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THE TWO-LEVEL CORRELATION FUNCTION AND THE FORM-FACTOR

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The disordered quantum systems are considered in the one-electron approximation. The discrete spectrum is calculated using the Anderson model for orthogonal and unitary symmetry. The electron spectral fluctuations are studied in terms of the two-level correlation function and the form-factor. The universality at the metal-insulator transition is demonstrated. It is shown that asymptotic of the two-level correlations at large energies vanish faster as expected.

Keywords: quantum systems, two-level correlation function, metal-insulator transition, form-factor.

Introduction

There is a well known relation between the electronic transport in the nanostructured materials and their spectral statistical properties. For example, the conductance of a disordered metallic or semiconductor sample exhibits fluctuation nature at the low temperatures. At the same time, good-resolved quasi-discrete energy levels of the electron spectrum follow some chaotical behavior obeying similar statistical laws. From the theoretical point of view the conductivity, as a thermodynamical quantity, is usually described in terms of the two-point correlation function or the linear response. It is imperative therefore to put forward an analogous theoretical approach towards the statistics of the electron spectrum and investigate the spectral correlations along the energy scale, instead of in space.

The advantage of this approach is supported by the fact, that both- the conductance statistics and the spectral statistics undergo the sharp crossover exactly at critical point of the disorder-induced metal-insulator transition. By other words, both quantities can be served as indicators of the mobility edge. Their dependence on the sample size can be treated by the finite-size scaling analysis to be conventionally applied for all standard critical phenomena, where the correlation length diverges.

Despite that the relation between averaged values of the conductance and the energy level sensitivity with respect to the boundary conditions has been established, still many question remain open: what is the role of the low-order correlations of the electron spectrum? Whether or not are they responsible for the multi-fractal nature of the corresponding electron wave functions?

One of the important statistical measures, which characterizes the spectral correlations is the two level-density correlation function

$$R(s) = \frac{\langle \rho(\varepsilon + s)\rho(s) \rangle}{\rho_0^2}, \qquad \rho_0 = \frac{1}{\Delta L^d}. \tag{1}$$

Here the angular brackets denote averaging over various realizations of the random potential and ρ is the averaged density of states. The autocorrelation term $\delta(s)$ is properly taken into account. The explicit expressions for $R(s)$ for the conventional symmetry classes of the random matrix theory are well known [1]. As expected from mentioned above, they can be applied for describing the spectral correlations in a disordered metallic sample with the dimensionless conductance g that is much than unity (the ergodic limit). For instance at large s, the correlations fall off with increasing energy s according to the power law $R(s) \sim (\pi s)^{-2}$.

At the diffusive regime (e.g. for finite g) when the energy exceeds the Thouless energy (a typical energy scale of the problem) $s > E_T \sim g\Delta$ the correlation function decays as follows $R(s) \sim s^{d/2-2}$. This result has been found by using the perturbation impurity method [2]. One should note

that various non-perturbation approaches provide additional oscillating terms [3]. In the insulating regime, no level correlations exist and, therefore, *R(s) = 0*, since the electron states are strongly localized. As in the cases of both the level number variance and the level spacing distribution function *P(s)*, widely considered in the literature, it is interesting to study behaviour of *R(s)* at the Anderson transition. It should be noted that, in contrast to *R(s)* the spacing distribution function *P(s)* contains information about all higher-order level correlations, whereas the level number variance is completely defined by the two-level correlation function *R(s)* via the direct relation:

$$\left\langle \delta N^2(E) \right\rangle = \int_{-E}^{E} (E - |s|) R(s) ds = E - 2\int_{0}^{E} R(s) ds. \qquad (2)$$

We have computed the correlations in spectra of 3D systems with linear sizes ranging from L = 8 to 20 at $W = Wc$ = 16,5 for large number of realizations of the random potential. For L = 12 the total number of realizations was $6 \cdot 10^5$. Numerically, it is easier to calculate *R(s)* by using direct relation to the next-next level spacing distribution *p(n, s)*, by using:

$$R(s) = 1 - \sum_{n_l=0}^{n_u=\infty} p(n,s), \qquad p(0,s) = P(s), \qquad (3)$$

where *p(n, s)* is the distribution functions of the separations s between the levels ε_{i+n+1} and ε_i.

Orthogonal Symmetry

Figure 1 shows the results of the critical two-level correlation function *R(s)* for the orthogonal symmetry. In the calculations the upper limit n_u = 100 of Eq. (2) is assumed. This value is sufficient for a reliable convergence of the sum within the given accuracy. As expected, our numerical results convincingly show that the empirical *R(s)* at the disorder-induced metal-insulator transition is a scale-invariant and it differs from the both results of the Gaussian Orthogonal Ensemble (GOE) and Poisson limits. The former reflects the universality of spectral statistics at the mobility edge; the latter signalizes a presence a new fundamental statistics, characteristic of the critical phenomena.

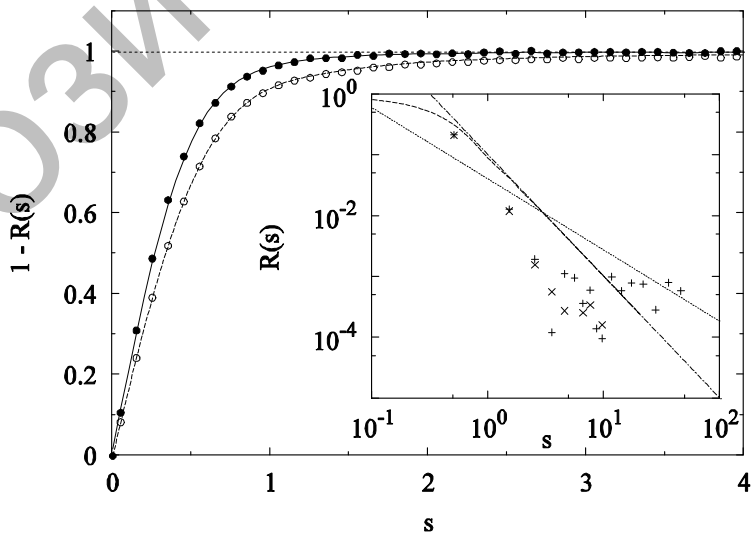

Fig. 1. Two-level correlation function R(s) at the critical disorder W = 16.5 for linear sizes L = 12 by the solid line and L = 20 (•). Dashed line: GOE result, short-dashed line: Poisson limit, *R(s)* = 0. Inset: large-*s* behaviour of *R(s)* for L = 8 (+) and 12 (x); dashed line: exact GOE result for R(s); dashed-dotted line: GOE asymptotic $R(s) \sim (\pi s)^{-2}$. Eq. (4) with α = 1,2 is drawn by dotted line for comparison.

One of the particular significance is the large-*s* behaviour of the critical *R*(*s*). It has been suggested [4] that the correlation function at the metal-insulator transition depends only on dimensionality *d* and the symmetry class (in our case orthogonal and unitary classes). In the asymptotic region, where s is much larger than 1, the authors of [4] on the basis of self-consistent considerations have obtained,

$$R(s) = \frac{A}{s^{\alpha}}, \qquad \text{with} \qquad \alpha = 1 + \frac{1}{\nu d} \tag{4}$$

A famous Harris inequality $\nu d > 2$ defines a simple upper bound for $\alpha < \alpha_u = 3/2$. From the available scaling result for the localization length critical exponent in the 3D orthogonal case $\nu = 1.5$ one obtains $\alpha = 1{,}2$ in 3D. While previous numerical calculations [5] were in agreement with the proposed power law yielding $\alpha = 1.17$, our present high-precision results do not confirm the above analytical expression Eq. (4). Despite that the system size $L = 12$ and number of realizations are considerably larger than those in previous calculations, thus giving a better relative accuracy, our results for the tail of *R(s)* have not been found to be reliably consistent with the proposed power law Eq.(4). See the inset of Figure 1.

At the first approximation one can take a few points between *s*=1 and *s*=10 and interpolate them with the power law. The best fit would yield $\alpha \approx 4.6$ meaning a strong deviation from both the GOE-result and the power law Eq. (4). Further the data of computer simulations fluctuate very strongly when *s* becomes larger ($s > 10$), so that no definite conclusion about the asymptotic behaviour could be drawn.

We note here in passing that the statistical uncertainties of the simulated data are controlled by the central limit theorem. This means that the accuracy of the calculated *R(s)* for large *s* is restricted by the total number *N* of the computed eigenvalues: $\delta R(s)/R(s) \sim 1/N^{1/2}$. Obviously, there is a certain maximal value of $s_{\max}$ up to which a fitting procedure for the asymptotic form is meaningful. A simple estimate yields $s_{\max} \approx 10$ -12 for the total number of eigenvalues $N = 10^8$. Any interpolation beyond this value does not make any sense.

Even though one could interpolate the decay of 1 - *R(s)* only in the region $s < 10$ by the power law, the fitted value of $\alpha = 2.3$ deviates strongly from the expected theoretical estimate $\alpha = 1+1/(3\nu)$ and is obviously in disagreement with the Harris criterion. In this view the conclusion about the agreement between analytical expression Eq. (4) and previous numerical results [5,6] seem to be a bit optimistic.

Unitary symmetry

Now we consider the unitary class of symmetry and calculate the two-point correlation function by direct diagonalization of the Anderson model with a constant magnetic field. The Hamiltonian is complex due to inclusion of the phase of the electron collected while travelling across the lines of the magnetic field. This is in contrast to the previous case with the orthogonal symmetry. This leads to an additional degree of freedom. As a consequence, the growth of 1-*R(s)* at small *s* is more suppressed. Indeed, one observes a quadratic increase at small energies *s*, while without magnetic field the two-level correlation function grows linearly.

Figure 2 shows the numerical results of the critical two-level correlation function for the unitary ensemble. For comparison the data for the metallic regime (*W*=5) are also provided, which fully obey the predictions of the random matrix theory. In contrast to the level number variance and to the form-factor described in the following section we do not concentrate here on the quantitative analysis of weak corrections to the ergodic expressions in the limit of large, but finite conductance *g*.

As expected, *R*(*s*) for the unitary ensemble at the critical point of the Anderson transition is scale-invariant and markedly differs from both the RMT and the Poissionian limits. In the unitary

case the shape of the critical $R(s)$ depends not only on the magnitude, but also on the number of components of the external Aharonov- Bohm flux.

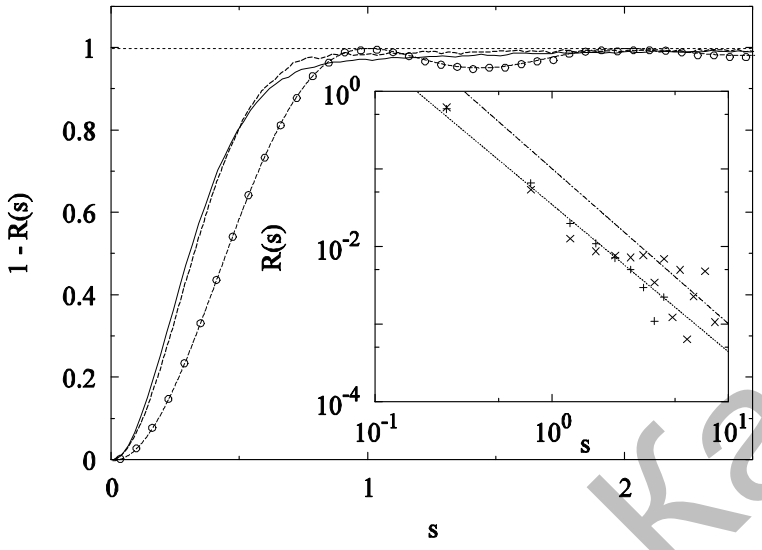

Fig. 2. Two-level correlation function $R(s)$ of the unitary symmetry at the critical disorder $W = 16.5$ for linear sizes $L = 12$ for the one-component (thin line) and the three-component (thick line) Aharonov-Bohm flux $\varphi = 1/4$; and at the 'metallic' disorder $W = 5$ (°) for the three-component flux. Dashed line: GUE result $R(s) = \sin^2 \pi s/(\pi s)^2$, short-dashed line: Poisson limit, $R(s) = 0$. *Inset*: large-s behaviour of the critical $R(s)$ for one- (+) and two-component (x) flux; dashed-dotted line: overall GUE asymptotic $R(s) = (\pi s)^{-2}$. Dotted line: power-law $R(s) = cs^{-\alpha}$ with fitted constant $c = 0.035$ and $\alpha = 1.9$.

One can see in Fig. 3 a slight difference between results for systems in which the T-invariance is broken by the one-component and the three-component flux of the same magnitude $\varphi = 0.25$. For small energies the short-range correlations are entirely controlled by $P(s)$, i.e. $1 - R(s) = Bs^{\beta}$ for $s << \Delta$. This is valid for arbitrary disorder, including the transition point W_c, because the basic symmetry is decoupled from the dimensionality d.

We have also studied the asymptotic behaviour of the two-level density correlation function in the unitary case (see the inset of Figure 3). One observes that the decay of the critical $R(s)$ is somewhat slower, than in the orthogonal case, thereby facilitating our calculations. Although the data for $s < 10$ look like to follow a power-law, the obtained ν is far from being correct. Indeed, the best fit results to $\alpha = 1.9 \pm 0.5$ with a prudent accuracy. This means if the prediction Eq. (4) were correct, the critical exponent were about $\nu \simeq 0.4 \div 1.0$, while high-precision findings for the unitary case [6] yield $\nu = 1.45 \pm 0.03$. The results for the symplectic symmetry, where the spin-orbit coupling plays an important role, are not known yet.

Despite that the system size $L = 12$ and number of realizations are considerably larger than in previous calculations in the literature, thus leading to a better accuracy, our results for the tail of $R(s)$ for the both symmetries have not been found to be reliably consistent with the existing analytical proposal [3]. The exact asymptotic behaviour of the correlation function at the Anderson transition has to be considered, in our opinion, as an open problem and deserves a serious numerical check.

The form-factor

Another important entity for investigating the level statistics is the spectral form-factor, which gives insight into the dynamics of disordered electrons and provides a deeper understanding of the

third universal (critical) statistics. By definition, the *spectral two-level form factor* is the energy Fourier transform of $R(s)$,

$$S(t) = \int_{-E}^{E} R(s) \exp(2\pi i s t) ds, \qquad s = \left|\varepsilon_i - \varepsilon_j\right|, \tag{5}$$

The exact expressions of the form-factor in the RMT are known for all three universality classes. For instance, for the GUE one has, leaving aside the δ-function term in Eq. (1):

$$S_{GUE}(t) = \begin{cases} |t| - 1, & |t| \le 1 \\ 0, & |t| \ge 1. \end{cases} \tag{6}$$

The point $t = 1$ corresponds to the Heisenberg time $\tau_H = h/\Delta$, where the form factor has a singularity (the divergence starting from the second derivative). According to the semiclassical formalism [7], the form-factor can be associated with the probability $B(t)$ for a diffusing electron to return to the initial point within time interval t. In the metallic regime for times much smaller than τH ($t << 1$) one has

$$S(t) = \frac{|t| P(t)}{2\pi}, \tag{7}$$

In a finite system beside the Heisenberg time τ_H there are three additional important time scales:

the Thouless time $\tau_T = L^2/D$, with the diffusion constant D;

the elastic scattering time τ_{sc} (for metals $\tau_{sc} << \tau_T$); and

the shortest time of the problem $\tau_F = h/\epsilon_F$, the Fermi time with the Fermi energy $/\epsilon_F$.

For times larger than the Thouless time τ_T the electrons diffuse homogeneously all over the sample, therefore, $B(t)$ is constant and $S(t)$ grows linearly with t. For times shorter than τ_T the time dependence of the classical return probability is defined by the dimensionality d of the system $B(t) = (Dt)^{-d/2}$. Then the form-factor behaves as follows

$$S(t) = \frac{1}{|t|^{d/2-1}}, \tag{8}$$

that is consistent with the perturbation result. It is clear that for $d = 3$ there is a time domain $\tau_{sc} < \tau < \tau_T$, where $S(t)$ decreases as $1/\sqrt{|t|}$ when t increases according to Eq.(8).

Here we focus our attention on the critical unitary symmetry with the maximal magnetic flux. Figure 3 shows the results of numerical calculations of the form-factor for the linear size $L = 12$ at the fixed three-component AB-flux $\varphi = 1/4$. The data have been averaged over an ensemble of large number of samples. One observes that the "knee" of $S(t)$ at the Heisenberg time $t = 1$ is washed out at nonzero disorder. The whole function changes from the GOE limit Eq. (69) to the Poissonian $S(t) = 0$.

At weak disorder W, the Thouless time τ_T separates the ergodic and the diffusive regimes. For smaller times, $t < \tau_T/\tau_H$, we observe for metallic samples ($W \le 14$) a pronounced decrease of $S(t)$. It corresponds to the diffusive regime, although the detailed quantitative comparison with $1/\sqrt{|t|}$-behaviour has to be done. The smallest time scale accessible for the given system's size is $\tau = \tau_F/2 \approx 0.001\tau_H$. After passing minima around τ_T the form-factor $S(t)$ for $t \ge \tau_T/\tau_H$ indeed increases proportionally to t, with negligible weak localization corrections to $S_{GOE}(t)$.

Basically, the positions of the minima yield good estimates of the Thouless time, and consequently the conductance. The fact that the character of the deviation of the form-factor from the ergodic limit in the mesoscopic conductor is governed only by the large, but finite conductance $g = G/(e^2/h) >> 1$ has been established earlier [8]. This analytical consideration is valid only in the limit of large g, where the numerically obtained dependences of g on disorder W and size L of the

electron system corresponds to the weak-localization regime. We however did not observe here the predicted non-perturbation oscillations of $S(t)$ in the region of $t = 1$ [8]. In approaching the point of the Anderson transition one obviously gets a deviation from the weak-localization behaviour.

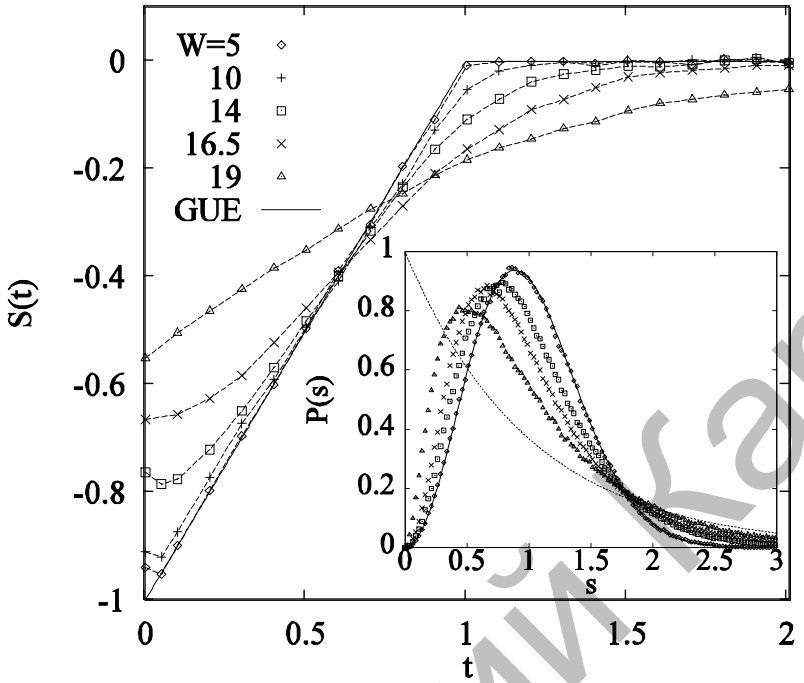

Fig. 3. The form-factor $S(t)$ of the unitary case, $\varphi = 1/4$, for system size $L = 12$ and different disorders W. The GUE-result shown by the solid line corresponds to Eq. (69) and the Poissonian case to $S(t) = 0$. *Inset*: level spacing distribution $P(s)$ for different disorders W. Solid line: exact $P_{\mathrm{GUE}}(s)$; dashed line: Poissonian distribution.

When approaching the critical region ($\xi(W) \geq L$), both the random matrix theory and the perturbation approximations are no longer valid, because the Thouless and the Heisenberg times become of the same order of magnitude, $\tau_T \approx \tau_H$, i. e. $g \simeq 1$, whereas the scattering time τ_{sc} decreases towards τ_F. Precisely at transition $W = W_c$ where the localization length ξ diverges, the data for $S(t)$ do not depend on L. Obviously, this corresponds to the *critical* form-factor and renders the third universal level statistics in the thermodynamic limit observed and discussed above for the two-point correlation function *R(s).*

For comparison, the level spacing distribution $P(s)$ for the same values of W is shown in the inset of Fig. 3. The disorder-driven crossover of $P(s)$ from $P_{\mathrm{GUE}}(s)$ to $P_{\mathrm{P}}(s)$ at the fixed non-zero value of the magnetic flux φ is similar to that for the orthogonal symmetry. No common crossing point s_0 has been found.

Conclusions

By linking spectral fluctuations with the spatial correlations of the eigenstates, a modified form for the spectral form-factor has been derived [9]. As a consequence, the new relationship between the spectral compressibility κ and the fractal dimension $D2$ characterizing the scaling of the inverse participation ratio of the wave functions can be proposed

$$1 - S(0) = \kappa = \frac{d - D_2}{2d}, \tag{9}$$

The latter expression, albeit being obtained under severe simplifications, provides a fundamental message, that at the metal-insulator transition the spectral rigidity in terms of the level

number fluctuations [10] contains the information about the multi-fractal nature of the critical wave functions. This defines an important practical tool for numerical analysis of transport properties at the Anderson transition without requiring the 'time-and-memory consuming' knowledge of the exact eigenfunctions [11] and/or the conductivity. A detailed quantitative comparison of the analytical prediction Eq.(9) with the independently simulated D_2 for various dimensions as well as for different universality classes would be desirable.

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