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Geometric properties of the Minkowski operator

M.Sh. Mamatov[1], J.T. Nuritdinov[2,3,*], Kh.Sh. Turakulov[3], S.M. Mamazhonov[2]

[1] *National University of Uzbekistan, Tashkent, Uzbekistan;*
[2] *Kokand University, Kokand, Uzbekistan;*
[3] *Kokand State Pedagogical Institute, Kokand, Uzbekistan*
(E-mail: mamatovmsh@mail.ru, nuritdinovjt@gmail.com, hamidtsh87@gmail.com, sanjarbekmamajonov@gmail.com)

This article is about Minkowski difference of sets, which is one of the Minkowski operators. The necessary and sufficient conditions for the existence of the Minkowski difference of given regular polygons in the plane were derived. The method of finding the Minkowski difference of given regular tetrahedrons in the Euclidean space $\mathbb{R}^3$ was explained. At the end of the article, the obtained results were summarized and a geometric method for finding the Minkowski difference of the convex set M and compact set N given in $\mathbb{R}^n$ was shown. The theory of foliations was applied to find the Minkowski difference of sets. New geometric concepts such as "dense embedding" and "completely dense embedding" were introduced. An important geometric property of the Minkowski operator was introduced and proved as a theorem.

Keywords: Minkowski sum, Minkowski difference, orthogonal projection, foliation, dense embedding in a foliation.

2020 Mathematics Subject Classification: 52B11, 53C12.

Introduction

Not all operations on sets may have a geometric meaning. For sets with elements of any kind, we can perform operations such as union, intersection, and difference.

So, the above operations do not necessarily mean geometrically in some cases. The Minkowski sum and difference on the sets were introduced precisely for the purpose of solving geometric problems, and these operations depend on the nature of the elements that make up the sets. That is why Minkowski operations are not performed for the sets given in the above example.

Definitions and some properties of Minkowski operators are presented in works [1, 2]. Among the known scientific works, the Minkowski difference was first used in [3] to solve the problem of pursuit in differential games under the name "geometric difference". Later, in other works such as [4, 5], various properties of this "geometric difference" were studied, and with their help, the conditions for solving the problem of chasing were eased. Also, many geometric properties of Minkowski difference and sum are presented in [6–9]. To date, several scientific researches have been conducted to find algorithms for calculating the Minkowski sum. Y. Yan, D.S. Chirikjian, A. Baram, E. Fogel, D. Halperin, M. Hemmer, S. Morr, O. Eduard, M. Sharir, A. Kaul, M.A. O'Connor, V. Srinivasan, S. Das, S.D. Ranjan, S. Sarvottamananda, W. Cox, L. While, M. Reynolds and other scientists obtained fundamental results on the calculation of the Minkowski sum of polygons in the plane [10–15].

Finding the Minkowski difference of sets is more complicated than finding their Minkowski sum. There are also not many works on finding the Minkowski difference of given sets [16, 17]. Several properties and calculation methods of the Minkowski difference are presented in the works of specialists such as L.A. Tuan, L. Yang, H. Zhang, J.B. Jeannin, N. Ozay, Y.T. Feng, Y. Tan, Y. Zhang, W. Qilin [18–21]. However, so far, the conditions for the Minkowski difference of an arbitrary given set to be empty or non-empty have not been obtained.

*Corresponding author. *E-mail: nuritdinovjt@gmail.com*
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The theory of foliation is one of the developing branches of modern geometry, and it has applications to many areas of geometry [22–28]. In summarizing the obtained results in this article, the foliation theory was also used. Through new geometrical concepts, an efficient method for finding the Minkowski difference of given compact sets in $\mathbb{R}^n$ has been created.

This article presents important geometric properties of the Minkowski operator and geometric ways to find the Minkowski difference of some sets using these properties. In this article, we solved the following problems:

1) a new geometric method and exact formula for finding the Minkowski difference of given regular polygons in the plane $\mathbb{R}^2$;
2) finding the Minkowski difference of two given regular tetrahedrons in the Euclidean space $\mathbb{R}^3$;
3) a new geometric property for finding the Minkowski difference of arbitrary sets;
4) applying foliation theory to finding the Minkowski difference.

1 Research Methodology

Definition 1. Let the sets A and B be non-empty sets of the n dimensional Euclidean space $\mathbb{R}^n$. Their Minkowski sum is the set of points formed by adding each point of set A to each point of set B, i.e.

$$A + B = \{c \in \mathbb{R}^n : c = a + b,\ a \in A,\ b \in B\}.$$

Using this introduced operation, the Minkowski difference of two sets is defined as follows.

Definition 2. Let the sets A and B be non-empty sets of the n dimensional Euclidean space $\mathbb{R}^n$. The following set is called their Minkowski difference:

$$D = A \overset{*}{-} B = \{d \in \mathbb{R}^n : d + B \subset A\}.$$

Definition 3. The Minkowski operators of a multi-valued mapping $G : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ are the operators $A_G : 2^{\mathbb{R}^n} \to 2^{\mathbb{R}^n}$ and $B_G : 2^{\mathbb{R}^n} \to 2^{\mathbb{R}^n}$ given by the formulas

$$A_G S = \bigcup_{x \in S} (x + G(x)),$$

$$B_G S = \mathbb{R}^n \backslash (A_G(\mathbb{R}^n \backslash S)),$$

for any set S.

If, in particular, we take the multi-valued mapping G to be constant $G(x) = G_0$ for all $x \in S$, the Minkowski operators correspond to Minkowski sum and difference, respectively:

$$A_G S = S + G_0, \qquad B_G S = S \overset{*}{-} (-G_0).$$

Minkowski sum and Minkowski difference have been used to obtain sufficient conditions for ending the game in differential games [3–5]. Today, the approximate calculation of Minkowski sum and difference takes an important place in solving practical problems with the help of differential games. At the same time, it is one of the most important issues to evaluate the Minkowski difference from below and above in theoretical studies.

Minkowski operator were first applied to the study of differential games in the works of L.S. Pontryagin [3, 4]. He called this operator geometric difference and marked it as ($\overset{*}{-}$). In [17], a necessary and sufficient condition for the Minkowski difference of two squares to be non-empty was obtained. Formulas for calculating Minkowski differences are also presented in these works.

2 Minkowski Difference of Regular Polygons

On the Euclidean plane $\mathbb{R}^2$, let regular n-sided polygons P^A and P^B be given by vertices $A_1, A_2, ..., A_n$ and $B_1, B_2, ..., B_n$, respectively. Using these points, we can express vectors corresponding to the sides of regular polygons P^A and P^B:

$$\overrightarrow{A_1A_2} = \vec{a}_1, \overrightarrow{A_2A_3} = \vec{a}_2, ..., \overrightarrow{A_nA_1} = \vec{a}_n,$$
$$\overrightarrow{B_1B_2} = \vec{b}_1, \overrightarrow{B_2B_3} = \vec{b}_2, ..., \overrightarrow{B_nB_1} = \vec{b}_n.$$

Theorem 1. In order for the Minkowski difference $P^A \overset{*}{-} P^B$ of regular polygons P^A and P^B given on the Euclidean plane $\mathbb{R}^2$ to be non-empty, the following relation is necessary and sufficient:

$$\frac{|\vec{a}_1|}{2\tan\frac{\pi}{n}} \geq \frac{\left|\vec{b}_1\right|}{2\sin\frac{\pi}{n}} \cdot \cos\left(\frac{\pi}{n} - \alpha_i\right). \tag{1}$$

Here $\alpha_i = \min\limits_{i=\overline{1,n}} \left\{ \arccos\left(\frac{\langle \vec{a}_1, \vec{b}_i \rangle}{|\vec{a}_1| |\vec{b}_i|} \right) \right\}$ is the smallest angle between vectors $\vec{a}_1$ and $\vec{b}_i$, $i = \overline{1,n}$.

Proof. Since P^A is a regular polygon, the centers of the circumcircle and incircles of this polygon are at the same point. Let's denote this point as O^A. In the same way, we mark the center of circumcircle and incircles of the polygon P^B as O^B. $P^A \overset{*}{-} P^B \neq \varnothing$ means that the set P^B can be nested inside the set P^A. For this, we move the set P^B parallel until the point O^B falls on the point O^A, that is, we move the set P^B parallel along the vector $\overrightarrow{O^BO^A}$. There can be two cases.

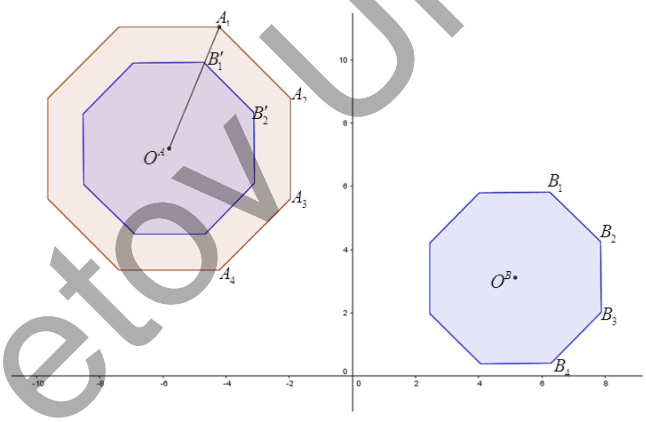

Figure 1. The Minkowski difference of regular polygons with parallel sides

In the first case, it can be $\vec{a}_1 \uparrow\uparrow \vec{b}_1$, $\vec{a}_2 \uparrow\uparrow \vec{b}_2, ..., \vec{a}_n \uparrow\uparrow \vec{b}_n$ (Fig. 1). In such a situation, the images of points $B_1', B_2', ..., B_n'$ formed by parallel displacement of points $B_1, B_2, ..., B_n$ along vector $\overrightarrow{O^BO^A}$ will be located on straight lines O^AA_i, $i = \overline{1,n}$. In order for the points $B_1', B_2', ..., B_n'$ to belong to the regular polygon P^A (here, the points inside the polygon are also considered to belong to the polygon), it is necessary and sufficient to satisfy the relation

$$\left|O^AA_i\right| \geq \left|O^AB_i'\right|, \; i = \overline{1,n}. \tag{2}$$

The length of the segments O^AB_i', $i = \overline{1,n}$ is equal to the radius of the circumcircle of the P^B polygon, i.e

$$\left|O^AB_i'\right| = \frac{\left|\vec{b}_1\right|}{2\sin\frac{\pi}{n}}, \; i = \overline{1,n}. \tag{3}$$

The length of the segment $O^A A_i$, $i = \overline{1,n}$ is equal to the radius of the circumcircle of polygon P^A, but if we express it by the radius of the incircle of the polygon P^A, it will be in the form of

$$\left|O^A A_i'\right| = \frac{|\vec{a}_1|}{2\tan\frac{\pi}{n}} \cdot \frac{1}{\cos\frac{\pi}{n}},\ i = \overline{1,n}. \tag{4}$$

Since $\vec{a}_1 \uparrow\uparrow \vec{b}_1$, follows that $\alpha_i = \min\limits_{i=\overline{1,n}} \left\{ \arccos\left(\frac{\langle \vec{a}_1, \vec{b}_i\rangle}{|\vec{a}_1||\vec{b}_i|}\right)\right\} = 0$. From this we can write equation(4) as

$$\left|O^A A_i'\right| = \frac{|\vec{a}_1|}{2\tan\frac{\pi}{n}} \cdot \frac{1}{\cos\left(\frac{\pi}{n} - \alpha_i\right)},\ i = \overline{1,n}. \tag{5}$$

If we put equations (5) and (3) to relation (2), condition (1) is obtained.

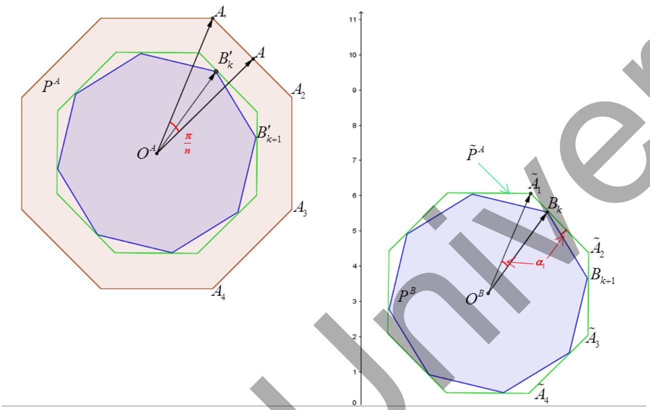

Figure 2. The Minkowski difference of regular polygons with corresponding sides not parallel

In the second case, relations $\vec{a}_i \nparallel \vec{b}_j;\ i, j = \overline{1,n}$ are appropriate, that is, none of the sides of the polygons P^A and P^B are parallel to each other (Fig. 2). In studying this situation, we must first determine the smallest angle between the vectors $\vec{a}_1$ and $\vec{b}_i$, $i = \overline{1,4}$ and we denote this angle as α_i and calculate it as follows

$$\alpha_i = \min_{i=\overline{1,n}} \left\{ \arccos\left(\frac{\left\langle \vec{a}_1, \vec{b}_i\right\rangle}{|\vec{a}_1|\left|\vec{b}_i\right|}\right)\right\}.$$

Suppose this angle is the angle between the vector $\overrightarrow{A_1A_2}$ and the vector $\overrightarrow{B_kB_{k+1}}$, $k = \overline{1,n}(B_{n+1} = B_1)$. In that case, we construct the vector $\overrightarrow{O^A A}$, whose beginning is at the point O^A, and whose end is at the point A, the middle of the segment A_1A_2. This vector forms an angle $\frac{\pi}{n} - \alpha_i$, $i = \overline{1,n}$ with the vector $\overrightarrow{O^A B_k'}$, whose beginning is at point O^A and whose end is at point B_k'. In order for the points to belong to the regular polygon P^A, it is necessary and sufficient that the length of the orthogonal projection of the vector $\overrightarrow{O^A B_k'}$ onto the vector $\overrightarrow{O^A A}$ is not greater than the length of the vector $\overrightarrow{O^A A}$ (Fig. 3), i.e

$$\left|\overrightarrow{O^A A}\right| \geq \left|\overrightarrow{O^A B_k'}\right| \cdot \cos\left(\frac{\pi}{n} - \alpha_i\right). \tag{6}$$

The length of the vector $\overrightarrow{O^A A}$ is equal to the radius of the incircle of the regular polygon P^A,

$$\left|\overrightarrow{O^A A}\right| = \frac{|\vec{a}_1|}{2\tan\frac{\pi}{n}}. \tag{7}$$

The length of the vector $\overrightarrow{O^A B_k'}$ is equal to the radius of the circumcircle of the regular polygon P^B,

$$\left|\overrightarrow{O^A B'_k}\right| = \frac{\left|\vec{b}_1\right|}{2\sin\frac{\pi}{n}}. \tag{8}$$

If we put equations (8) and (7) to relation (6), condition (1) is obtained. This completes the proof.

3 Minkowski Difference of Regular Tetrahedrons

We know that a polyhedron is called a regular polyhedron, if all its faces are congruent regular polygons and all dihedral angles are also congruent. Since at least three edges of the polyhedron pass through each vertex, the sum of all plane angles at that end is less than 2π. A regular tetrahedron is a pyramid with all faces consisting of equilateral triangles, and it has 4 vertices, 4 faces and 6 edges. The spheres drawn inside and outside a regular tetrahedron have their centers at the same point. To define a tetrahedron in a three-dimensional Euclidean space, it is enough to give the coordinates of its vertices.

Let's say that the points corresponding to the vertices of the tetrahedron T^A are given by $A_i = \left\{\alpha_i^1, \alpha_i^2, \alpha_i^3\right\}$, $i = \overline{1,4}$ coordinates, and the points corresponding to the vertices of the tetrahedron T^B are given by $B_i = \left\{\beta_i^1, \beta_i^2, \beta_i^3\right\}$, $i = \overline{1,4}$ coordinates. Then the coordinate of the center of the circumsphere and insphere of the tetrahedron T^A is in the form

$$O^A = \{a_1, a_2, a_3\}, \quad a_j = \frac{1}{4}\sum_{i=1}^{4}\alpha_i^j, \quad j = \overline{1,3}.$$

Similarly, the coordinate of the center of the circumsphere and insphere of the tetrahedron T^B is also in the form

$$O^B = \{b_1, b_2, b_3\}, \quad b_j = \frac{1}{4}\sum_{i=1}^{4}\beta_i^j, \quad j = \overline{1,3}.$$

We denote the vectors starting at point O^A and ending at the points where the medians of the faces of the tetrahedron T^A intersect as $\vec{r}_i^A$, $i = \overline{1,4}$ and the coordinates of these vectors are in the form

$$\vec{r}_i^A = \frac{1}{3}\{a_1 - \alpha_i^1, a_2 - \alpha_i^2, a_3 - \alpha_i^3\}, \quad i = \overline{1,4}.$$

The lengths of these vectors are the same and equal to the radius of the insphere of the tetrahedron T^A, i.e.

$$\left|\vec{r}_i^A\right| = \frac{\sqrt{6}}{12}\left|\vec{a}_1\right|, \ \ i = \overline{1,4}.$$

Where $\vec{a}_1 = \overrightarrow{A_1A_2}$ and represents the vector corresponding to the edge of the tetrahedron T^A.

Let's denote the vectors starting at O^B and ending at points B_i, $i = \overline{1,4}$ as $\vec{R}_i^B$, $i = \overline{1,4}$ respectively, and the coordinates of these vectors are in the form

$$\vec{R}_i^B = -\{b_1 - \beta_i^1, b_2 - \beta_i^2, b_3 - \beta_i^3\}, \quad i = \overline{1,4}.$$

The lengths of these vectors are equal to the radius of the circumsphere of the tetrahedron T^B:

$$\left|\vec{R}_i^B\right| = \frac{\sqrt{6}}{4}\left|\vec{b}_1\right|, \ \ i = \overline{1,4},$$

where $\vec{b}_1 = \overrightarrow{B_1B_2}$ and represents the vector corresponding to the edge of the tetrahedron T^B. By α we denote the smallest angle between $\vec{r}_i^A$, $i = \overline{1,4}$ vectors and $\vec{R}_i^B$, $i = \overline{1,4}$ vectors.

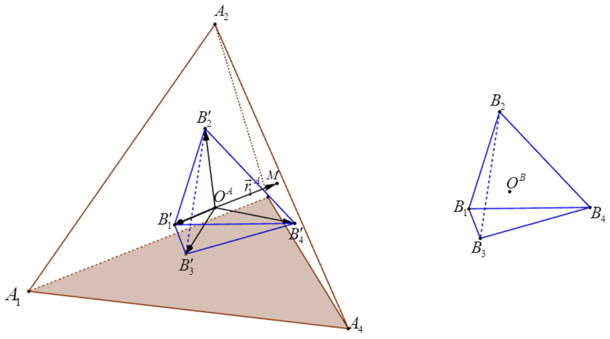

Figure 3. Minkowski difference of tetrahedrons

Theorem 2. In order for the Minkowski difference $T^A \overset{*}{-} T^B$ of regular tetrahedrons T^A and T^B given in Euclidean space $\mathbb{R}^3$ to be non-empty, the following relation is necessary and sufficient:

$$|\vec{a}_1| \geq 3|\vec{b}_1| \cos\alpha. \tag{9}$$

Proof. To calculate the difference $T^A \overset{*}{-} T^B$, we move the tetrahedron T^B parallel to the vector $\overrightarrow{O^B O^A}$. Let us denote the images of points $B_i = \{\beta_i^1, \beta_i^2, \beta_i^3\}$, $i = \overline{1,4}$ in this parallel displacement as B_i', $i = \overline{1,4}$ respectively (Fig. 3). In order for the difference $T^A \overset{*}{-} T^B$ not to be empty, these points must lie inside the tetrahedron T^A or at most on its faces.

Let the points B_i', $i = \overline{1,4}$ lie on the faces of the tetrahedron T^A. The radius of the insphere of the tetrahedron T^A drawn from the point O^A to the face formed by the vertices $A_2 = \{\alpha_2^1, \alpha_2^2, \alpha_2^3\}$, $A_3 = \{\alpha_3^1, \alpha_3^2, \alpha_3^3\}$, $A_4 = \{\alpha_4^1, \alpha_4^2, \alpha_4^3\}$ of the tetrahedron T^A falls on the point where the medians of the triangle $\triangle\, A_2A_3A_4$ intersect and is perpendicular to this face. Let's designate the vector corresponding to this radius as $\vec{r}_1^{\,A}$, its coordinate will be in the form

$$\vec{r}_1^{\,A} = \frac{1}{3}\{a_1 - \alpha_1^1, a_2 - \alpha_1^2, a_3 - \alpha_1^3\}.$$

The length of the orthogonal projection of all vectors starting from O^A and ending at points lying on the face $A_2A_3A_4$ onto the vector $\vec{r}_1^{\,A}$ is equal to $\left|\vec{r}_1^{\,A}\right|$. Hence, if any point B_i', $i = \overline{1,4}$ belongs to face $A_2A_3A_4$, equality

$$proj_{\vec{r}_1^{\,A}} \overrightarrow{O^A B_i'} = \left|\vec{r}_1^{\,A}\right|, \quad i = \overline{1,4} \tag{10}$$

holds. Points B_i', $i = \overline{1,4}$ can also be located inside the tetrahedron T^A, so we generalize equation (10) and write it in the form

$$proj_{\vec{r}_1^{\,A}} \overrightarrow{O^A B_i'} \leq \left|\vec{r}_1^{\,A}\right|, \quad i = \overline{1,4}. \tag{11}$$

We can write the same relation for other faces of the tetrahedron T^A:

$$\begin{aligned} proj_{\vec{r}_2^{\,A}} \overrightarrow{O^A B_i'} &\leq \left|\vec{r}_2^{\,A}\right|, \quad i = \overline{1,4}, \\ proj_{\vec{r}_3^{\,A}} \overrightarrow{O^A B_i'} &\leq \left|\vec{r}_3^{\,A}\right|, \quad i = \overline{1,4}, \\ proj_{\vec{r}_4^{\,A}} \overrightarrow{O^A B_i'} &\leq \left|\vec{r}_4^{\,A}\right|, \quad i = \overline{1,4}. \end{aligned} \tag{12}$$

Summarizing relations (11) and (12), we can write as follows

$$proj_{\vec{r}_j^{\,A}} \overrightarrow{O^A B_i'} \leq \left|\vec{r}_j^{\,A}\right|, \quad i = \overline{1,4}, \quad j = \overline{1,4}. \tag{13}$$

We know that the lengths of vectors $\overrightarrow{O^A B_i'}$ are the same and equal to the radius of the circumsphere of the tetrahedron T^B. $\vec{r}_j^{\,A}$ vectors have the same length and are equal to the radius of the insphere of the tetrahedron T. Based on these, we write relation (13) in form (9), were α is the smallest of the angles between vectors $\vec{r}_j^{\,A}, j=\overline{1,4}$ and vectors $\overrightarrow{O^A B_i'}$, $i=\overline{1,4}$. Because the cosine of a smaller angle is greater than the cosine of a larger angle. This means that if relation (9) holds for the smallest angle, it holds for the rest of the angles as well. Therefore, (9) is considered a necessary and sufficient condition for the relation $T^A \overset{*}{-} T^B$ not to be empty.

During the proof of the theorem, we derived the algorithm for finding the Minkowski difference of two tetrahedrons given by their vertices in the Euclidean space $\mathbb{R}^3$. According to it, the following should be done in sequence:

1) Let's say that the points corresponding to the vertices of the tetrahedron T^A are given by $A_i = \left\{\alpha_i^1, \alpha_i^2, \alpha_i^3\right\}$, $i=\overline{1,4}$ coordinates, and the points corresponding to the vertices of the tetrahedron T^B are given by $B_i = \left\{\beta_i^1, \beta_i^2, \beta_i^3\right\}$, $i=\overline{1,4}$ coordinates. First of all we determine the Minkowski difference of tetrahedrons T^A and T^B is not empty. For this we check relation (10) according to the above theorem. The numbers $|\vec{a}_1|$ and $\left|\vec{b}_1\right|$ in relation (10) are lengths of vectors $\overrightarrow{A_1 A_2}$ and $\overrightarrow{B_1 B_2}$ respectively, and they are founded by following equality:

$$|\vec{a}_1| = \left|\overrightarrow{A_1 A_2}\right| \sqrt{\left(\alpha_2^1-\alpha_1^1\right)^2+\left(\alpha_2^2-\alpha_1^2\right)^2+\left(\alpha_2^3-\alpha_1^3\right)^2},$$

$$\left|\vec{b}_1\right| = \left|\overrightarrow{B_1 B_2}\right| \sqrt{\left(\beta_2^1-\beta_1^1\right)^2+\left(\beta_2^2-\beta_1^2\right)^2+\left(\beta_2^3-\beta_1^3\right)^2}.$$

2) Suppose that as a result of the check, equality $|\vec{a}_1| = 3|\vec{b}_1|\cos\alpha$ is satisfied. This means that difference $T^A \overset{*}{-} T^B$ consists only one point and this point is in the form $O^A - O^B$.

3) Suppose that as a result of the check, relation $|\vec{a}_1| > 3|\vec{b}_1|\cos\alpha$ is satisfied. In this case to calculate the difference $T^A \overset{*}{-} T^B$, we construct a tetrahedron $\tilde{T}^B$ such that, the edges are parallel to the edges of the tetrahedron T^A, and the vertices of the tetrahedron T^B lie on the faces of this tetrahedron. Such a tetrahedron is unique, the center of the insphere of this is at point O^B and the radius is equal to $\frac{\sqrt{6}}{4}\left|\vec{b}_1\right| \cdot \cos\alpha$. If we designate the vertices of the tetrahedron $\tilde{T}^B$ as $\tilde{B}_i$, $i=\overline{1,4}$ the directions of the vectors $\overrightarrow{O^B \tilde{B}_i}$, $i=\overline{1,4}$ are the same as the directions of the vectors $\overrightarrow{O^A A_i}$, $i=\overline{1,4}$, and their lengths are equal to the radius of the circumsphere of the tetrahedron $\tilde{T}^B$. Since the radius of the circumsphere of the regular tetrahedron is three times longer than the radius of its insphere, the lengths of the vectors $\overrightarrow{O^B \tilde{B}_i}$, $i=\overline{1,4}$ are equal to the number $\frac{3\sqrt{6}}{4}\left|\vec{b}_1\right| \cdot \cos\alpha$. The coordinates of the vectors $\overrightarrow{O^A A_i}$, $i=\overline{1,4}$ are as follows:

$$\overrightarrow{O^A A_i} = \left\{\alpha_i^1 - a_1, \alpha_i^2 - a_2, \alpha_i^3 - a_3\right\}, \quad i=\overline{1,4}.$$

From these we can find the coordinates of vectors $\overrightarrow{O^B \tilde{B}_i}, i=\overline{1,4}$:

$$\overrightarrow{O^B \tilde{B}_i} = M \cdot \overrightarrow{O^A A_i}, \quad i=\overline{1,4}, \quad M = \frac{3\left|\vec{b}_1\right| \cdot \cos\alpha}{|\vec{a}_1|}.$$

Using these vectors, we find the points $\tilde{B}_i$, $i=\overline{1,4}$ the vertices of the tetrahedron $\tilde{T}^B$:

$$\tilde{B}_i = M\left\{\alpha_i^1 - a_1 + b_1, \alpha_i^2 - a_2 + b_2, \alpha_i^3 - a_3 + b_3\right\}, \quad i=\overline{1,4}.$$

4) We find the vertices of the tetrahedron formed as a result of difference $T^A \overset{*}{-} T^B$ by subtracting the corresponding coordinates of the points found from the coordinates of points $A_i = \left\{\alpha_i^1, \alpha_i^2, \alpha_i^3\right\}$, $i=\overline{1,4}$:

$$A_i - \tilde{B}_i.$$

4 Generalization of the results

In this section, we summarize the results obtained above [23–26]. Let, we are given convex set M and compact set N in $\mathbb{R}^n$. We denote by $\partial M_0 = L_0$ the boundary of a convex compact set $M = M_0$. $M_\alpha,\ \partial M_\alpha = L_\alpha,\ \alpha \in A$ are chosen in such a way that: 1) $\bigcup\limits_{\alpha\in A} L_\alpha = M$; 2) $M_\alpha \overset{*}{-} M_\beta \neq \varnothing$ for arbitrary $\alpha, \beta \in A$ and $\alpha \leq \beta$. Based on I. Tamura [23], we call $F = \ \{L_\alpha : L_\alpha = \partial M_\alpha,\ \alpha \in A\}$ a foliation and $L_\alpha,\ \alpha \in A$ a leaves of the foliation. Let $\partial \left(M_\alpha \overset{*}{-} M_\beta\right) \in F$ be for arbitrary $\alpha, \beta \in A$.

Theorem 3. If the condition $N \subset M_\alpha$ is satisfied for the convex compact sets M, M_α and compact set N given in $\mathbb{R}^n$, the equality

$$M \overset{*}{-} N = \left(M \overset{*}{-} M_\alpha\right) + \left(M_\alpha \overset{*}{-} N\right)$$

holds.

Proof. Let be $z \in M \overset{*}{-} N$, then we show that there are elements $z_1 \in M \overset{*}{-} M_\alpha$ and $z_2 \in M_\alpha \overset{*}{-} N$ such that $z = z_1 + z_2$. We can write the relation $z + N \subset M$ using the definition of the Minkowski difference for the element $z \in M \overset{*}{-} N$. Therefore, for any $c \in N$, there is an element $a \in M$ such that the equality $z + c = a$ holds. From this we get the expression

$$c = a - z \in N. \tag{14}$$

By condition, since $N \subset M_\alpha$, relation $M_\alpha \overset{*}{-} N \neq \varnothing$ is valid. Let $z_2 \in M_\alpha \overset{*}{-} N$. It follows that $z_2 + N \subset M_\alpha$. This relation holds for all elements of the set N. Hence, according to (14), we can write the relation

$$z_2 + a - z \in M_\alpha. \tag{15}$$

According to the condition, $M \overset{*}{-} M_\alpha \neq \varnothing$. Let $z_1 \in M \overset{*}{-} M_\alpha$. Then, $z_1 + M_\alpha \subset M$ is appropriate. Since this relation holds for all elements of the set M_α, it also holds for the element $z_2 + a - z$ in expression (15)

$$z_2 + z_1 + a - z \in M.$$

Since $a \in M,\ z_1 + z_2 - z = 0$ and hence, the equality $z_1 + z_2 = z$ holds true.

Now, let $z \in \left(M \overset{*}{-} M_\alpha\right) + \left(M_\alpha \overset{*}{-} N\right)$, then there are elements $z_1 \in M \overset{*}{-} M_\alpha$ and $z_2 \in M_\alpha \overset{*}{-} N$ such that $z_1 + z_2 = z$. According to the definition of Minkowski difference from relation $z_1 \in M \overset{*}{-} M_\alpha$, we can write relation $z_1 + M_\alpha \subset M$, similarly, we get the expression $z_2 + N \subset M_\alpha$ from the relation $z_2 \in M_\alpha \overset{*}{-} N$. From these two expressions we get $z_1 + z_2 + N \subset M$, which leads to $z_1 + z_2 \subset M \overset{*}{-} N$. The theorem is proved.

Definition 4. A compact set N is said to be embedded in a foliation F, if such a leaf $L_\alpha = \partial M_\alpha$, $\alpha \in A$ and an element $z \in \mathbb{R}^n$ are found for which the relation $z + N \subset M_\alpha$ holds.

Definition 5. A compact set N is said to be densely embedded in a foliation F, if $z + N \subset M_{\alpha_0}$ and the index α_0 is the smallest among the numbers $\alpha \in A$ for which the relation $z + N \subset M_\alpha$ holds.

It is easy to understand from this definition that if the compact set N is densely embedded in foliation F, the dimension of the geometric difference $M_\alpha \overset{*}{-} N$ is smaller than the dimension of the space $\mathbb{R}^n$.

Definition 6. A compact set N is said to be completely densely embedded in a foliation F, if Minkowski difference $M_\alpha \overset{*}{-} N = \{a\}$ consists of a single point.

Theorem 4. If compact set N completely densely embedded in a foliation F, then the equality

$$M \overset{*}{-} N = \left(M \overset{*}{-} M_\alpha\right) + a$$

holds.

Using the concept of "complete dense embedding", we can write the following results for cases where the "subtrahend" set in the theorem 1 and theorem 2, above is an arbitrary compact set N.

Theorem 5. For polygons P^A and P^B in the Euclidean plane $\mathbb{R}^2$, condition (1) holds. If compact set N is completely dense embedded in set P^B, then the equality $P^A \overset{*}{-} N = P^A \overset{*}{-} P^B$ holds.

Theorem 6. For tetrahedrons T^A and T^B in the Euclidean space $\mathbb{R}^3$, condition (9) holds. If compact set N is completely dense embedded in set T^B, then the equality $T^A \overset{*}{-} N = T^A \overset{*}{-} T^B$ holds.

Conclusion

The Minkowski difference is actually useful as a research and conceptual tool. But, unfortunately, it is well known that there are serious difficulties in finding the Minkowski difference for given arbitrary forms of sets. This is the main obstacle for using the Minkowski difference in various practical applications. The results of the analysis of the work done by experts so far on finding the Minkowski difference and sum have shown that the Minkowski sum of sets is sufficiently studied, but there is a lack of data and literature on the Minkowski difference and its calculation.

Above, we introduced new methods for finding Minkowski differences of regular polygons given by vertices in the plane $\mathbb{R}^2$, regular tetrahedron given by vertices in space $\mathbb{R}^3$. Taking these results, we came to the conclusion that the form of the Minkowski difference of these sets will be similar to the "minuend" set.

But we cannot state this conclusion for the Minkowski difference of n-dimensional cubes in $\mathbb{R}^n$. Because the Minkowski difference of two cubes can also be a rectangular parallelepiped edges of which are parallel to the edges of the "minuend" cube. At the end of the article, we stated a theorem that helps to calculate the Minkowski difference of arbitrary convex compact sets in $\mathbb{R}^n$ using the elements of the theory of foliation.

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Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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Author Information

Mashrabjon Shaxabutdinovich Mamatov — Doctor of physical and mathematical sciences, Professor, National University of Uzbekistan, 4 University street, Tashkent, 100174, Uzbekistan; e-mail: *mamatovmsh@mail.ru*; https://orcid.org/0000-0001-8455-7495

Jalolxon Tursunboy ugli Nuritdinov (*corresponding author*) — Teacher, Kokand University and Kokand State Pedagogical University, 15 Naymancha street, Kokand, 150700, Uzbekistan; e-mail: *nuritdinovjt@gmail.com*; https://orcid.org/0000-0001-8288-832X

Khamidullo Shamsidinovich Turakulov — Doctor of Philosophy in physical and mathematical sciences (PhD), Associate professor, Kokand State Pedagogical University, 17 Doimobod Village, Dangara, 150500, Uzbekistan; e-mail: *hamidtsh87@gmail.com*; https://orcid.org/0000-0001-8503-7256

Sanzharbek Mirzaevich Mamazhonov — Doctor of Philosophy in physical and mathematical sciences (PhD), Associate professor, Kokand University, 52 Kichik Kashkar Village, Uchkuprik, 151606, Uzbekistan; e-mail: *sanjarbekmamajonov@gmail.com*; https://orcid.org/0000-0001-7878-8932