a certain class of mixed-type equations with singular coefficients. The degenerate equations with singular coefficients considered in this article differ from the well-known classical problems in that the correctness of the known Cauchy problem (in the hyperbolic region) and Holmgren's problem (in the elliptic domain) does not always hold. In the considered domains, these problems in their standard formulation may turn out to be unsolvable if the mixed-type equation degenerates along a line that is simultaneously a characteristic (an envelope of a family of characteristics) or if coefficients of the equation at lower-order terms are singular. Therefore, in these cases, it is natural to consider modified Cauchy and Holmgren problems, where the condition on the degeneration line is given with weight functions. Therefore, in the problem formulation of the study, along with local and nonlocal boundary conditions on the degeneration line of the equation, discontinuous matching conditions for the normal derivatives of the sought function with weight functions are specified. The problem conditions are given in a combined form: locally on the boundary of the ellipticity of the equation and on one part of the boundary characteristic, while on the other part, nonlocal conditions are imposed, establishing pointwise relationships between the values of the sought function at different sections of the characteristic boundary using the fractional differentiation operator in the sense of Riemann-Liouville. At the same time, part of the boundary of the hyperbolic domain remains free from boundary conditions.

By modifying A.V. Bitsadze's extremum principle for a mixed-type equation with singular coefficients, the uniqueness of the combined problem has been proven.

The solvability of the problem is reduced to the study of non-standard singular Tricomi integral equations with a numerical parameter in the non-singular part of the kernel and a non-Fredholm operator on the right-hand side of the equation.

The obtained singular integral equation is characterized by the following properties:

- generalizes the singular integral equation of F. Tricomi. In a particular case, this equation is reduced to the equation studied by F. Tricomi;
- the "nonsingular" part of the kernel has non-Carleman shifts;
- the non-characteristic part of the singular integral equation contains non-Fredholm integral operators; more precisely, the kernels of these operators have isolated singularities of the first order.

An algorithm has been developed for solving such non-standard integral equations: first, temporarily assuming the non-characteristic part of the equation as a known quantity, a singular integral equation of Tricomi with a shift is obtained; then, by regularizing it using the Carleman method developed by S.G. Mikhlin, the Wiener-Hopf equation is derived, which, through the Fourier transform, is reduced to a Riemann boundary value problem in the theory of functions of a complex variable.

Furthermore, it is proved that the index of the Riemann problem is equal to zero, which ensures the unique regularization of the Wiener-Hopf equation into a Fredholm integral equation of the second kind, whose unique solvability follows from the uniqueness of the solution to the formulated problems.

1 *Statement of the problem A*

In a finite simply connected domain D of the plane of independent variables x, y, bounded for $y > 0$ by a regular curve

$$\sigma_0: \quad x^2 + 4(m+2)^{-2} y^{m+2} = 1$$

with endpoints $A = A(-1, 0)$ and $B = B(1, 0)$, and for $y < 0$ with characteristics AC and BC of a mixed-type equation with singular coefficients of the following form

$$(signy)|y|^m u_{xx} + u_{yy} + \frac{\beta_0}{y} u_y = 0 \tag{1}$$

is considered, where $m > 0$, $-\frac{m}{2} < \beta_0 < 1$.

Let D^+ and D^- denote the parts of the domain D, located, respectively, in the half-planes $y > 0$ and $y < 0$, and let C_0 and C_1, respectively represent the intersection points of the characteristics AC and BC with the characteristic originating from the point $E(c, 0)$, where $c \in I = (-1, 1)$ is the interval of the $y = 0$ axis.

Problem A. Find the generalized solution $u(x, y)$ of equation (1), that satisfies the following conditions:

$$u(x, y)|_{\sigma_0} = \varphi(x), \quad -1 \le x \le 1, \tag{2}$$

$$u|_{AC_0} = \psi(x), \quad -1 \le x \le \frac{c-1}{2}, \tag{3}$$

$$\begin{gathered} a_0(x)(1+x)^\beta D_{c,x}^{1-\beta} u[\theta_0(x)] + b_0(x)(1-x)^\beta D_{x,1}^{1-\beta} u[\theta_1(x)] = c_0(x) u(x, 0) + \\ + d_0(x) \lim_{y \to -0} (-y)^{\beta_0} \frac{\partial u}{\partial y} + f_0(x), \quad c < x < 1, \end{gathered} \tag{4}$$

where $D_{c,x}^{1-\beta}$, $D_{x,1}^{1-\beta}$ are fractional differentiation operators of an order $1-\beta$; $\beta = \frac{m+2\beta_0}{2(m+2)}$; $\theta_0(x)$ and $\theta_1(x)$ represent the corresponding affixes of the intersection points of characteristics AC and BC with a characteristic originating from a point $M(x_0, 0)$, where $x_0 \in [c, 1]$:

$$\theta_0(x_0) = \frac{x_0 - 1}{2} - i\left(\frac{m+2}{4}(1 + x_0)\right)^{\frac{2}{m+2}},$$

$$\theta_1(x_0) = \frac{x_0 + 1}{2} - i\left(\frac{m+2}{4}(1 - x_0)\right)^{\frac{2}{m+2}}.$$

Given functions $\psi(x)$, $a_0(x)$, $b_0(x)$, $c_0(x)$, $d_0(x)$, $f_0(x)$ are continuously differentiable on the closure of their definition's domain, with the conditions:

$$a_0^2(x) + b_0^2(x) \ne 0, \quad c_0(x) \ge 0, \quad d(x) = a_0(x) + b_0(x) - d_0(x) > 0, \quad x \in (c, 1),$$

$$d(c) + \lambda\pi \, \mathrm{ctg} 3\alpha\pi \, (a_0(c) - b_0(c) \ne 0, \quad \lambda = \frac{\cos\beta\pi}{\pi(1 + \sin\beta\pi)}, \quad \alpha = \frac{1}{4}(1 - 2\beta)$$

and the function $\varphi(x)$ is represented in the form

$$\varphi(x) = (1 - x^2)^{2(1-\beta_0)} \tilde{\varphi}(x), \tag{5}$$

where $\tilde{\varphi}(x) \in C^1(\bar{I})$, $\psi(-1) = 0$.

Under the generalized solution of Problem A in the domain D, we refer to a function $u(x,y) \in C\left(\overline{D}\right) \cap C^2\left(D^+\right)$, which satisfies equation (1) in the domain D^+, while in the domain D^-, it is a generalized solution of a class R_1 [5, 6] and in the degeneration interval I it satisfies the following conjugation condition

$$\lim_{y\to -0}(-y)^{\beta_0}\frac{\partial u}{\partial y} = \lim_{y\to +0} y^{\beta_0}\frac{\partial u}{\partial y}\ , \quad x \in I\backslash \{c\}\,,$$

and these limits at $x = \pm 1,\ x = c$ may have singularities of an order no higher than $1 - 2\beta$, while satisfying conditions (2)–(4).

Note that Problem A is a generalization of the problem by F. Tricomi [7] and, for the limiting value $c = -1$ from the Problem A it reduces to the problem of A.M. Nakhushev [8], and for the limiting value $c = 1$, with the additional conditions $c_0(x) = 0,\ d_0(x) = 0$,

$$a_0(x)(1+x)^{\beta} D_{c,x}^{1-\beta}\ \psi(0)|_{\substack{x=1\\ c=1}}\ + b_0(x)(1-x)^{\beta} D_{x,1}^{1-\beta}\,\varphi(1)|_{x=1} = c_0(1),$$

it leads to the problem of F. Tricomi [7]. Problem A in a particular case, was studied in [9].

For degeneration on the boundary of the hyperbolic domain with singular coefficients, the generalized Tricomi problem with Goursat conditions was studied in [10]. Solvability issues and spectral properties of local and nonlocal problems for model mixed-type equations were investigated in [1, 5–8, 11–14].

2 The Extremum Principle and Uniqueness of the Solution to Problem A

Before proceeding to the proof of solution uniqueness for Problem A, we present, without proof, the extremum principle and the local properties of the solution to equation (1) in the domain D^+.

Let us consider the Gellerstedt equation with a singular coefficient

$$E(u) = y^m u_{xx} + u_{yy} + \frac{\beta_0}{y}\ u_y = 0,\ y > 0, \tag{6}$$

where $m > 0,\ -\frac{m}{2} < \beta_0 < 1$, in a finite simply connected domain Ω of the complex plane $z = x + iy$ which is limited by a simple Jordan arc Γ with endpoints $A(-1,0),\ B(1,0)$ lying on the half plane $y > 0$ and a segment AB of the axis $y = 0$.

Lemma 1. (The Extremum Principle) [1,14] Any regular solution $u(x,y)$ of equation (6), continuous in Ω, does not achieve its positive maximum or negative minimum at the interior points of the domain Ω.

Let a regular solution $u(x,y)$ of equation (6) achieve its positive maximum in the domain $\bar{\Omega}$ at the point $(b,\,0)$ along the axis $y = 0$.

We derive the inequality in the neighborhood of the point $(b,\,0)$ for the function

$$\nu(x) = \lim_{y\to +0} y^{\beta_0}\frac{\partial u}{\partial y},\ \ x \in (-1,1). \tag{7}$$

Let limit (7) exists at the point $(b,\,0)$ [13, 14].

Lemma 2. (Analog of the Zaremba-Giraud Principle) [1, 14] Let 1) the function $u(x,y) \in C(\bar{\Omega}) \cap C^2(\Omega)$ continuous in $\bar{\Omega}$ satisfy the inequality $E\left(u\right) \geq 0\ (\leq 0)$ and take its maximum positive value (minimum negative value) at some point $(b,\,0)$, $b \in (-1,1)$;

2) the value $u\left(x,y\right)$ on the curve Γ is less (greater), than at the point $(b,\,0)$. Then

$$\lim_{y\to +0} y^{\beta_0}\frac{\partial u}{\partial y} < 0\ \ (> 0)\,,$$

provided that this limit exists.

Suppose that the limit (7) does not exist at the point $(b,\,0)$ and in the neighborhood of this point, the partial derivatives of the solution $u(x,y)$ to equation (6) are allowed to have the following order of singularities

$$\left|y^{\beta_0}u_y(x,y)\right| \le O\left(\rho^{2\beta-1+\varepsilon_0}\right), \qquad |u_x(x,y)| \le O\left(\rho^{\varepsilon_0-1}\right), \tag{8}$$

where ε_0 is a sufficiently small positive constant, $\rho=(x-b)^2+\frac{4}{(m+2)^2}y^{m+2}$.

Lemma 3. [6] If the solution of equation (6) in the domain Ω achieves its positive maximum (negative minimum) at the point $(b,\,0)$ on the axis $y=0$ and the estimates (8) are valid at this point, then there exists a neighborhood $(b-r_1,\ b+r_1)$ of the point $(b,\,0)$ for which $\int\limits_{b-r_1}^{b+r_1}\nu(x)dx<0\ \ (>0)$.

By virtue of Darboux's formula [9]

$$u(x,y)=\gamma_1\int\limits_{-1}^{1}\tau\left[x+\tfrac{2t}{m+2}(-y)^{\frac{m+2}{2}}\right](1-t)^{\beta-1}(1+t)^{\beta-1}dt+$$
$$+\gamma_2(-y)^{1-\beta_0}\int\limits_{-1}^{1}\nu\left[x+\tfrac{2t}{m+2}(-y)^{\frac{m+2}{2}}\right](1-t)^{-\beta}(1+t)^{-\beta}dt,$$

where

$$\beta=\frac{m+2\beta_0}{2(m+2)},\ \gamma_1=\frac{\Gamma(2\beta)}{\Gamma^2(\beta)}2^{1-2\beta},\ \gamma_2=-\frac{\Gamma(2-2\beta)}{(1-\beta_0)\,\Gamma^2(1-\beta)}2^{2\beta-1},$$

giving a solution to the modified Cauchy problem

$$u(x,0)=\tau(x),\ \ x\in I,\ \ \lim_{y\to-0}(-y)^{\beta_0}\frac{\partial u}{\partial y}=v(x)\,,\quad x\in I,$$

from the boundary conditions (3) and (4), respectively, we have

$$\nu\left(x\right)=\gamma D_{-1,x}^{1-2\beta}\tau\left(x\right)+\Psi\left(x\right),\quad x\in\left(-1,\ c\right), \tag{9}$$

$$d(x)\nu(x)=\ \gamma a_0(x)\ D_{-1,x}^{1-2\beta}\ \tau(x)+\gamma b_0(x)\,D_{x,1}^{1-2\beta}\ \tau(x)\ +c_0(x)\tau(x)+f(x),\ x\in(c,\ 1), \tag{10}$$

where

$$\gamma=\frac{2\Gamma\left(2\beta\right)\Gamma\left(1-\beta\right)}{\Gamma\left(\beta\right)\Gamma\left(1-2\beta\right)}\left(\frac{m+2}{4}\right)^{2\beta},\ \Psi\left(x\right)=\frac{-\gamma\Gamma\left(\beta\right)}{\Gamma\left(2\beta\right)}D_{-1,x}^{1-\beta}\psi\left(\frac{x-1}{2}\right),\quad x\in(-1,\ c),$$

$$f(x)=\frac{2^{1-2\beta}}{\gamma_2\Gamma(1-\beta)(m+2)^{1-2\beta}}\left[f_0\left(x\right)+\frac{a_0\left(x\right)}{\Gamma(\beta)}\frac{d}{dx}\int\limits_{-1}^{c}\frac{\psi((t-1)/2)\,dt}{(x-t)^{1-\beta}}\right],\quad x\in(c,\ 1),$$

here $\Psi(x)\in C[-1,\ c]\cap C^1(-1,\ c),\ f(x)\in C[c,1]\cap C^1(c,\ 1)$.

Theorem 1. (Analogue of A.V. Bitsadze's Extremum Principle) [1] The solution $u(x,y)$ to Problem A under the conditions: $\psi\left(x\right)\equiv 0\ ,\ f_0(x)\equiv 0$,

$$a_0(x)\ge 0,\quad b_0(x)\ge 0,\quad c_0(x)\ge 0,\quad a_0(x)+b_0(x)\ge d_0(x), \tag{11}$$

achieves its positive maximum or negative minimum in the closed domain D^+ only at points on the arc σ_0.

Proof. Let $u(x, y)$ be a solution to Problem A, satisfying the conditions of Theorem 1. Clearly, by the extremum principle, the solution $u(x, y)$ in the domain D^+ cannot achieve its extreme. Assume the function $u(x, y)$ achieves its positive maximum in the closed domain $\bar{D}^+$ at the point $P(x_0, 0)$, $x_0 \in I\backslash\{c\}$, i.e. $\max\limits_{(x,y)\in\bar{D}^+} u(x, y) = u(x_0, 0) = \tau(x_0) > 0$. Using the fact that the fractional derivatives $D_{c,x}^{1-2\beta}\tau(x)$, $D_{x,1}^{1-2\beta}\tau(x)$ at the point of the positive maximum of the function $\tau(x)$ are strictly positive from (9) and (10) based on (11), we have $\nu(x_0) > 0$, it contradicts the known analogue of the Zaremba-Giraud principle, stating that at the point of positive maximum $\nu(x_0) < 0$ (Lemma 2) and hence it follows that $x_0 \notin I\backslash\{c\}$.

Now suppose that the solution $u(x, y)$ achieves its positive maximum (negative minimum) at the point $E(c, 0)$. Then, by Theorem 2 there exists $r_1 > 0$, such that for the interval the following holds

$$\int\limits_{c-r_1}^{c+r_1} \nu(x)dx < 0 \ \ (> 0). \tag{12}$$

On the other hand, using (9) and (10), for the specified r_1, we have

$$\int\limits_{c-r_1}^{c+r_1} \nu(x)dx = \int\limits_{c-r_1}^{c} \nu(x)dx + \int\limits_{c}^{c+r_1} \nu(x)dx > 0 \ \ (< 0). \tag{13}$$

Inequality (13) contradicts inequality (12), i.e., the function $u(x, y)$ does not achieve its positive maximum (negative minimum) at the point $E(c, 0)$.

Therefore, the function $u(x, y)$ achieves its positive maximum in the domain $\bar{D}^+$ at points on the curve σ_0.

It can also be shown that the function $u(x, y)$, which satisfies the conditions of Theorem 1, attains its negative minimum within the domain $\bar{D}^+$, including at points on the curve σ_0. Theorem 1 is proved.

From Theorem 1 follows

Corollary. Problem A under condition (11) has no more than one solution.

3 *Existence of a solution to problem A*

Theorem 2. Let the following conditions be hold:

$$\frac{\sin\alpha\pi\cos\beta\pi\,[b_0(c) - a_0(c)]\,[(2a_0(c) - d_0(c))\sin\beta\pi + 2b_0(c) - d_0(c)]}{[ch2\pi y + \cos 2\alpha\pi]\left\{[d(c)(1 + \sin\beta\pi)]^2 + [(b_0(c) - a_0(c))\cos\beta\pi]^2\right\}} < 1, \tag{14}$$

where $\alpha = \frac{1-2\beta}{4}$, $y = \frac{1+x}{1+t}$. Then there exists a solution to Problem A.

The proof of Theorem 2 will be carried out in several stages. From the known solution to the modified problem N (6), (2) and (7) [14], we obtain a functional relationship between the unknown functions $\tau(x)$ and $v(x)$ carried over to I from the domain D^+

$$\tau(x) = -k_1 \int\limits_{-1}^{1} \left[|x - t|^{-2\beta} - (1 - xt)^{-2\beta}\right] \nu(t)dt + \Phi(x)\,, \quad x \in \bar{I}, \tag{15}$$

where $k_1 = \left(\frac{4}{m+2}\right)^{2\beta} \frac{\Gamma^2(\beta)}{4\pi\Gamma(2\beta)}$,

$$\Phi(x) = 2k_1\left(\frac{m+2}{2}\right)^{2\beta}(1 - x^2)\int\limits_{-1}^{1} (1 - t^2)^{\beta-1/2}(1 - 2xt + x^2)^{-1-\beta}\varphi(t)dt.$$

From relationships (9), (10) and (14), excluding the function $\tau(x)$, and taking into account that the resulting equations have singularities at $x \in (-1,\ c)$ and $x \in (c, 1)$, we obtain

$$\begin{gathered}\nu\left(x\right)=-\lambda\int\limits_{-1}^{c}\left(\frac{1+s}{1+x}\right)^{1-2\beta}\left(\frac{1}{s-x}-\frac{1}{1-xs}\right)\nu\left(s\right)ds-\\ -\lambda\int\limits_{c}^{1}\left(\frac{1+s}{1+x}\right)^{1-2\beta}\left(\frac{1}{s-x}-\frac{1}{1-xs}\right)\nu\left(s\right)ds+\tilde{F}_{0}\left(x\right),\quad x\in\left(-1,c\right),\end{gathered}\tag{16}$$

$$\begin{gathered}d(x)v(x)=-\lambda a_0(x)\int\limits_{-1}^{c}\left(\frac{1+s}{1+x}\right)^{1-2\beta}\left(\frac{1}{s-x}-\frac{1}{1-xs}\right)v(s)ds-\\ -\lambda a_0(x)\int\limits_{c}^{1}\left(\frac{1+s}{1+x}\right)^{1-2\beta}\left(\frac{1}{s-x}-\frac{1}{1-xs}\right)v(s)ds+\lambda b_0(x)\int\limits_{-1}^{c}\left(\frac{1-s}{1-x}\right)^{1-2\beta}\left(\frac{1}{s-x}+\frac{1}{1-xs}\right)v(s)ds+\\ +\lambda b_0(x)\int\limits_{c}^{1}\left(\frac{1-s}{1-x}\right)^{1-2\beta}\left(\frac{1}{s-x}+\frac{1}{1-xs}\right)v(s)ds-\frac{k_1c_0(x)}{1+\sin\beta\pi}\int\limits_{-1}^{c}\left[|x-s|^{-2\beta}-(1-xs)^{-2\beta}\right]\nu(s)ds-\\ -\frac{k_1c_0(x)}{1+\sin\beta\pi}\int\limits_{c}^{1}\left[|x-s|^{-2\beta}-(1-xs)^{-2\beta}\right]\nu(s)ds+\tilde{F}_1(x),\quad x\in(c,1),\end{gathered}\tag{17}$$

where

$$\tilde{F_0}(x)=\frac{\gamma D_{-1,x}^{1-2\beta}\Phi\left(x\right)+\Psi\left(x\right)}{1+\sin\beta\pi},\quad \tilde{F_1}(x)=\frac{\gamma a_0(x)D_{-1,x}^{1-2\beta}\Phi\left(x\right)+\gamma b_0(x)D_{x,1}^{1-2\beta}\Phi\left(x\right)+f(x)+c_0(x)\Phi\left(x\right)}{1+\sin\beta\pi}.$$

Proceeding similarly to the approaches in works [6, 14], due to the conditions imposed on the given functions of the problem, particularly condition (5), in the case of a normal curve, it can be shown that

$$\begin{gathered}\tilde{F_1}\left(x\right)\in C\left(c,\ 1\right]\cap C^{(0,2\beta)}\left(c,\ 1\right),\\ \tilde{F_1}\left(x\right)\in C\left(c,\ 1\right]\cap C^{(0,2\beta)}\left(c,\ 1\right),\ \ \tilde{F_1}\left(x\right)=O\left(\left(x-c\right)^{2\beta-1}\right).\end{gathered}$$

Note that relations (16) and (17) hold for $x \in (-1,\ c)$ and $x \in (c, 1)$ respectively. To consider them in a single interval $I = (-1, 1)$ in (15) replace x with $ax - b$, and in (17), replace x with $bx + a$, where $a = (1 + c)/2$, $b = (1 - c)/2$,$a + b = 1$, $a - b = c$ and then perform the substitution of variables $s = at - b$ for integrals over the interval $(-1, c)$ and $s = bt + a$ for integrals over the interval $(c, 1)$, where $t \in [-1, 1]$. By isolating the characteristic part in the integrals with singular properties and performing some transformations, we have

$$\begin{gathered}\nu_0\left(t\right)+\lambda\int\limits_{-1}^{1}\left(\frac{1+t}{1+x}\right)^{1-2\beta}\left(\frac{1}{t-x}-\frac{a}{(ax-b)(at-b)}\right)\nu_0\left(t\right)dt=\\ =\lambda\int\limits_{-1}^{1}\frac{b\,v_1(t)dt}{bt-ax+1}+\mathrm{T}_1[\nu_1]+F_0(x),\ x\in(-1,1),\end{gathered}\tag{18}$$

$$\begin{gathered}D(x)\nu_1(x)+K(x)\int\limits_{-1}^{1}\left(\frac{1+t}{1+x}\right)^{1-2\beta}\left(\frac{1}{t-x}-\frac{b}{1-(bx+a)(bt+a)}\right)\nu_1\left(t\right)dt=\\ =\mu\int\limits_{-1}^{1}\frac{av_0(t)dt}{at-bx-1}+H_0[v_0]+H_1[v_1]+F_1(x),\ x\in(-1,1),\end{gathered}\tag{19}$$

where

$$\nu_0(x)=\nu(ax-b),\ \nu_1(x)=\nu(bx+a),\ A\left(x\right)=\lambda a_0\left(bx+a\right),\ B\left(x\right)=\lambda b_0\left(bx+a\right),$$

$$D\left(x\right)=d(bx+a),\ F_0\left(x\right)=\tilde{F_0}\left(ax-b\right),\ F_1\left(x\right)=\tilde{F_1}\left(bx+a\right),\ \lambda=\frac{\cos\beta\pi}{\pi(1+\sin\beta\pi)},$$

$$\mathrm{T}_1[\nu_1]=\lambda\int\limits_{-1}^{1}\left[\left(\frac{1+a+bt}{a\,(1+x)}\right)^{1-2\beta}-1\right]\frac{b\,v_1(t)dt}{bt-ax+1}+\lambda\int\limits_{-1}^{1}\left(\frac{1+a+bt}{a\,(1+x)}\right)^{1-2\beta}\frac{b\,v_1(t)dt}{1-(ax-b)(bt+a)},$$

$$K(x,t)=A(x)-B(x)\left(\frac{1-x}{1-t}\right)^{2\beta}\left(\frac{1+x}{1+t}\right)^{1-2\beta}\frac{1+bt+a}{1+bx+a},$$

$$K(x)=K(x,x)=A(x)-B(x),\;\;\mu=-K(-1),$$

$$H_1[v_1]=A\,(x)\int\limits_{-1}^{1}\left\{\left[\left(\tfrac{1+a+bt}{1+a+bx}\right)^{1-2\beta}-\left(\tfrac{1+t}{1+x}\right)^{1-2\beta}\right]\left(\tfrac{1}{t-x}-\tfrac{b}{1-(bx+a)(bt+a)}\right)\right\}\nu_1\,(t)\,dt-$$
$$-\tfrac{bk_1c_0(bx+a)}{1+\sin\beta\pi}\int\limits_{c}^{1}\left[(|b(x-t))|^{-2\beta}-(1-(bx+a)(bt+a))^{-2\beta}\right]v_1(t)dt-$$
$$-\int\limits_{-1}^{1}\left(\tfrac{1+t}{1+x}\right)^{1-2\beta}\left(\tfrac{1}{t-x}-\tfrac{b}{1-(bx+a)(bt+a)}\right)(K(x,t)-K(x,x))\nu_1(t)dt$$

are regular operators.

3.1 Regularization of the System of Singular Integral Equations of the Tricomi Problem with a Shift

Let us proceed to the regularization of the system of singular integral equations (18), (19). In equation (19) we will perform the following step-by-step:

- Temporarily consider the right-hand side of equation (19) as a known function from the class $L_\rho(-1,1)$, $p>1$ satisfying Hölder's condition. By regularizing equation (19) using the Carleman-Vekua method [15–17] in the class of functions H, where $(1+x)^{1-2\beta}\nu_1(x)$ is bounded on the left end and may be unbounded on the right end of the interval $\bar{I}$, we transform equation (19) with respect to the function $(1+x)^{1-2\beta}\nu_1(x)$.
- In the obtained equation, considering assumptions introduced above for the right-hand side of (19) and after some transformations, isolating the characteristic part of the equation, it is easy to establish that the kernel of the obtained integral equation when the condition $M(-1)\neq 0$ is fulfilled, where

$$M(x)=\mu\left(D^*(x)+\pi ctg3\alpha\pi K^*(x)\right),$$

$$M(-1)=\mu\;\left(D^*(-1)+\pi ctg3\alpha\pi\,K^*(-1)\right)=$$
$$=\tfrac{\mu(D(-1)+\pi ctg3\alpha\,K(-1))}{D^2(-1)+\pi^2K^2(-1)}=\tfrac{\mu[(a_0(c)+b_0(c)-d_0(c)-\lambda\mu\pi ctg3\alpha\pi\,)]}{(a_0(c)+b_0(c)-d_0(c))^2+\lambda\pi^2\mu^2},$$

$$\mu=\lambda\left[b_0(c)-a_0(c)\right],\;D^*(x)=\frac{D(x)}{D^2(x)+\pi^2K^2(x)},\;K^*(x)=\frac{K(x)}{D^2(x)+\pi^2K^2(x)}$$

for $t=1,\;x=-1$ has a first-order singularity [17,18], therefore, this operator is non-Fredholm. In the obtained equations, after some simple transformations, we have

$$\nu_1(x)=-\int\limits_{-1}^{1}n\left(\frac{1+x}{1+t}\right)\frac{\tilde{\nu}_0(t)}{1+t}dt+\bar{N}_0[\tilde{\nu}_0]+N_1[\nu_1]+F_2(x),\tag{20}$$

where $\tilde{\nu}_0(x)=\nu_0(-x)$

$$n(y)=\frac{M(-1)}{1+by/a},\tag{21}$$

$$N_0[\nu_0] = (M(x) - M(-1)) \int\limits_{-1}^{1} \frac{a\nu_0(t)dt}{at-bx-1} + \int\limits_{-1}^{1} K(x,t)\nu_0(t)dt +$$
$$+D^*(x)\tilde{H}_0[\nu_0] - K^*(x) \int\limits_{-1}^{1} \left(\frac{1+t}{1+x}\right)^{3\alpha} \left(\frac{1-t}{1-x}\right)^{2\alpha} \left(\frac{1-(bx+a)c}{1-(bt+a)c}\right)^{\alpha} \times$$
$$\times \frac{\omega(x)(D(x)+i\pi K(x))}{\omega(t)(D(t)+i\pi K(t))} \left(\frac{1}{t-x} - \frac{b}{1-(bx+a)(bt+a)}\right) H_0[\nu_0]dt -$$
$$-\mu\, aK^*(x) \int\limits_{-1}^{1} \nu_0(s)ds \int\limits_{-1}^{1} \left(\frac{1+t}{1+x}\right)^{3\alpha} \left(\frac{1-t}{1-x}\right)^{2\alpha} \left[\left(\frac{1-(bx+a)c}{1-(bt+a)c}\right)^{\alpha} \frac{\omega(x)(D(x)+i\pi K(x))}{\omega(t)(D(t)+i\pi K(t))} - 1\right] \times$$
$$\times \left(\frac{1}{t-x} - \frac{b}{1-(bx+a)(bt+a)}\right) \frac{dt}{as-bt-1}.$$

The operator $\bar{N}_0[\tilde{\nu}_0]$ is obtained from the operator N_0 by substituting $\nu_0(-x)$ with $\tilde{\nu}_0(x)$.

$$N_1[\nu_1] = D^*(x)H_1[\nu_1] - K^*(x) \int\limits_{-1}^{1} \left(\frac{1+t}{1+x}\right)^{3\alpha} \left(\frac{1-t}{1-x}\right)^{2\alpha} \left(\frac{1-(bx+a)c}{1-(bt+a)c}\right)^{\alpha} \times$$
$$\times \frac{\omega(x)(D(x)+i\pi K(x))}{\omega(t)(D(t)+i\pi K(t))} \left(\frac{1}{t-x} - \frac{b}{1-(bx+a)(bt+a)}\right) H_1[\nu_1]dt,$$

$N_1[\nu_1]$ is a regular operator,

$$F_2(x) = D^*(x)F_1(x) - K^*(x) \int\limits_{-1}^{1} \left(\frac{1+t}{1+x}\right)^{3\alpha} \left(\frac{1-t}{1-x}\right)^{2\alpha} \left(\frac{1-(bx+a)c}{1-(bt+a)c}\right)^{\alpha} \times$$
$$\times \frac{\omega(x)(D(x)+i\pi K(x))}{\omega(t)(D(t)+i\pi K(t))} \left(\frac{1}{t-x} - \frac{b}{1-(bx+a)(bt+a)}\right) F_1(t)dt, \;\; \omega(x) = \frac{1+a-ax}{bx+a}.$$

Now consider equation (18). Equation (18) will be considered as a singular integral equation with a shift in the non-summable part of the kernel relative to an unknown function. Here, proceeding similarly to the case of equation (19) and performing analogous calculations, we obtain

$$\tilde{\nu}_0(x) = -\int\limits_{-1}^{1} m\left(\frac{1+x}{1+s}\right) \frac{v_1(s)\,ds}{1+s} + T_2^*[v_1] + F_3(-x), \tag{22}$$

where

$$m(y) = A\frac{b^{1+\alpha}}{a^{\alpha}(b+ay)y^{\alpha}}, \quad A = \sin\alpha\pi/\pi, \tag{23}$$

$$T_2[v_1] = \lambda\frac{\cos\beta\pi}{2\pi} \int\limits_{-1}^{1} b\,\nu_1(s)ds \int\limits_{-1}^{1} \left(\frac{1+t}{1+x}\right)^{2a} \left(\frac{1-t}{1-x}\right)^{a} \left(\frac{1+a+bs}{a(1+t)}\right)^{4a} \left[\left(\frac{1-c(at-b)}{1-c(ax-b)}\right)^{a} - 1\right] \times$$
$$\times \left(\frac{1}{t-x} - \frac{a}{1-(ax-b)(at-b)}\right) dt + \left(\frac{\sin\alpha\pi}{\pi}\right)^2 T_1[v_1] + \frac{\sin\alpha\pi}{\pi} \int\limits_{-1}^{1} \left(\frac{b(1+s)}{a(1+x)}\right)^{\alpha} \frac{b\nu_1(s)ds}{1+(ax+b)(bs+a)},$$

$$T_1[v_1] = -\frac{\pi}{\sin\alpha\pi} \int\limits_{-1}^{1} \left(\frac{b(1+s)}{a(1-x)}\right)^{\alpha} \left[\left(\frac{1+a+bs)}{a(1+x)}\right)^{2\alpha} - 1\right] \left(\frac{1}{bs-ax+1} - \frac{1}{1-(ax-b)(bs+a)}\right) b\nu_1(s)ds +$$
$$+\int\limits_{-1}^{1} \left(\frac{1+a+bs}{a}\right)^{4\alpha} \frac{1}{(1+x)^{2\alpha}(1-x)^{\alpha}} \times$$
$$\times \left\{-\frac{2^{\alpha}B(1-2\alpha,\alpha)}{(1+x)^{2\alpha}} \left[F\left(1-2\alpha,-\alpha,1-\alpha;\frac{1-x}{2}\right) - \left(\frac{a(1+x)}{1+a+bs}\right)^{2\alpha} F\left(1-2\alpha,-\alpha,1-\alpha;-\frac{b}{a}\frac{1-s}{2}\right)\right] + \right.$$
$$\left. + a(bs+a)M_1(s) - a(ax-b)M_2(x)\right\} b_1 v(s)ds,$$

are regular operators,

$$M_1\left(s\right)=\int\limits_{-1}^{1}\frac{\left(1-t\right)^{\alpha}}{\left(1+t\right)^{2\alpha}}\frac{dt}{1-(bs+a)(at-b)}=\frac{2^{1-\alpha}B\left(1-2\alpha,\alpha\right)}{1-c(bs+a)}F\left(1+\alpha,1,2-\alpha;-\frac{2a(bs+a)}{1-c(bs+a)}\right),$$

$$M_2\left(x\right)=\int\limits_{-1}^{1}\frac{(1-t)^{\alpha}}{(1+t)^{2\alpha}}\frac{dt}{1-(ax-b)(at-b)}=\frac{2^{1-\alpha}B(1+\alpha,1-2\alpha)}{1-c(ax-b)}F\left(1-2\alpha,1,2-\alpha;-\frac{2a(ax-b)}{1-c(ax-b)}\right).$$

Here $B\left(\alpha,\beta\right)$ and $F\left(a,\,b,\,c,\,z\right)$ are beta and hypergeometric Gauss functions respectively.

$$F_3(x)=\frac{1+\sin\beta\pi}{2}F_0(x)-\frac{\cos\beta\pi}{2\pi}\int\limits_{-1}^{1}\left(\frac{1+t}{1+x}\right)^{2a}\left(\frac{1-t}{1-x}\right)^{a}\left(\frac{1-c(at-b)}{1-c(ax-b)}\right)^{a}\times$$
$$\times\left(\frac{1}{t-x}-\frac{a}{1-(ax-b)(at-b)}\right)F_0(t)dt,$$

where $\tilde{v}_0\left(x\right)=v_0\left(-x\right)$, and the operator $T_2^*\left[v_1\right]$ is obtained from operator $T_2\left[v_1\right]$ by substituting x with $-x$.

3.2 Derivation and Analysis of the Wiener-Hopf Integral Equation

Thus, equation (22) together with equation (20) form a system of integral equations with respect to unknown functions $\tilde{v}_0\left(x\right)$ and $v_1\left(x\right)$ with a singular feature in the kernel [19]. From equations (20) and (22), excluding the function $\tilde{v}_0\left(x\right)$ with respect to the function $v_1\left(x\right)$, we obtain the equation

$$v_1(x)=\int\limits_{-1}^{1}\frac{\Omega(x,t)v_1(t)dt}{1+t}+L\left[v_1\right]+F_4(x), \tag{24}$$

where $L\left[v_1\right]$ is a regular operator and

$$\Omega\left(x,t\right)=\int\limits_{-1}^{1}n\left(\frac{1+x}{1+s}\right)m\left(\frac{1+s}{1+t}\right)\frac{ds}{1+s}, \tag{25}$$

$$L\left[v_1\right]=-\int\limits_{-1}^{1}n\left(\frac{1+x}{1+t}\right)\frac{T_1^*\left[v_1\right]}{1+t}dt+\bar{N}_0\left[-\int\limits_{-1}^{1}m\left(\frac{1+x}{1+s}\right)\frac{v_1\left(s\right)ds}{1+s}+T_2^*\left[v_1\right]\right]+N_1\left[v_1\right],$$

$$F_4(x)=-\int\limits_{-1}^{1}n\left(\frac{1+x}{1+t}\right)\frac{F_3(-t)}{1+t}dt+\bar{N}_0[F_3(-x)]+F_2(x).$$

We evaluate the kernel $\Omega\left(x,t\right)$, for this in (25) making a substitution $\frac{1+s}{1+t}=r$, we have

$$\Omega\left(x,t\right)=\int\limits_{0}^{2/(1+t)}n\left(\frac{1+x}{r\left(1+t\right)}\right)m\left(r\right)\frac{dr}{r}=\Omega_1\left(x,t\right)-\Omega_2\left(x,t\right), \tag{26}$$

where

$$\Omega_1\left(x,t\right)=\int\limits_{0}^{\infty}n\left(\frac{1+x}{r\left(1+t\right)}\right)m\left(r\right)\frac{dr}{r},\quad \Omega_2\left(x,t\right)=\int\limits_{2/(1+t)}^{\infty}n\left(\frac{1+x}{r\left(1+t\right)}\right)m\left(r\right)\frac{dr}{r}.$$

Based on (21) and (23) it is easy to establish the estimate

$$|\Omega_2(x,t)| = \int\limits_{2/(1+t)}^{\infty} \left| n\left(\frac{1+x}{r(1+t)}\right) \right| |m(r)| \frac{dr}{r} \leq M(-1) A \left(\frac{b}{a}\right)^{1+\alpha} \left(\frac{1+t}{r}\right)^{\alpha}.$$

From this, it follows that $\frac{\Omega_2(x,t)}{1+t}$ is a regular kernel. Now, by direct computation, we have

$$\Omega_1(x,t) = \int\limits_0^{\infty} n\left(\frac{1+x}{r(1+t)}\right) m(r) \frac{dr}{r} = M(-1) \frac{y^\alpha - 1}{y^\alpha (y-1)}, \tag{27}$$

where $y = (1+x)/(1+t)$. Now, based on (26) and (27), we rewrite equation (24) in the form

$$v_1(x) = M(-1) \int\limits_{-1}^{1} \left(\frac{1+t}{1+x}\right)^{\alpha} \frac{\left(\frac{1+x}{1+t}\right)^{\alpha} - 1}{\frac{1+x}{1+t} - 1} \frac{v_1(t)\,dt}{1+t} + L_1[v_1] + F_4(x), \tag{28}$$

where $L_1[v_1] = L[v_1] - \int\limits_{-1}^{1} \frac{\Omega_2(x,t) v_1(t) dt}{1+t}$ is a regular operator.

Now, introducing the notation in (28) $\rho(z) = e^{(\alpha-1)z/2} \nu_1(2e^{-z} - 1)$, after some transformations, we obtain

$$\rho(z) = M(-1) \int\limits_0^{\infty} \frac{sh(\alpha(z-s)/2)\rho(s)ds}{sh((z-s)/2)} + L_3[\rho] + F_5(z), \tag{29}$$

where $L_3[\rho] = L_2\left[e^{(1-\alpha)z/2}\rho(z)\right]$, $F_5(z) = e^{(\alpha-1)z/2} F_4(2e^{-z} - 1)$. Equation (29) is an integral Wiener-Hopf equation [20]. Under the condition

$$\frac{M(-1)\pi \sin \alpha\pi}{ch 2y\pi + \cos 2\alpha\pi} < 1 \tag{30}$$

it is easy to calculate that the index of equation (29) is equal to 0. Therefore, equation (29) is uniquely reduced to a Fredholm integral equation of the second kind, and the unique solvability of this equation follows from the uniqueness of the solution to Problem A. From condition (30) due to the entered notations, by straight calculations we obtain (14).

Thus, Theorem 2 is proved.

Conclusion

The paper investigates the existence and uniqueness of a solution to a combined problem with local and nonlocal conditions for one class of mixed elliptic-hyperbolic type equations with singular coefficients.

The studied problem differs from known problems in that the values of the sought function are specified on one part of the characteristic boundary, while, nonlocal conditions are imposed on the other part. These nonlocal conditions relate the values of the sought function on one part of the characteristic boundary pointwise to the values on the characteristic boundary of another family using the fractional differentiation operator in the sense of Riemann-Liouville. At the same time, a part of the boundary of the hyperbolic region of the domain is freed from boundary conditions.

The uniqueness of the solution to formulated problem is proved using an analogue of extremum principle by A.V. Bitsadze for a mixed type equation with singular coefficients.

The proof of the existence of a solution is reduced to solving a system of singular Tricomi integral equations with a shift in the non-summable part of the kernel and a non-Fredholm operator with an isolated first-order singularity in the kernel of the operator. By the Carleman-Vekua regularization method, the obtained singular integral equations with a non-zero operator on the right are reduced to the Wiener-Hopf integral equation. It is proved that the index of the Wiener-Hopf integral equation is zero. Consequently, the Wiener-Hopf equation is uniquely reduced to the Fredholm integral equation of the second kind, the unambiguous solvability of which follows from the uniqueness of the solution of the problem A.

An algorithm has been developed for solving non-standard singular integral Tricomi equations with a shift in the non-summable part of the kernel and a non-Fredholm operator with an isolated first-order singularity in the operator's kernel.

Thus, the issues of the unique solvability of a combined problem with local and nonlocal conditions for a certain class of mixed-type equations with singular coefficients have been formulated and studied. It has been established that the well-posedness of the combined problem, defined by local and nonlocal conditions on a single characteristic boundary, significantly depends on the ratio of the coefficients of the nonlocal conditions at the junction point of the local and nonlocal conditions, which lies on the degeneration line of the equation.

In conclusion, we note that the developed methods for studying non-standard singular integral equations can be applied to a broader class of partial differential equations with singular coefficients, including for other values of the parameter β_0 in equation (1).

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Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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