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Absolute Convergence of Fourier Series of Superpositions of Functions

G. A. Akishev[1*]

[1] *Karaganda State University, ul. Universitetskaya 28, Karaganda, 100028 Republic of Kazakhstan*

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Abstract—In this paper we establish conditions for the absolute convergence of series of Fourier coefficients with respect to a generalized Haar system of a superposition of two functions.

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In this paper we consider the problem stated by P. L. Ul'yanov about conditions for the absolute convergence of Fourier series for superposition of functions. Let us recall the necessary definitions.

Let $L[0,1] = L_1$ be a Lebesgue space with the norm

$$\|f\|_1 = \int_0^1 |f(x)|dx$$

and let $L_\infty[0,1] = L_\infty$ be the space of essentially bounded Lebesgue measurable functions on the segment $[0,1]$ with the norm ([1], P. 12)

$$\|f\|_\infty = \sup_{x\in[0,1]} \text{vrai}\,|f(x)|.$$

N. Ya. Vilenkin [2] was first to consider the class χ of complete orthogonal systems. Each system $\chi\{p_n\}$ of this class is defined by a sequence of natural primes $\{p_n\}$. B. I. Golubov [3] has defined the class of systems $\chi\{p_n\}$, where $\{p_n\}$ are arbitrary primes no less than two. In [4] B. I. Golubov and A. I. Rubinshtein considered such a system for a bounded sequence $\{p_n\}$.

Let us recall the definition of this system. Let a sequence $\{p_n\}$ of natural numbers such that $p_n \geq 2$ $(n = 1, 2, \dots)$ be given. A generalized Haar system $\chi\{p_k\} = \{\chi_n(t)\}$ on the segment $[0,1]$ is defined as follows: $\chi_1(t) \equiv 1$ on $[0,1]$. If $n \geq 2$, then $n = m_k + r(p_{k+1} - 1) + s$, where $m_0 = 1$ and $m_k = p_1 p_2 \dots p_k$; $k = 1, 2, \dots$; $r = 0, 1, \dots, m_k - 1$; $s = 1, 2, \dots, p_{k+1} - 1$. Denote by A the set of points in the form $\frac{l}{m_k}$ on the segment $[0,1]$. If $t \in B$, where $B \equiv [0,1] \setminus A$, then the expansion

$$t = \sum_{k=1}^{\infty} \frac{\alpha_k(t)}{m_k}, \quad \alpha_k(t) = 0, 1, \dots, p_k - 1,$$

is unique.

Now the function $\chi_n(t) \equiv \chi_{k,r}^s(t)$ is defined as follows:

$$\chi_n(t) = \chi_{k,r}^{(s)}(t) = \begin{cases} \sqrt{m_k} \exp \frac{2\pi i s \alpha_{k+1}(t)}{p_{k+1}}, & t \in \left(\frac{r}{m_k}, \frac{r+1}{m_k}\right) \cap B; \\ 0, & t \notin \left[\frac{r}{m_k}, \frac{r+1}{m_k}\right]. \end{cases}$$

Since the set B is everywhere dense on $[0,1]$, the function $\chi_n(t)$ is extended by continuity onto the interval $(\frac{r}{m_k}, \frac{r+1}{m_k})$. After that at points of discontinuity we set the function $\chi_n(t)$ to the half-sum of its

*E-mail: akishev@ksu.kz.

right and left limit values, and at the end points of the segment $[0,1]$ we set it to its limit values inside the segment.

The Fourier coefficients of a function $f \in L$ with respect to the system $\chi\{p_n\}$ are defined by the formula

$$a_n(f) = \int_0^1 f(t)\overline{\chi_n(t)}dt.$$

If $n = m_k + r(p_{k+1} - 1) + s$ ($k = 0, 1, \ldots; r = 0, 1, \ldots, m_k - 1; s = 1, 2, \ldots, p_{k+1} - 1$), then together with $a_n(f)$ one uses the denotation $a_{k,r}^{(s)}(f)$.

Definition. Let a function $\varphi(z)$ of complex variable be defined on the complex plane $\mathbb{C}$. Then the function $\varphi(z) \in \mathrm{Lip}_D\, 1$, if $D > 0$ exists such that

$$|\varphi(z_1) - \varphi(z_2)| \le D|z_1 - z_2| \ \ \forall z_1, z_2 \in \mathbb{C}.$$

A finite nonnegative function $g(t)$ is almost increasing on an interval (a, b) with a constant $C > 0$, if $g(t_1) \le Cg(t_2)$ for any $t_1, t_2 \in (a, b)$, $t_1 < t_2$.

Denote by Ω the set of functions $\omega(t)$ that are modules of continuity for $t \in [0, +\infty)$.

If $\omega \in \Omega$ and a constant $B > 0$ exists such that

$$\int_0^\delta \omega(t)\frac{dt}{t} \le B\omega(\delta), \quad \delta \in [0, 1],$$

then the function $\omega(\delta)$ satisfies the Bari condition ($\omega \in B$) on the segment $[0, 1]$ (see [5]).

Let a module of continuity $\omega(\delta)$ and a nonnegative sequence $\tau = \{\tau(m)\}_{m=1}^\infty$ be given. For a function $f \in L$ we set

$$A_{\omega,\tau}(f) = \sum_{n=1}^\infty \omega(|a_n(f)|)\tau(n).$$

Consider the class

$$A_{\omega,\tau} = \{f \in L : A_{\omega,\tau}(f) < +\infty\}.$$

In the case $\tau = 1$ instead of $A_{\omega,\tau}(f)$, $A_{\omega,\tau}$, respectively, we write $A_\omega(f)$, A_ω.

In the theory of trigonometric series, the Levi theorem is well-known ([6], pp. 92–97); it states that if a 2π-periodic continuous function f, $m \le f(x) \le M$, $x \in R$, has the absolutely convergent Fourier series, then so does the function $\varphi(f)$, as soon as the function φ is analytic on $[m, M]$.

The converse proposition was proved in 1958 by Katsnel'son: If on $[m, M]$ a function φ is defined such that for any function $f \in C[0, 2\pi]$, $f(x) \in [m, M]$, with the absolutely convergent Fourier series the function $\varphi(f)$ also has the absolutely convergent Fourier series, then φ is analytic.

In the case $\omega(t) = t^\alpha$ this question was studied by Marcinkiewicz, Rivier, and Sager. They proved that $\varphi(f) \in A_{t^\alpha} = A_\alpha$ if and only if $\varphi \in J_{\frac{1}{\alpha}}$ is the Gevrer class, $0 < \alpha < 1$ ([6], P. 186).

P. L. Ul'yanov [7] has formulated the problem in a general form: *For which functions φ the superposition $\varphi(f)$ belongs to A_ω with any $f \in A_\omega$?*

For the Fourier–Haar coefficients this problem was solved by P. L. Ul'yanov in [8–13].

Theorem (by P. L. Ul'yanov). *Let $\omega(\delta)$ be a module of continuity; assume that $\tau(t)$ is an almost increasing function on $[1, +\infty)$ with a constant $C \ge 1$, and $\tau(2^{n+1}) \le C_3\tau(2^n + 1)$, $n = 0, 1, \ldots$ The inequality*

$$A_{\omega,\tau}(\varphi(f)) \le A_{\omega,\tau}(f)$$

is fulfilled with all $f \in L[0, 1]$ and all $\varphi \in \mathrm{Lip}_D\, 1$ for some $C_{\omega,\tau}$ if and only if functions $\omega(\delta), \tau(t)$ satisfy the (ω, τ)-condition

$$\sum_{j=1}^N \omega(\delta 2^{-\frac{j}{2}})\tau(2^{N-j} + 1) \le C_4\tau(2^N + 1)\omega(\delta), \ \ \delta \ge 0.$$

Here

$$A_{\omega,\tau}(f) = \sum_{n=1}^{\infty} \omega(|a_n(f)|)\tau(n).$$

In this paper we consider the problem stated above for the Fourier coefficients with respect to a generalized Haar system $\chi\{p_n\}$ defined by a bounded sequence $\{p_n\}$.

Let us first prove several auxiliary assertions.

Lemma 1. *Let a system $\chi\{p_n\}$ be defined by a bounded sequence $\{p_n\}$ and $\varphi \in \mathrm{Lip}_D\, 1$, $f \in L$. Then the inequality*

$$|a_k(\varphi(f))| \le D \int_0^1 \Big| \sum_{\nu=m_n+1}^{\infty} a_\nu(f)\chi_\nu(t) \Big| |\chi_{n,r}^{(s)}(t)| dt$$

takes place for $k = m_n + r(p_{n+1} - 1) + s$; $r = 0, 1, \dots, m_n - 1$; $s = 1, 2, \dots, p_{n+1} - 1$.

Proof. By the assumption of the lemma $\{p_n\}$ is bounded. Therefore due to the E. A. Vlasova theorem [14] the Fourier series for the function $f \in L$ with respect to the system $\chi\{p_n\}$ converges almost everywhere (a. e.) on $[0, 1]$, i.e.,

$$f(t) = \sum_{\nu=1}^{\infty} a_\nu(f)\chi_\nu(t) = \sum_{\nu=1}^{m_n} a_\nu(f)\chi_\nu(t) + \sum_{\nu=m_n+1}^{\infty} a_\nu(f)\chi_\nu(t)$$

a. e. on $[0, 1]$. Due to the condition $\varphi \in \mathrm{Lip}_D\, 1$ we have

$$\Big| \varphi(f(t)) - \varphi\Big(\sum_{\nu=1}^{m_n} a_\nu(f)\chi_\nu(t) \Big) \Big| \le D \Big| \sum_{\nu=m_n+1}^{\infty} a_\nu(f)\chi_\nu(t) \Big| \tag{1}$$

a.e. on $[0, 1]$.

Now, taking into account the definition of the Fourier coefficient and properties of the integral, we get

$$|a_k(\varphi(f))| = \Big| \int_0^1 \varphi(f(t))\overline{\chi_k(t)} dt \Big| \le \int_0^1 \Big| \varphi(f(t)) - \varphi\Big(\sum_{\nu=1}^{m_n} a_\nu(f)\chi_\nu(t) \Big) \Big| \overline{\chi_k(t)} dt$$
$$+ \Big| \int_0^1 \varphi\Big(\sum_{\nu=1}^{m_n} a_\nu(f)\chi_\nu(t) \Big) \overline{\chi_k(t)} dt \Big|. \tag{2}$$

It is known (see [4]) that

$$\sum_{\nu=1}^{m_n} a_\nu(f)\chi_\nu(t) = m_n \int_{\frac{r-1}{m_n}}^{\frac{r}{m_n}} f(t) dt, \quad t \in \Big[\frac{r-1}{m_n}, \frac{r}{m_n} \Big], \quad r = 1, 2, \dots, m_n.$$

Therefore, the intervals of constancy of the function $\varphi\Big(\sum\limits_{\nu=1}^{m_n} a_\nu(f)\chi_\nu(t) \Big)$ have the length $\frac{1}{m_n}$. Hence, since the system $\chi\{p_n\}$ is orthogonal,

$$\int_0^1 \varphi\Big(\sum_{\nu=1}^{m_n} a_\nu(f)\chi_\nu(t) \Big) \overline{\chi_k(t)} dt = 0. \tag{3}$$

Now correlations (1)–(3) give

$$|a_k(\varphi(f))| \le D \int_0^1 \Big| \sum_{\nu=m_n+1}^{\infty} a_\nu(f)\chi_\nu(t) \Big| |\overline{\chi_\nu(t)}| dt$$

for $k = m_n + r(p_{n+1} - 1) + s$. □

Lemma 2. *Let* $f_0(t) = \chi_{n,0}^{(1)}(t)$, $t \in [0,1]$. *Then*

$$|f_0(t)| = \frac{1}{\sqrt{m_n}} \sum_{k=0}^{n-1} \sum_{s=1}^{p_{k+1}-1} \chi_{k,0}^{(s)}(t)\sqrt{m_k}.$$

Proof. Let $m = m_k + r(p_{k+1}-1) + s$, $r = 0,1,\dots,m_k-1$; $s = 1,2,\dots,p_{k+1}-1$. Calculate coefficients $a_m(|f_0|)$. By the definition of the Fourier coefficient and the function $\chi_{n,0}^{(1)}(t) = f_0(t)$ we have

$$a_m(|f_0|) = \int_0^1 |f_0(t)|\overline{\chi_m(t)}dt = \sqrt{m_n}\int_0^{\frac{1}{m_n}} \overline{\chi_m(t)}dt. \tag{4}$$

Since $\chi_1(t) \equiv 1$, taking into account equality (4) (with $m = 1$), we obtain

$$a_1(|f_0|) = \int_0^1 |f_0(t)|\overline{\chi_1(t)}dt = \int_0^1 |f_0(t)|dt = \sqrt{m_n}\int_0^{\frac{1}{m_n}} dt = \frac{1}{\sqrt{m_n}}.$$

Let $k < n$. Then $\frac{1}{m_n} < \frac{r}{m_k}$ $\forall r = 1,2,\dots,m_k-1$, i.e., $(0,1) \cap \left(\frac{r}{m_k}, \frac{r+1}{m_k}\right) = \emptyset$ $\forall r = 1,2,\dots,m_k-1$. Therefore, equality (4) gives $a_m(|f_0|) = a_{k,r}^{(s)}(|f_0|) = 0$ $\forall r = 1,2,\dots,m_k-1$; $s = 1,2,\dots,p_{k+1}-1$.

But if $r = 0$, then $\left(0, \frac{1}{m_n}\right) \subset \left(0, \frac{1}{m_k}\right)$. Since $\frac{1}{m_n} < \frac{1}{m_{k+1}}$ $\forall k = 1,2,\dots,n-1$, we have $\overline{\chi_{k,0}^{(s)}(t)} = \sqrt{m_k}$, $t \in \left(0, \frac{1}{m_n}\right)$. Therefore from (4) we get

$$a_{k,r}^{(s)}(|f_0|) = \sqrt{m_n}\sqrt{m_k}\int_0^{\frac{1}{m_n}} dt = \frac{\sqrt{m_k}}{\sqrt{m_n}}$$

for $s = 1,2,\dots,p_{k+1}-1$.

Thus, if $k < n$, then

$$a_m(|f_0|) = a_{k,r}^{(s)}(|f_0|) = \begin{cases} \sqrt{\frac{m_k}{m_n}}, & r = 0, \quad s = 1,2,\dots,p_{k+1}-1; \\ 0, & r = 1,2,\dots,m_k-1, \quad s = 1,2,\dots,p_{k+1}-1. \end{cases} \tag{5}$$

Let $k = n$, i.e., $m = m_n + r(p_{n+1}-1) + s$. Then the support of the function $\overline{\chi_m(t)}$ does not intersect $\left(0, \frac{1}{m_n}\right)$ for $r = 1,2,\dots,m_n-1$. Then (see (4)) for $r = 1,2,\dots,m_n-1$; $s = 1,2,\dots,p_{n+1}-1$,

$$a_m(|f_0|) = a_{n,r}^{(s)}(|f_0|) = 0.$$

If $r = 0$, then $[0, \frac{1}{m_n}]$ is the support of the function $\overline{\chi_{n,0}^{(0)}(t)}$. Therefore, due to the orthogonality we have (see (4))

$$a_m(|f_0|) = a_{n,r}^{(s)}(|f_0|) = 0 \ \ \forall s = 1,2,\dots,p_{n+1}-1.$$

Thus, if $k = n$, then

$$a_m(|f_0|) = a_{n,r}^{(s)}(|f_0|) = 0 \tag{6}$$

$\forall r = 1,2,\dots,m_n-1$; $s = 1,2,\dots,p_{n+1}-1$.

Let $k > n$. Then $\left(0, \frac{1}{m_k}\right) \subset \left(0, \frac{1}{m_n}\right)$. Hence, $\frac{1}{m_n} = \frac{p_{n+1}\cdots p_k}{m_k} \le \frac{r}{m_k}$ $\forall r \ge p_{n+1},\dots,p_k$, i.e., $(0, \frac{1}{m_n}) \cap (\frac{r}{m_k}, \frac{r+1}{m_k}) = \emptyset$. Consequently (see (4)),

$$a_m(|f_0|) = a_{k,r}^{(s)}(|f_0|) = 0 \ \ \forall r \ge p_{n+1}\cdots p_k, \ \ \forall s = 1,2,\dots,p_{k+1}-1.$$

If $r = 0,1,\dots,p_{n+1}\cdots p_k - 1$, then $\frac{r+1}{m_k} \le \frac{1}{m_n}$. Therefore in view of the definition of the function $\chi_m(t)$ we have (see (4))

$$a_m(|f_0|) = \int_{\frac{r}{m_k}}^{\frac{r+1}{m_k}} \overline{\chi_{k,r}^{(s)}}(t)dt = 0, \quad r = 0,1,\dots,p_{n+1}\cdots p_k - 1; \quad s = 1,2,\dots,p_{k+1}-1,$$

i.e.,

$$a_m(|f_0|) = a_{k,r}^{(s)}(|f_0|) = 0 \tag{7}$$

with $k > n \ \ \forall r = 0, 1, \ldots, m_k - 1, \forall s = 1, 2, \ldots, p_{k+1} - 1$.

Let $k = 0$, then $m = m_k + r(p_{n+1} - 1) + s = 1 + s$ and

$$a_m(|f_0|) = a_{0,0}^{(s)}(|f_0|) = \sqrt{m_n} \int_0^{\frac{1}{m_n}} \overline{\chi_{0,0}^{(s)}}(t)dt = \frac{1}{\sqrt{m_n}}. \tag{8}$$

Equalities (5)–(8) give

$$|f_0(t)| = \frac{1}{\sqrt{m_n}} \sum_{k=0}^{n-1} \sum_{s=1}^{p_{k+1}-1} \sqrt{m_k} \overline{\chi_{k,0}^{(s)}}(t). \qquad \square$$

Lemma 3. *Let $\omega(\delta)$ be a module of continuity. If for some $C_\omega \geq 1$ the following inequality is fulfilled:*

$$A_\omega(|f|) \leq C_\omega A_\omega(f) \ \ \forall f \in L_\infty, \tag{9}$$

then

$$\int_0^1 \frac{\omega(t\delta)}{t} dt \leq C\omega(\delta)$$

for any $\delta > 0$.

Proof. Let inequality (9) be fulfilled. Consider the function $f_1 = \delta\chi_{n,0}^{(1)}(t)$, $t \in [0, 1]$ and $\forall \delta > 0$. Then in view of Lemma 2 and inequality (9) we have

$$A_\omega(|f_1|) = \sum_{k=0}^{n-1} \sum_{s=1}^{p_{k+1}-1} \omega\left(\delta \frac{\sqrt{m_k}}{\sqrt{m_n}}\right) \leq C_\omega \omega(\delta). \tag{10}$$

Since $\omega(t)$ is non-decreasing, we get

$$\int_{\sqrt{\frac{m_{k-1}}{m_n}}}^{\sqrt{\frac{m_k}{m_n}}} \frac{\omega(t\delta)}{t} dt \leq \omega\left(\delta \frac{\sqrt{m_k}}{\sqrt{m_n}}\right) \left[\ln \sqrt{\frac{m_k}{m_n}} - \ln \sqrt{\frac{m_{k-1}}{m_n}}\right]$$
$$= \omega\left(\delta \frac{\sqrt{m_k}}{\sqrt{m_n}}\right) \ln \sqrt{p_k} = \frac{1}{2}\omega\left(\delta \frac{\sqrt{m_k}}{\sqrt{m_n}}\right) \ln(p_k)$$

for $k = 1, 2, \ldots, n$.

Summing these inequalities and taking into account the boundedness of $\{p_n\}$ and (10), we obtain

$$\int_{\frac{1}{\sqrt{m_n}}}^1 \frac{\omega(t\delta)}{t} dt \leq \frac{1}{2} \sum_{k=1}^n (\ln p_k) \omega\left(\delta \frac{\sqrt{m_k}}{\sqrt{m_n}}\right) \leq \frac{1}{2} \ln C_0 \left[\sum_{k=0}^{n-1} \omega\left(\delta \frac{\sqrt{m_k}}{\sqrt{m_n}}\right) + \omega(\delta)\right]$$
$$\leq \frac{1}{2} \ln C_0 \left[C_\omega \omega(\delta) + \omega(\delta)\right] = \frac{1}{2} \ln C_0 \left[C_\omega + 1\right] \omega(\delta).$$

Proceeding to the limit with $n \to +\infty$ in this inequality, we obtain the assertion of the lemma.

Theorem 1. *Let $\omega(\delta) \in \Omega$; assume that the function $\tau(t)$ is almost increasing on $[1, +\infty)$ with constant C. Then any functions $f \in L$ and $\varphi \in \mathrm{Lip}_D\, 1$ satisfy the inequality*

$$A_{\omega,\tau}(\varphi(f)) \leq C \sum_{\nu=0}^{\infty} \sum_{k=m_\nu+1}^{m_{\nu+1}} \sum_{n=0}^{\nu} (p_{n+1} - 1) \cdot \tau(m_{n+1}) \omega(D\sqrt{m_n}|a_k(f)|m_\nu^{-\frac{1}{2}}).$$

Proof. By the condition of the theorem, $\varphi \in \mathrm{Lip}_D 1$. Therefore, applying Lemma 1 and the properties of the module of continuity of the function $\tau(t)$, we obtain

$$A_{\omega,\tau}(\varphi(f)) = \sum_{n=0}^{\infty}\sum_{r=0}^{m_n-1}\sum_{s=1}^{p_{n+1}-1} \tau(m_n + r(p_{n+1}-1)+s)\omega\left(|a_{n,r}^{(s)}(\varphi(f))|\right)$$
$$\le \sum_{n=0}^{\infty}\sum_{r=0}^{m_n-1}\sum_{s=1}^{p_{n+1}-1} \tau(m_n + r(p_{n+1}-1)+s)\omega\left(D\sqrt{m_n}\int_{\frac{r}{m_n}}^{\frac{r+1}{m_n}}\left|\sum_{k=m_n+1}^{\infty} a_k(f)\chi_k(t)\right|dt\right)$$
$$\le C\sum_{n=0}^{\infty}\sum_{r=0}^{m_n-1}(p_{n+1}-1)\tau(m_{n+1})\sum_{k=m_n+1}^{\infty}\omega\left(D\sqrt{m_n}\,|a_k(f)|\int_{\frac{r}{m_n}}^{\frac{r+1}{m_n}}|\chi_k(t)|dt\right). \quad (11)$$

Since $k > m_n$, the support of the function $\chi_k(t)$ belongs to only one of segments $\left[\frac{r}{m_n}, \frac{r+1}{m_n}\right]$. Therefore with fixed $k = m_\nu + 1, \dots, m_{\nu+1}$,

$$\sum_{r=0}^{m_n-1}\omega\left(D\sqrt{m_n}\,|a_k(f)|\int_{\frac{r}{m_n}}^{\frac{r+1}{m_n}}|\chi_k(t)|dt\right) = \omega\left(D\sqrt{m_n}\,|a_k(f)|\int_{\frac{r}{m_n}}^{\frac{r+1}{m_n}}|\chi_k(t)|dt\right)$$
$$= \omega(D\sqrt{m_n}\,|a_k(f)|\,m_\nu^{-\frac{1}{2}}).$$

Consequently, from (11) we obtain

$$A_{\omega,\tau}(\varphi(f)) \le \sum_{n=0}^{\infty}(p_{n+1}-1)\,\tau(m_{n+1})\sum_{k=m_n+1}^{\infty}\omega(D\sqrt{m_n}\,|a_k(f)|\,m_\nu^{-\frac{1}{2}})$$
$$= \sum_{\nu=0}^{\infty}\sum_{k=m_\nu+1}^{m_{\nu+1}}\sum_{n=0}^{\nu}(p_{n+1}-1)\tau(m_{n+1})\omega(D\sqrt{m_n}\,|a_k(f)|\,m_\nu^{-\frac{1}{2}}). \qquad \square$$

Theorem 2. *Let $\omega(\delta)$ be a module of continuity. The inequality*

$$A_\omega(\varphi(f)) \le C_\omega A_\omega(f) \quad (12)$$

is fulfilled with all $f \in L$ and all $\varphi \in \mathrm{Lip}_D 1$ for some positive constant C_ω if and only if the function $\omega(\delta)$ satisfies the Bari condition.

Proof. Let $\omega(\delta)$ satisfy the Bari condition. By Theorem 1 with $\tau = 1$ we have

$$A_\omega(\varphi(f)) \le C\sum_{\nu=0}^{\infty}\sum_{k=m_\nu+1}^{m_{\nu+1}}\sum_{n=0}^{\nu}\omega(D\sqrt{m_n}|a_k(f)|m_\nu^{-\frac{1}{2}}). \quad (13)$$

For $k = m_\nu + 1, \dots, m_{\nu+1}$ we estimate the following sum from above:

$$\sum_{n=0}^{\nu}\omega(D\sqrt{m_n}|a_k(f)|m_\nu^{-\frac{1}{2}}).$$

Taking into account the property of the module of continuity and the inequality $p_n \ge 2$ for all $n = 1, 2, \dots$, we get

$$\int_{\sqrt{\frac{m_n}{m_\nu}}}^{\sqrt{\frac{m_{n+1}}{m_\nu}}}\omega(D|a_k(f)|t)\frac{dt}{t} \ge \omega\left(D|a_k(f)|\sqrt{\frac{m_n}{m_\nu}}\right)\int_{\sqrt{\frac{m_n}{m_\nu}}}^{\sqrt{\frac{m_{n+1}}{m_\nu}}}\frac{dt}{t}$$
$$= \omega\left(D|a_k(f)|\sqrt{\frac{m_n}{m_\nu}}\right)\ln\sqrt{p_{n+1}} \ge \frac{\ln 2}{2}\omega\left(D|a_k(f)|\sqrt{\frac{m_n}{m_\nu}}\right), \quad n = 0, 1, \dots, \nu-1.$$

We sum these inequalities and obtain

$$\sum_{n=0}^{\nu}\omega(D\sqrt{m_n}|a_k(f)|m_\nu^{-\frac{1}{2}}) \le \frac{2}{\ln 2}\sum_{n=0}^{\nu-1}\int_{\sqrt{\frac{m_n}{m_\nu}}}^{\sqrt{\frac{m_{n+1}}{m_\nu}}}\omega(D|a_k(f)|t)\frac{dt}{t}+\omega(D|a_k(f)|)$$
$$=\frac{2}{\ln 2}\int_{\frac{1}{\sqrt{m_\nu}}}^{1}\omega(D|a_k(f)|t)\frac{dt}{t}+\omega(D|a_k(f)|). \quad (14)$$

Now, changing the variable $u = D|a_k(f)|t$ and using the B-condition, we get

$$\int_{\frac{1}{\sqrt{m_\nu}}}^{1}\omega(D|a_k(f)|t)\frac{dt}{t}=\int_{\frac{D|a_k(f)|}{\sqrt{m_\nu}}}^{D|a_k(f)|}\omega(u)\frac{du}{u}\le C\omega(D|a_k(f)|).$$

Consequently, from (14) we obtain

$$\sum_{n=0}^{\nu}\omega(D\sqrt{m_n}|a_k(f)|m_\nu^{-\frac{1}{2}}) \le C_1\omega(D|a_k(f)|).$$

Therefore inequality (13) gives

$$A_\omega(\varphi(f)) \le CC_1\sum_{\nu=0}^{\infty}\sum_{k=m_\nu+1}^{m_{\nu+1}}\omega(D|a_k(f)|) = CC_1\sum_{k=2}^{\infty}\omega(D|a_k(f)|) \le CC_1A_\omega(f).$$

Inequality (12) is proved.

Assume that (12) is fulfilled. In particular, it is true for the function $\varphi(z) = |z|$, i.e.,

$$A_\omega(|f|) \le CA_\omega(f).$$

Consequently, due to Lemma 3 the function $\omega(\delta)$ satisfies the Bari condition. □

Theorem 3. 1) *Let a generalized Haar system $\chi\{p_n\}$ be defined by a bounded sequence $\{p_n\}$. If $\omega(\delta) \in \Omega$ satisfies the Bari condition on $[0,1]$, then*

$$A_\omega(f^2) \le C_\omega A_\omega(f) \quad (15)$$

for any functions $f \in A_\omega^\infty$ with $\|f\|_\infty = 1$.

2) *If the system $\chi\{p_n\}$ is defined by numbers $p_n = p$, $n = 1, 2, \ldots$, then for the fulfillment of inequality (15) it is necessary that the function ω should satisfy the Bari condition.*

Proof. One can prove the first item just in the same way as in theorem 1 from [9], using Theorem 2. For the proof of the second item one should use Lemma 2 and repeat the reasoning of the necessity proof in theorem 1 in [9].

Remark. In the case when the system $\chi\{p_n\}$ is defined by an unbounded sequence $\{p_n\}$, the Fourier series of the function $f \in L_1$ does not necessarily converge a. e. (see [14]). However, if we consider the subset S_L that represents the set of all functions $f \in L_1$ for which the Fourier series by the system $\{\chi p_n\}$ converges a. e. on [0,1], then the assertion of Theorem 1 remains valid.

Note that the above assertions for the Fourier–Haar series were proved by P. L. Ul'yanov [8–13]. In the case $\tau = 1$ Theorem 1 is proved in [15]. The absolute convergence a. e. of the Fourier–Haar series of a superposition of two functions were studied by V. M. Bugadze [16].

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