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General bounded multiperiodic solutions of linear equation with differential operator in the direction of the main diagonal

In this article we determine the structure of the general solution of a n-th order linear equation with differential operator in the direction of the main diagonal in a space of independent variables, and with coefficients being constant on the characteristic of this operator under some condition on its eigenvalues. It is assumed that the coefficients and a given vector-function have the properties of periodicity and smoothness, where periods are rationally incommensurable positive constants. First, we study the homogeneous equation that reduces to a homogeneous linear system. Moreover, on this base, in terms of eigenvalues we establish conditions of existence of solutions being periodic with respect to all independent variables (so-called *multiperiodic* solutions). We give an integral representation of the multiperiodic solution of nonhomogeneous equation. The conditions for existence and uniqueness of the bounded and multiperiodic solutions of the n-th order linear nonhomogeneous equation are established. It is shown that the bounded solution of the nonhomogeneous equation is periodic in all variable solutions with a variable bounded period. This is one of the specific features of the equations with differential operator in the direction of the main diagonal.

Keywords: linear equation, differential operator, multiperiodic solution, integral representation.

Introduction

Let $x(\tau,t)$ be a function of variables $\tau\in(-\infty,+\infty)=\mathbb{R}$ and $t=(t_1,\dots,t_m)\in\mathbb{R}\times\dots\times\mathbb{R}=\mathbb{R}^m$, $D_e=\frac{\partial}{\partial\tau}+\left\langle e,\frac{\partial}{\partial t}\right\rangle$ the operator determined by the scalar product $\langle\cdot,\cdot\rangle$ of m-dimensional vectors $e=(1,\dots,1)$ and $\frac{\partial}{\partial t}=\left(\frac{\partial}{\partial t_1},\dots,\frac{\partial}{\partial t_m}\right)$.

The operator D_e is called the differential operator in the direction of the main diagonal or of vector field $\frac{dt}{d\tau}=e$ with characteristic $t=e(\tau-s)+\sigma$, where $(s,\sigma)\in\mathbb{R}\times\mathbb{R}^m$, $\sigma=(\sigma_1,\dots,\sigma_m)$. Obviously, $\sigma=t-e(\tau-s)$ is a base integral of the vector field, consequently, $D_eh(\sigma)=0$ for any differentiable function $h(\sigma)$. In particular, we have $D_e\sigma=0$. A function of the type $h(\sigma)$ is called a function constant on characteristic.

The study of oscillation solutions of systems of first order partial differential equations are importance in mathematics as in theoretical and applied aspect. For example, (θ,ω)-periodic systems in (τ,t) of the form

$$D_ex=f(\tau,t,x)$$

with differential operator D_e are closely connected with the theory of multifrequency oscillations [1–3], where $\omega=(\omega_1,\dots,\omega_m)$, $\omega_0=\theta,\omega_1,\dots,\omega_m$ are positive incommensurable constants. (θ,ω,ω)-periodicity in (τ,t,σ) of solutions of systems has been studied in [4–6].

In [7] a method of studying (θ,ω)-periodic solutions of such systems has been offered. A further study of these problems has brought forth the systems with characteristic $\sigma=t-e(\tau-s)$, of the form

$$D_ex=g(\tau,t,\sigma,x),$$

see [8]. After substitution $\tau-s\mapsto\tau$ we have $\sigma=t-e\tau$, with ω-periodicity in t of the systems being still valid. Consequently, $g(\tau,t,\sigma,x)$ is ω-periodic as well in σ. Let us remark that, generally speaking, with any fixed value of t, this system is quasi-periodic in τ.

A system of such form can be obtained from n-order linear equations

$$D_e^nx+a_1(\sigma)D_e^{n-1}x+...+a_n(\sigma)x=b(\tau,t,\sigma), \tag{1}$$

with operators $D_e^jx=D_e(D_e^{j-1}x)$, $j=\overline{1,n}$, where $a_j(\sigma)$, $j=\overline{1,m}$, and given vector-functions $b(\tau,t,\sigma)$ have the properties:

$$a_j(\sigma+q\omega)=a_j(\sigma)\in C_\sigma^{(1)}(\mathbb{R}^m),\quad j=\overline{1,m},\quad q\in\mathbb{Z}^m; \tag{2}$$

$$b(\tau+\theta, t+q\omega, \sigma+q\omega) = b(\tau, t, \sigma) \in C^{(0,1,1)}_{\tau,t,\sigma}(\mathbb{R}\times\mathbb{R}^m\times\mathbb{R}^m), \quad q\in\mathbb{Z}^m \tag{3}$$

with multiple vector-period $q\omega = (q_1\omega_1, \ldots, q_m\omega_m)$, $q = (q_1, \ldots, q_m)$, $q_j \in \mathbb{Z}$, where $\mathbb{Z}$ is the set of integers, $j = \overline{1,m}$, $\sigma = t - e\tau$.

Our basic objects of study here are the structure of the general solution to the equation (1) and, on this base, of its (θ, ω, ω)-periodic solutions.

Linear homogeneous equation

If $b = 0$, (1) becomes a homogeneous equation

$$D_e^n x + a_1(\sigma)D_e^{n-1}x + \ldots + a_n(\sigma)x = 0 \tag{4}$$

with corresponding characteristic polynomial equation in λ

$$H_n(\sigma,\lambda) = \lambda^n + a_1(\sigma)\lambda^{n-1} + \ldots + a_n(\sigma) = 0. \tag{5}$$

Suppose that all roots $\lambda = \lambda(\sigma)$ of (5) are in $\mathbb{R}^m$ and have the following properties.

1^0. Roots are either zero everywhere in $\mathbb{R}^m$, or roots are different from zero in $\mathbb{R}^m$: $\lambda(\sigma) = 0$, $\sigma\in\mathbb{R}^m$ or $\lambda(\sigma)\neq 0$, $\sigma \in \mathbb{R}^m$.

2^0. Roots are separated: $\inf\left|\lambda'(\sigma) - \lambda''(\sigma)\right| \geq \delta = const > 0$ for any pair of roots $\lambda'(\sigma)$ and $\lambda''(\sigma)$.

3^0. Roots are periodic with period ω: $\lambda(\sigma + q\omega) = \lambda(\sigma)$, $\sigma\in\mathbb{R}^m$.

4^0. Roots are continuously differentiable: $\lambda(\sigma) \in C^{(1)}_\sigma(\mathbb{R}^m)$.

It is not difficult to notice that the property 2^0 implies there exist exactly n roots of the equation (5) counted with multiplicity.

Assuming (2) and $1^0 - 4^0$, we aim to describe the structure of the set of solutions of the equation (4).

For this purpose, having put $x = y_1$, $D_e y_1 = y_2$, $D_e y_2 = y_3$, ..., $D_e y_{n-1} = y_n$, we obtain the equation (4) in the form of a linear system

$$D_e y = A(\sigma) y, \tag{6}$$

where $y = (y_1, \ldots, y_n)$ a vector, $A(\sigma)$ an $(n\times n)$-matrix of the form

$$A(\sigma) = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ -a_n(\sigma) & -a_{n-1}(\sigma) & -a_{n-2}(\sigma) & \ldots & -a_2(\sigma) & -a_1(\sigma) \end{pmatrix}.$$

Obviously, the characteristic polynomials of the equation (4) and system (6) coincide $\det[\lambda E - A(\sigma)] = H_n(\sigma,\lambda)$. Consequently, the roots $\lambda = \lambda(\sigma)$ of equation (5) are eigenvalues of the matrix $A(\sigma)$.

We shall describe the structure of the set solutions of the system (6) and, consequently, of equations (4) in dependence on multiplicity of eigenvalues.

The case of simple eigenvalues. $\lambda_i(\sigma) \neq \lambda_j(\sigma)$, $i\neq j$, $i,j = \overline{1,n}$ and $\lambda_i(\sigma)$ are real-valued functions. It is easy to check that the Vandermonde matrix $T(\sigma)$, formed with eigenvalues $\lambda_j = \lambda_j(\sigma)$, $j = \overline{1,n}$, $\sigma\in\mathbb{R}^m$, and the diagonal matrix $J(\sigma) = \mathrm{diag}\left[\lambda_1(\sigma), \ldots, \lambda_n(\sigma)\right]$ satisfy the identity

$$A(\sigma)T(\sigma) = T(\sigma)J(\sigma).$$

Moreover, $\det T(\sigma) = \prod\limits_{1\leq j<i\leq n}[\lambda_i(\sigma) - \lambda_j(\sigma)] \neq 0$, $\sigma\in\mathbb{R}^m$, thanks to 1^0 and 2^0. Consequently, the matrix $T(\sigma)$ is reversible for $\sigma\in\mathbb{R}^m$ and with substitution $y = T(\sigma)z$ the system (6) boils down to the system

$$D_e z = J(\sigma) z, \tag{7}$$

which in scalar form can be written in the form $D_e z_\alpha = \lambda_\alpha(\sigma) z_\alpha$, $\alpha = \overline{1,n}$, where $z = (z_1, \ldots, z_n)$. From this system we get

$$z_\alpha = C_\alpha(\sigma)\exp\left[\tau\lambda_\alpha(\sigma)\right] \tag{8}$$

with arbitrary differentiable function $C_\alpha(\sigma)$. Taking into account (8), the general solution $z = z(\tau,\sigma)$ of the system (7) is of the form

$$z(\tau,\sigma) = Z(\tau,\sigma)C(\sigma), \tag{9}$$

where $Z(\tau,\sigma) = \operatorname{diag}\left[e^{\tau\lambda_1(\sigma)},\ldots,e^{\tau\lambda_n(\sigma)}\right]$ is a matrix, $C(\sigma) = (C_1(\sigma),...,C_n(\sigma))$ an arbitrary differentiable vector-function.

Then, substituting (9) into $y = T(\sigma)z$ we obtain the general solution of the system (6) in the form

$$y(\tau,\sigma) = T(\sigma)Z(\tau,\sigma)C(\sigma). \tag{10}$$

Further, from the relation (10) we determine the structure of the solution of the equation (4):

$$x(\tau,\sigma) = y_1(\tau,\sigma) = \sum_{j=1}^{n} e^{\lambda_j(\sigma)\tau}C_j(\sigma), \tag{11}$$

where it is taken into account that t_{1j}, the entries in the first row of the matrix $T(\sigma)$, are equal 1, $C_j(\sigma)$ are components of arbitrary differentiable vector-function $C(\sigma)$.

We remark that, in view of 3^0 and 4^0, the matrix $T(\sigma)$ is ω-periodic and continuously differentiable.

Theorem 1. Assume the conditions (2) and $1^0 - 4^0$ hold. Then in the case of simple eigenvalues, the solution x of equation (4) satisfying the initial condition

$$x|_{\tau=0} = u_1(\sigma), \quad D_e x|_{\tau=0} = u_2(\sigma), \quad \ldots, \quad D_e^{n-1}x|_{\tau=0} = u_n(\sigma) \tag{12}$$

can be presented in the form (11), where $u_j(\sigma)$, $j = \overline{1,n}$, are given differentiable ω-periodic functions.

Proof. Indeed, to determine the solution of the problem (4), (12) we act step by step with the operator D_e on the solution (11) and use the condition (12). Then we shall obtain a linear system of algebraic equations of the form $\sum\limits_{j=1}^{n}\lambda_j^{\alpha}(\sigma)C_j(\sigma) = u_2(\sigma)$, $(\alpha = \overline{1,n-1})$ with coefficients matrix being the Vandermonde matrix $T(\sigma)$. Consequently, $C(\sigma) = T^{-1}(\sigma)u(\sigma)$, where $u(\sigma) = (u_1(\sigma),\ldots,u_n(\sigma))$.

Therefore, the initial problem (4), (12) is uniquely solvable and its solution can be presented in the form (11). Consequently, the relation (11) is itself the general solution of equation (4).

The case of multiple eigenvalues. In view of $1^0 - 2^0$ $\lambda_i(\sigma)$, $i = \overline{1,r}$ have multiplicity k_i independent of $\sigma \in \mathbb{R}^m$ (we shall assume the eigenvalues are real-valued), where $k_1 + \cdots + k_n = n$.

A) We start with the particular case when $\lambda(\sigma)$ is a unique, with multiplicity n, root of the equation (4). We introduce Jordan block $J(\sigma)$, corresponding to the eigenvalue $\lambda(\sigma)$: $J(\sigma) = \lambda(\sigma)E + I$, where E is the identity matrix of order n, I is the matrix with units on the superdiagonal and the rest of entries being zero.

Let $T(\sigma)$ be the matrix with entries $t_{ij}(\sigma)$, $i,j = \overline{1,n}$, of the form

$$t_{ij}(\sigma) = \begin{cases} \sum\limits_{k=1}^{j} C_{i-1}^{k-1}\lambda^{i-k}(\sigma), & j \le i; \\ \sum\limits_{k=1}^{i} C_{i-1}^{k-1}\lambda^{i-k}(\sigma) = t_{ii}, & j > i, \end{cases}$$

where $C_i^j = \frac{i(i-1)\ldots(i-j+1)}{j!}$, $j \le i$ is a binomial coefficient.

It is not difficult to check that $A(\sigma)T(\sigma) = T(\sigma)J(\sigma)$, moreover $\det T(\sigma) = 1$. Consequently, a transformation of system (6) of the form (7) produces the system $D_e z = (\lambda(\sigma)E + I)z$. Then general solution $z(\tau,\sigma)$ of this system can be presented as

$$z(\tau,\sigma) = e^{\tau\lambda(\sigma)}Z(\tau)C(\sigma), \tag{13}$$

where $Z(\tau)$ is an $(n\times n)$-matrix of the form

$$Z(\tau) = \begin{pmatrix} 1 & \frac{\tau}{1!} & \frac{\tau^2}{2!} & \ldots & \frac{\tau^{n-1}}{(n-1)!} \\ 0 & 1 & \frac{\tau}{1!} & \ldots & \frac{\tau^{n-2}}{(n-2)!} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix},$$

$C(\sigma) = (C_1(\sigma),\ldots,C_n(\sigma))$ an arbitrary differentiable vector-function.

Therefore, by (7) and (13), we have a general solution $x(\tau,\sigma)$ of equation (4) in the form

$$x(\tau,\sigma) = \sum_{j=1}^{n} C_j(\sigma)\frac{\tau^{j-1}}{(j-1)!}e^{\tau\lambda(\sigma)} \tag{14}$$

with arbitrary differentiable functions $C_j(\sigma)$, $j=\overline{1,n}$. Obviously, with initial problem (4), (12) we have $C_j(\sigma)=u_j(\sigma)$, $j=\overline{1,n}$.

Consequently, one can formulate the result on the structure of the general solution (14) of the equation (4) in the case A).

Theorem 2. Assume (2) and 1^0-4^0. Then in the case of one eigenvalue $\lambda(\sigma)$ of multiplicity n, the solution $x(\tau,\sigma)$ of (4), (12) has the form (14).

Proof. Indeed, acting with operator D_e on relation (14), then using condition (12) we get an equation $(E+\lambda I_1)C(\sigma)=u(\sigma)$, where E is the identity matrix, I_1 is the matrix with units on the subdiagonal.

Obviously, $\det(E+\lambda I_1)=1$. Consequently, $C(\sigma)$ is uniquely defined by $u(\sigma)$. Theorem 2 is proved.

Before we pass to the general case, under the conditions as in Theorem 2 we first by study nonhomogeneous equation corresponding to equation (4) with quasilinear polynomial in τ of the form

$$D_e^n x+a_1(\sigma)D_e^{n-1}x+...+a_n(\sigma)x=\sum_{j=0}^{k}C_j^*(\sigma)\tau^j e^{\tau\mu(\sigma)}, \tag{15}$$

where coefficients $C_j^*(\sigma)$, $j=\overline{0,k}$, and index $\mu(\sigma)$ are differentiable ω-periodic functions with $\mu(\sigma)\neq\lambda(\sigma)$.

Since $\lambda(\sigma)$ is an n-multiple root of the characteristic equation (5), using the symbolic operator $D_e-\lambda(\sigma)$ the equation (15) can be given in the following form:

$$[D_e-\lambda(\sigma)]^n x=\sum_{j=0}^{k}C_j^*(\sigma)\tau^j e^{\tau\mu(\sigma)}. \tag{16}$$

In order to solve the equation (16), first we make the substitution $x=e^{\tau\mu(\sigma)}y$, and bring it to the form

$$\sum_{j=0}^{n}C_n^j[\mu(\sigma)-\lambda(\sigma)]^j D_e^{n-j}y=\sum_{j=0}^{k}C_j^*(\sigma)\tau^j, \tag{17}$$

where C_n^j is, as before, the binomial coefficient.

We set a particular solution $y^*(\tau,\sigma)$ of (17) with undetermined coefficients $v_j(\sigma)$, $j=\overline{0,k}$, as

$$y^*(\tau,\sigma)=\sum_{j=0}^{k}v_j(\sigma)\tau^j. \tag{18}$$

Then these coefficients are defined by recurrence relations and, as $\mu(\sigma)\neq\lambda(\sigma)$, they have unique presentation through coefficients $C_j^*(\sigma)$, $j=\overline{0,k}$:

$$v_j(\sigma)=\frac{1}{j!}v_j^*(\sigma)\equiv[\mu(\sigma)-\lambda(\sigma)]^{-j}\pi_j(\sigma,C_0^*(\sigma),\dots,C_j^*(\sigma)), \tag{19}$$

where $\pi_j(\sigma,C_0^*(\sigma),\dots,C_j^*(\sigma))$ are linear with respect to $C_0^*(\sigma),\dots,C_j^*(\sigma)$ and ω-periodic in σ, differentiable functions. Having substituted (19) into (18) we get a particular solution $y^*(\tau,\sigma)$ of equation (17):

$$y^*(\tau,\sigma)=\sum_{j=0}^{k}v_j^*(\sigma)\frac{\tau^j}{j!}, \tag{20}$$

while in view of $x=e^{\tau\mu(\sigma)}y$ we have a particular solution $x^*(\tau,\sigma)$ of equation (16), consequently, equations (15) in the form of

$$x^*(\tau,\sigma)=y^*(\tau,\sigma)e^{\tau\mu(\sigma)} \tag{21}$$

with multipliers (20).

Since equation (15) is linear, its general solution $x(\tau,\sigma)$ is the sum of general solution (14) of the homogeneous equation (4) and a particular solution (21) of nonhomogeneous equation (15):

$$x(\tau,\sigma)=\sum_{j=1}^{n}C_j(\sigma)\frac{\tau^{j-1}}{(j-1)!}e^{\tau\lambda(\sigma)}+\sum_{j=0}^{k}v_j^*(\sigma)\frac{\tau^j}{j!}e^{\tau\mu(\sigma)} \tag{22}$$

with arbitrary differentiable coefficients $C_j(\sigma)$, $j=\overline{1,n}$.

As it was shown in the proof Theorem 2, analogously one can prove uniquely the initial problem for equation (15) with condition (12). Consequently, the relation (22) describes the structure of the general solution of the equation (15).

Corollary. Under the same conditions as in Theorem 2, the general solution $x(\tau,\sigma)$ of (15) is of the form (22).

B) We pass with our considerations to the general case, when roots $\lambda_1(\sigma),\ldots,\lambda_r(\sigma)$ have multiplicities $k_1,\ldots,k_r$ respectively, $k_1+\cdots+k_n=n$, and satisfy conditions 1^0-4^0.

In this case we will determine the structure of the general solution $x(\tau,\sigma)$ of equation (4) provided condition (2) holds. Using the symbolic operator

$$L(D_e)=D_e^n+a_1(\sigma)D_e^{n-1}+...+a_n(\sigma)=[D_e-\lambda_1(\sigma)]^{k_1}\ldots[D_e-\lambda_r(\sigma)]^{k_r},$$

we shall present (4) in the form

$$[D_e-\lambda_1(\sigma)]^{k_1}\ldots[D_e-\lambda_{r-1}(\sigma)]^{k_{r-1}}[D_e-\lambda_r(\sigma)]^{k_r}x=0. \tag{23}$$

We shall prove that the general solution $x(\tau,\sigma)$ of (4), and so of (23), has the form

$$x(\tau,\sigma)=\sum_{j=1}^{k_1}C_j(\sigma)\frac{\tau^{j-1}e^{\tau\lambda_1(\sigma)}}{(j-1)!}+...+\sum_{j=1}^{k_r}C_{n-k_r+j}(\sigma)\frac{\tau^{j-1}e^{\tau\lambda_r(\sigma)}}{(j-1)!}. \tag{24}$$

For the proof we will use the induction method. For $r=1$ the formula (24) holds due to Theorem 2. We shall assume that it true for $r-1$ and prove it for r. For this purpose, in equation (24) we put $[D_e-\lambda_r(\sigma)]^{k_r}x=z$ and get an equation

$$[D_e-\lambda_1(\sigma)]^{k_1}\ldots[D_e-\lambda_{r-1}(\sigma)]^{k_{r-1}}z=0. \tag{25}$$

Since the formula (24) holds for $r-1$ eigenvalues, the equation (25) has a general solution $z(\tau,\sigma)$ of the form

$$z(\tau,\sigma)=\sum_{j=1}^{k_1}C_j(\sigma)\frac{\tau^{j-1}e^{\tau\lambda_1(\sigma)}}{(j-1)!}+...+\sum_{j=1}^{k_{r-1}}C_{n-k_r-kr-1+j}(\sigma)\frac{\tau^{j-1}e^{\tau\lambda_{r-1}(\sigma)}}{(j-1)!}. \tag{26}$$

Further, having put the expression (26) into $[D_e-\lambda_r(\sigma)]^{k_r}x=z$ we get a nonhomogeneous equation

$$[D_e-\lambda_r(\sigma)]^rx=\sum_{j=1}^{k_1}C_j(\sigma)\frac{\tau^{j-1}}{(j-1)!}e^{\tau\lambda_1(\sigma)}+...+\sum_{j=1}^{k_{r-1}}C_{n-k_r-kr-1+j}(\sigma)\frac{\tau^{j-1}}{(j-1)!}e^{\tau\lambda_{r-1}(\sigma)}. \tag{27}$$

Now, in order to solve the equation (27) it is necessary to apply the corollary of Theorem 2 to solution of each of the equations

$$[D_e-\lambda_r(\sigma)]^rx=\sum_{j=1}^{k_1}C_j(\sigma)\frac{\tau^{j-1}}{(j-1)!}e^{\tau\lambda_1(\sigma)};$$

....

$$[D_e-\lambda_r(\sigma)]^rx=\sum_{j=1}^{k_{r-1}}C_{n-k_r-kr-1+j}(\sigma)\frac{\tau^{j-1}}{(j-1)!}e^{\tau\lambda_{r-1}(\sigma)}$$

and use by the superposition principle. Obviously, as coefficients of quasipolynomial solutions of these equations depend linearly on the arbitrary coefficients of their righthand parts, they are also arbitrary. The formula (24) is proved. Consequently, the solution of (4), (12) is of the form (24). This result will be formulated as the theorem.

Theorem 3. Assume the conditions (2) and 1^0-4^0 hold. Then the general solution $x(\tau,\sigma)$ of equation (4) has the form (24), with arbitrary differentiable coefficients $C_j(\sigma)$, $j=\overline{1,n}$.

The case of complex eigenvalues. Let the roots of equation (4) $\lambda_j^{\pm}(\sigma)=\alpha_j(\sigma)\pm i\beta_j(\sigma)$, $j=\overline{1,p}$ have multiplicity k_j, $j=\overline{1,p}$, $2k_1+\cdots+2k_p=n_1\le n$ and its have sets of real parts $\{\alpha_j(\sigma)\}$ and imaginary parts $\{\beta_j(\sigma)\}$ both of which possess the properties 1^0-4^0.

Since the complex solution $x(\tau,\sigma)=v(\tau,\sigma)+i\omega(\tau,\sigma)$ has real and imaginary parts, $\operatorname{Re}x(\tau,\sigma)=v(\tau,\sigma)$ and $\operatorname{Im}x(\tau,\sigma)=\omega(\tau,\sigma)$, being solutions of (4), to any pair of complex coupled roots $\lambda_j^{\pm}(\sigma)=\alpha_j(\sigma)\pm i\beta_j(\sigma)$ there corresponds the solution

$$x^j(\tau,\sigma)=\left[P_j(\tau,\sigma)\cos(\beta_j(\sigma)\tau)+Q_j(\tau,\sigma)\sin(\beta_j(\sigma)\tau)\right]e^{\tau\alpha_j(\sigma)}, \tag{28}$$

where $P_j(\tau,\sigma)=\sum\limits_{k=1}^{k_j} p_k^{(j)}(\sigma)\frac{\tau^{k-1}}{(k-1)!}$ and $Q_j(\tau,\sigma)=\sum\limits_{k=1}^{k_j} q_k^{(j)}(\sigma)\frac{\tau^{k-1}}{(k-1)!}$ with arbitrary coefficients $p_k^{(j)}(\sigma)$ and $q_k^{(j)}(\sigma)$, $k=\overline{1,k_j}$, $j=\overline{1,p}$.

Consequently, the general solution $x(\tau,\sigma)$ of equation (4) in the case of complex roots has the form

$$x(\tau,\sigma)=\sum_{j=1}^{p} x^{(j)}(\tau,\sigma)+\sum_{j=n_1+1}^{n} x^{(j)}(\tau,\sigma), \tag{29}$$

where $x^{(j)}(\tau,\sigma)$, $j>n_1$ are solutions corresponding to real roots and for $j=\overline{1,p}$ $x^{(j)}(\tau,\sigma)$ are defined by the relation (28).

Therefore, the following theorem is proved.

Theorem 4. Suppose that, under condition (2), the equation (4) has complex eigenvalues $\lambda_j(\sigma)=\alpha_j(\sigma)\pm i\beta_j(\sigma)$, $j=\overline{1,p}$ of multiplicity k_j and real eigenvalues satisfying the properties 1^0-4^0. Then the general solution $x(\tau,\sigma)$ of equation (4) is defined by relations (28) and (29).

Notice that in the case of Theorem 3 and Theorem 4, endowed with initial conditions (12), it is possible to show the unique solubility of the initial problem (4), (12).

Let $x^{(j)}(\tau,\sigma)$, $j=\overline{1,n}$ be solutions of the equation (4) satisfying the initial conditions

$$D_e^k x^{(j)}(\tau,\sigma)\big|_{\tau=\tau_0}=\begin{cases}0, & k\neq j-1;\\ 1, & k=j-1,\end{cases} \tag{30}$$

where $k=\overline{0,n-1}$.

Such a system of solutions we shall call a normalized fundamental system of solutions of (4).

Theorem 5. Under conditions as in Theorem 4 the unique solution $x(\tau,\sigma)$ of problem (4), (12) is defined by

$$x(\tau,\sigma)=\sum_{j=1}^{n} u_j(\sigma)x^{(j)}(\tau,\sigma), \tag{31}$$

where $x^{(j)}(\tau,\sigma)$, $j=\overline{1,n}$ is a normalized fundamental system of the solutions.

Indeed, it is not difficult to check that (31) fulfills (4) and in the view (30) the initial condition (12). Linear combinations of solutions from normalized fundamental system satisfying condition (12) are uniquely described the relation (31).

Linear nonhomogeneous equation

Assume the conditions (2), (3) and 1^0-4^0 hold. On the base of Theorem 5 we introduce the solution $X(\tau,t,\sigma,s,\sigma+es)$ of equation (4) satisfying the initial condition

$$D_e^k X(s,\sigma+es,\sigma,s,\sigma+es)=0,\quad (k=\overline{0,n-2}),\quad D_e^{n-1}X(s,\sigma+es,\sigma,s,\sigma+es)=1 \tag{32}$$

and function $x^0(\tau,t,\sigma)=\int\limits_0^\tau X(\tau,t,\sigma,s,\sigma+es)b(s,\sigma+es,\sigma)ds$. It is easy to check that $x^0(\tau,t,\sigma)$ in view of (32) satisfy the equation (1) with zero initial condition $D_e^k x^0(s,\sigma+es,\sigma)=0,\quad k=\overline{0,n-1}$.

Therefore, unique solution $x(\tau,t,\sigma)$ of equation (1) with initial condition (12) is defined by

$$x(\tau,t,\sigma)=\sum_{j=1}^{n} u_j(\sigma)x^{(j)}(\tau,\sigma)+x^0(\tau,t,\sigma). \tag{33}$$

This result will be formulated as Theorem.

Theorem 6. Assume the conditions (2) and (3) hold, sets of real eigenvalues and complex eigenvalues possess the properties 1^0-4^0. Then initial problem for equation (1) with condition (12) has unique solution (33).

The bounded and periodic solutions

When is known structures of the general solution to the equations (1) and (4) then can present conditions of existence of bounded and periodic solutions in terms of eigenvalues.

Let's start to consider homogeneous equation (4) with condition (2). It is limited solutions of problem of the form (4), (12).

Theorem 7. The solution $x(\tau,\sigma)$ of problem (4), (12) under condition (2) is ω-periodic in $\sigma \in R^m$.

Proof of the Theorem 7 follows from ω-periodic of initial functions $u_j(\sigma)$, $j=\overline{1,n}$ and eigenvalues $\lambda_j(\sigma)$, $j=\overline{1,n}$.

Theorem 8. Under the conditions (2), 1^0-4^0 and when real parts of eigenvalues are different from zero $Re\,\lambda_j(\sigma)\neq 0$, $j=\overline{1,n}$ then equation (4) hasn't bounded therefore periodic solutions except zero.

It is not difficult to show that under the conditions of Theorem 8 will be found constant $\gamma>0$, $\Gamma>0$ and any solution $x^j(\tau,\sigma)$ of equation (4) entering into the fundamental system is satisfied by estimation

$$|x^j(\tau,\sigma)| \le \Gamma e^{-\gamma|\tau|}. \tag{34}$$

Then as in Theorem 5 in view of (34) follows unbounded of all solutions $x(\tau,\sigma)$ of problem (4), (12) except zero.

Further allow that equation (4) has only imaginary eigenvalues $\lambda_{1,2}(\sigma)=\pm i\beta(\sigma)\neq 0$ which by the conditions on Theorem 4 satisfied bounded solution of the form $x^*(\tau,\sigma)=C_1(\sigma)\cos(\beta_j(\sigma)\tau)+C_2(\sigma)\sin(\beta_j(\sigma)\tau)$ with arbitrary ω-periodic in σ coefficients $C_1(\sigma)$ and $C_2(\sigma)$. Obviously that this solution (θ,ω)-periodic in (τ,σ), where $\theta=\frac{2\pi}{\beta(\sigma)}\equiv\theta(\sigma)$ is ω-periodic differentiable function.

Therefore in this case the bounded solution $x^*(\tau,\sigma)$ of equation being periodic in τ with variable bounded period $\theta(\sigma)=\omega_0(\sigma)$. We note that it is one of specific particularities of equation with operator D_e.

In this case (ω_0,ω)-periodic solutions consist double-parameter family where parameters are ω-periodic in σ functions $C_1=C_1(\sigma)$ and $C_2=C_2(\sigma)$. If equation (4) has zero eigenvalue $\lambda=0$ then it is in view of Theorem 3 satisfied one-parameter family ω-periodic in σ of solutions $x=C(\sigma)$.

Theorem 9. If under the conditions as in Theorem 4 the equation (4) has zero $\lambda_1=0$ and only imaginary eigenvalues $\lambda_j(\sigma)=\pm i\beta_j(\sigma)$, $j=\overline{2,p}$ then it is allowed bounded solutions

$$x(\tau,\sigma)=C_1(\sigma)+\sum_{j=2}^{p}C_{1j}(\sigma)\cos(\beta_j(\sigma)\tau)+C_{2j}(\sigma)\sin(\beta_j(\sigma)\tau),$$

where $p<n$, $C_1(\sigma)$, $C_{1j}(\sigma)$, $C_{2j}(\sigma)$ are differentiable arbitrary ω-periodic functions.

From Theorem 9 we see that presented solution here $x(\tau,\sigma)$ consists of line combinations $\theta_j(\sigma)=\frac{2\pi}{\beta_j(\sigma)}$-periodic in τ of functions $j=\overline{1,p}$.

Further we consider the case of Theorem 8. Let $x^j(\tau-s,\sigma)$, $j=\overline{1,n}$ – fundamental system of solutions of homogeneous equation (4) are satisfied by the conditions

$$D_e^{k-1}x^{(j)}(\tau-s,\sigma)|_{\tau=s}=\begin{cases}0, & k\neq j;\\ 1, & k=j.\end{cases}$$

Then solution $X(\tau-s,\sigma)$ of equation (4) is satisfied by the condition

$$D_e^{k-1}X(\tau-s,\sigma)|_{\tau=s}=0,\quad (k=\overline{1,n-2}),\quad D_e^{n-1}X(\tau-s,\sigma)|_{\tau=s}=1 \tag{35}$$

according to Theorem 5 is presented in the form $X(\tau-s,\sigma)=\sum\limits_{j=1}^{n}u_j(\sigma)x(\tau-s,\sigma)$. This solution fall into sum of solutions $X(\tau-s,\sigma)=X_-(\tau-s,\sigma)+X_+(\tau-s,\sigma)$, which are satisfied by estimations:

$$|X_-(\tau-s,\sigma)|\le\Gamma_-e^{-\gamma(\tau-s)},\quad \tau\ge s,\quad |X_+(\tau-s,\sigma)|\le\Gamma_+e^{\gamma(\tau-s)},\quad \tau<s \tag{36}$$

with several positive constants Γ_- and Γ_+ moreover in view of (35) have

$$D_e^{k-1}X_-(\tau-s,\sigma)|_{\tau=s}+D_e^{k-1}X_+(\tau-s,\sigma)|_{\tau=s}=0,\quad (k=\overline{1,n-2});$$

$$D_e^{n-1}X_-(\tau-s,\sigma)|_{\tau=s}+D_e^{n-1}X_+(\tau-s,\sigma)|_{\tau=s}=1. \tag{37}$$

Further we introduce function

$$x^*(\tau, t, \sigma) = \int\limits_{-\infty}^{\tau} X_-(\tau - s, \sigma) f(s, \sigma + es, \sigma) ds - \int\limits_{\tau}^{+\infty} X_+(\tau - s, \sigma) f(s, \sigma + es, \sigma) ds. \qquad (38)$$

Obviously that integrals in correlation (38) in view of (36) are converged evenly, allowed under the integral n-time differentiality and in view of (37) is satisfied equation (1). It is easy to check that in view of (3) $x^*(\tau, t, \sigma)$ has property (θ, ω, ω)-periodicity in (τ, t, σ). Condition (36) is provided unique bounded solution (38).

Therefore, the following Theorem is proved.

Theorem 10. Under the conditions (2), (3) and $1^0 - 4^0$ equation (1) has unique (θ, ω, ω)-periodic in (τ, t, σ) solution of the form (38).

This study is adjacent to the studies [9–11].

In conclusion we shall notice that in this work at research of basic object we used properties $1^0 - 4^0$ of eigenvalues and structures of general solutions are satisfied to being considered equations.

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Негізгі диагональ бойынша дифференциалдау операторлы сызықты теңдеудің жалпы, шенелген және көппериодты шешімдері

Мақалада тәуелсіз айнымалылар кеңістігінің негізгі диагоналінің бағыты бойынша дифференциалдау операторымен және меншікті мәндерге қойылатын кейбір шарттар кезінде осы оператордың сипаттамаларында тұрақты болатын коэффициенттермен n-ші ретті сызықты теңдеудің жалпы шешімінің құрылымы анықталды. Коэффициенттермен берілген вектор-функция периодтылық және тегістік қасиеттеріне ие деп ұйғарылады, мұндағы периодтар — рационалды өлшемдес емес, оң тұрақтылар. Әуелі біртекті сызықты жүйеге ауыстыру көмегімен келтірілетін, біртекті теңдеу зерттелді. Әрі қарай осы негізде меншікті мәндер терминінде, сызықты теңдеудің барлық тәуелсіз айнымалылар бойынша периодтылығының бар болу шарттары орнатылды. Біртексіз теңдеудің көппериодты шешімінің интегралды көрінісі берілген. n-ші ретті біртексіз сызықты теңдеудің шенелген және көппериодты шешімдерінің бар және жалғыз болу шарттары орнатылды. Біртексіз теңдеудің шенелген шешімі шенелген айнымалы периодымен барлық айнымалылар бойынша периодты шешім болатындығы көрсетілген. Бұл негізгі диагональ бағыты бойынша дифференциалдау операторлы теңдеудің спецификалық ерешеліктерінің бірі.

Кілт сөздер: сызықты теңдеу, дифференциалдау оператор, көппериодты шешім, интегралды көрініс.

А.А. Кульжумиева, Ж.А. Сартабанов

Общие, ограниченные и многопериодические решения линейного уравнения с дифференциальным оператором по главной диагонали

В статье определена структура общего решения линейного уравнения n-го порядка с дифференциальным оператором по направлению главной диагонали пространства независимых переменных и коэффициентами, постоянными на характеристике этого оператора при некоторых условиях на собственные значения. Предположено, что коэффициенты и заданная вектор-функция обладают свойствами периодичности и гладкости, где периоды – рационально несоизмеримые положительные постоянные. Сначала исследовано однородное уравнение, которое с помощью замены сводится к однородной линейной системе. Далее, на этой основе, в терминах собственных значений устанавливаются условия существования периодических по всем независимым переменным (многопериодическим) решений линейного уравнения. Дано интегральное представление многопериодического решения неоднородного уравнения. Установлены условия существования и единственности ограниченного и многопериодического решения линейного неоднородного уравнения n-го порядка. Показано, что ограниченное решение неоднородного уравнения является периодическим по всем переменным решением с переменным ограниченным периодом. Это есть одна из специфических особенностей уравнений с оператором дифференцирования по направлению главной диагонали.

Ключевые слова: линейное уравнение, дифференциальный оператор, многопериодическое решение, интегральное представление.

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