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FEATURES OF THE JOINT SOLUTION OF DEGENERATE HYPERGEOMETRIC SYSTEMS AND BESSEL-TYPE SYSTEMS

Tasmambetov Zh.N.[1], Ubayeva Zh.K.[2]

[1]M.Utemisov West Kazakhstan University, Oral, Kazakhstan
[1]E-mail: tasmam@rambler.ru
[2]Aktobe Regional University named after K. Zhubanov, Aktobe, Kazakhstan
[2]E-mail: Zhanar_ubaeva@mail.ru

Introduction. In the theory of degenerate hypergeometric functions of two variables, an important place is occupied by the study of the properties of 20 degenerate hypergeometric functions of two variables [1]. These were obtained from four Appell $F_1 - F_4$ functions [2] by limiting transitions.

The Italian mathematician Lauricella (1890) constructed systems of n equations: (F_A), (F_B), (F_C) and (F_D) and constructed their solutions F_A, F_B, F_C and F_D in the form of generalized power series in n variables [3].

In this paper, the subject of our study is degenerate hypergeometric systems

$$z_i\left(1-z_i\right)\frac{\partial^2 W}{\partial z_i^2}+\sum_{j=1,j\neq i}^{n} z_j\frac{\partial^2 W}{\partial z_i\partial z_j}+\left[\gamma-\left(\alpha_i+\beta_i+1\right)z_i\right]\frac{\partial W}{\partial z_i}-\alpha_i\beta_i W=0, \left(i=\overline{1,k}\right) \quad (1)$$

$$\sum_{j=1}^{n} z_j\frac{\partial^2 W}{\partial z_j\partial z_i}+\left(\gamma-z_i\right)\frac{\partial W}{\partial z_i}-\alpha'_{i-k}W=0, \quad \left(i=k+1,\ldots,k+l\right) \quad (2)$$

$$\sum_{j=1}^{n} z_j\frac{\partial^2 W}{\partial z_j\partial z_i}+\gamma\frac{\partial W}{\partial z_i}-W=0, \left(i=\overline{k+l+1,n}\right) \quad (3)$$

obtained by passing to the limit from the Lauricella system (F_B).

Studying the degenerate hypergeometric system (1)-(3), V.I. Khudozhnikov introduced a new function:

$$\Phi_{B,n}^{k,l}\left(\begin{matrix}(\alpha_k), & (\alpha'_l), & (\beta_k)\\ & \gamma & \end{matrix}\middle|\,(z_n)\right)=\sum_{i_1,\ldots,i_n}\frac{\prod_{j=1}^{k}(\alpha_j)_{i_j}(\beta_j)_{i_j}}{(\gamma)_{\sum_{j=1}^{n}i_j}}\prod_{j=k+1}^{k+l}\left(\alpha'_{j-k}\right)_{i_j}\prod_{j=1}^{n}\frac{(z_j)^{i_j}}{i_j!} \quad (4)$$

where the following abbreviations and notations are used [4]:

$$(a)_n=\left(a_1,a_2,\ldots,a_n\right), \quad \prod(\alpha_k)_{i_n}=\prod_{k=1}^{n}(\alpha_k)_{i_k}, \quad \sum_{i_1,\ldots,i_n}=\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\cdots\sum_{i_n=0}^{\infty} \quad (5)$$

(7) is a particular solution of the general system (4)-(6).

It should be noted that each equation of the system (1.1)-(1.3) itself is also a system of degenerate hypergeometric equations having solutions in the form of $\Phi_{B,n}^{k,l}\left(0 \leq k, l \leq n, n-1 \leq k+l \leq n,\right), n$ - number of variables..

The aim of this work is to study the features of the joint solution of these systems, emphasizing especially the connection with the solutions of system (1.3)—the Bessel-type system. To establish a connection between Khudozhnikov's functions and normally-regular solutions of such systems [4].

Further, we will consider various particular cases of degenerate systems and establish connections between their solutions and Bessel functions of one and several variables. If in (1)-(3) we take the values of indices $i = 1$ and $j = 1$, then we obtain three equations:

$$z_1\left(1-z_1\right)\frac{\partial^2 W}{\partial z_1^2}+\left[\gamma-\left(\alpha_1+\beta_1+1\right)z_1\right]\frac{\partial W}{\partial z_1}-\alpha_1\beta_1 W=0 \tag{6}$$

$$z_1\frac{\partial^2 W}{\partial z_1\partial z_1}+\left(\gamma-z_1\right)\frac{\partial W}{\partial z_1}-\alpha W=0 \tag{7}$$

$$z_1\frac{\partial^2 W}{\partial z_1\partial z_1}+\gamma\frac{\partial W}{\partial z_1}-W=0 \tag{8}$$

Their solutions are the well-known Gauss and Kummer functions and functions that can be reduced to the Bessel function [5-7].

Property 1.1. The system consisting of equation (1.6) and equations obtained by limiting transitions from the Gauss and Kummer equations are compatible, since they have a common solution $J\left(\gamma, z_1\right)$.

The compatibility property must always be considered.

The transformation

$$W = z_1^{-\nu} J_\gamma\left(z_1\right)$$

reduces Equation (6) to the Bessel equation and the equality holds:

$$J_\gamma\left(z_1\right)=\frac{\left(\frac{z_1}{2}\right)^\gamma}{\Gamma\left(\gamma+1\right)}J\left(\gamma+1;-\frac{z_1^2}{2^2}\right), \tag{9}$$

which establishes the connection between the degenerate hypergeometric function $J\left(\gamma, z_1\right)$ and the Bessel function $J_\gamma\left(z_1\right)$. Expanding the right-hand side of (9), after some transformations, we obtain the Bessel function of the first kind in the form:

$$J_\gamma\left(z_1\right)=\left(\frac{z_1}{2}\right)^\gamma\frac{1}{\Gamma\left(\gamma+1\right)}J\left(\gamma+1;-\frac{z_1^2}{4}\right)=\frac{\left(\frac{z_1}{2}\right)^\gamma}{\Gamma\left(\gamma+1\right)}\sum_{m_1=0}^{\infty}\frac{(-1)^{m_1}\left(\frac{z_1}{2}\right)^{2m_1}}{m_1!\Gamma\left(m_1+1\right)} \tag{10}$$

Based on the above reasoning, we can conclude that the following statement holds.

Theorem 1.1. The solution of the system consisting of equations (3) and the equations obtained by limiting from the Gauss equations (1) and Kummer (2) are expressed through the Bessel function (10).

2. Properties of solutions of Horn-type systems and their connection with degenerate functions reducing to Bessel functions.

Let us study the properties of solutions of Horn-type systems consisting of two second-order equations. In this case, the functions of M.P. Humbert and V.I. Khudozhnikov play an important role.

Definition 2.1. The Humbert-Khudozhnikov functions are called the function:

$$\Phi_2\left(\alpha_1,\alpha_2;\gamma;z_1,z_2\right)=\Phi_{B,2}^{(0,2)}\left(\begin{matrix}\alpha_1, & \alpha_2\\ \gamma & \end{matrix}\middle|\,(z_1,z_2)\right),\quad \Phi_1\left(\alpha_1;\gamma;z_1,z_2\right)=\Phi_{B,2}^{(0,1)}\left(\begin{matrix}\alpha_1\\ \gamma\end{matrix}\middle|\,(z_2)\right) \tag{11}$$

$$\Xi_1\left(\alpha_1,\alpha_2,\beta_1;\gamma;z_1,z_2\right)=\Phi_{B,2}^{(1,1)}\left(\begin{matrix}\alpha_1, & \alpha_2\\ \gamma & \end{matrix}\middle|\,(z_2)\right),\quad \Xi_2\left(\alpha_1,\beta_1;\gamma;z_1,z_2\right)=\Phi_{B,2}^{(1,0)}\left(\begin{matrix}\alpha_1, & \beta_1\\ \gamma & \end{matrix}\middle|\,(z_2)\right)$$

Humbert functions are special cases of the general Khudozhnikov function $\Phi_{B,n}^{k,l}$ and for $n=2$ the relations (11) hold.

All Horn systems with solutions in the form of function (11) are obtained from the degenerate system (1)-(3) and for X and Y from it we obtain the following three systems in succession:

$$\begin{gathered} z_1\left(1-z_1\right)W_{z_1z_1}+z_2W_{z_2z_1}+\left[\gamma-\left(\alpha_1+\beta_1+1\right)z_1\right]W_{z_1}-\alpha_1\beta_1W=0\\ z_2\left(1-z_2\right)W_{z_2z_2}+z_1W_{z_1z_2}+\left[\gamma-\left(\alpha_1+\beta_1+1\right)z_2\right]W_{z_2}-\alpha_2\beta_2W=0, \end{gathered} \tag{12}$$

$$\begin{gathered} z_1W_{z_1z_1}+z_2W_{z_2z_1}+\left(\gamma-z_1\right)W_{z_1}-\alpha_1W=0,\\ z_2W_{z_2z_2}+z_1W_{z_1z_2}+\left(\gamma-z_2\right)W_{z_2}-\alpha_2W=0 \end{gathered} \tag{13}$$

$$\begin{gathered} z_1W_{z_1z_1}+z_2W_{z_2z_1}+\gamma W_{z_1}-W=0,\\ z_2W_{z_2z_2}+z_1W_{z_1z_2}+\gamma W_{z_2}-W=0. \end{gathered} \tag{14}$$

System (2.2) is the well-known Horn system (F_3), with a solution in the form of the Appell function $F_3\left(\alpha_1,\alpha_2,\beta_1,\beta_2;\gamma;z_1,z_2\right)$. It is a special case of the Lauricella system (F_B) [8-12].

The solutions of system (13) are related to the Humbert-Khudozhnikov functions (11). (14) is a degenerate hypergeometric system reducible to a Bessel-type system. Let us consider some properties of such systems.

Property 2.1. A particular solution

$$W_1(z_1,z_2)=J\left(\gamma_1,z_1\right)J\left(\gamma_2,z_2\right) \tag{15}$$

of the system

$$\begin{gathered} z_1W_{z_1z_2}+\gamma_1W_{z_1}-W=0,\\ z_2W_{z_2z_1}+\gamma_2W_{z_2}-W=0 \end{gathered} \tag{16}$$

btained using limitting transitions from Horn-type systems (F_2) and (F_4), coincide and are represented as a generalized power series of two variables

$$\sum_{m_1,m_2=0}^{\infty}\frac{1}{\left(\gamma_1,m_1\right)\left(\gamma_2,m_2\right)}\frac{z_1^{m_1}}{m_1!}\frac{z_2^{m_2}}{m_2!}=J\left(\gamma_1,z_1\right)\cdot J\left(\gamma_2,z_2\right) \tag{17}$$

Considering the product of two Bessel functions and performing some transformations, we obtain a Bessel function of the first kind in two variables in the form:

$$J_{\gamma_1}(z_1)\,J_{\gamma_2}(z_2)=\left(\frac{z_1}{2}\right)^{\gamma_1}\frac{1}{\Gamma(\gamma_1+1)}J\left(\gamma_1+1;-\frac{z_1^2}{2^2}\right)\cdot\left(\frac{z_2}{2}\right)^{\gamma_2}\frac{1}{\Gamma(\gamma_2+1)}J\left(\gamma_2+1;-\frac{z_2^2}{2^2}\right)=$$

$$=\left(\frac{z_1}{2}\right)^{\gamma_1}\left(\frac{z_2}{2}\right)^{\gamma_2}\sum_{m_1,m_2=0}^{\infty}\frac{(-1)^{m_1+m_2}\left(\frac{z_1}{2}\right)^{2m_1}\left(\frac{z_2}{2}\right)^{2m_2}}{m_1!m_2!\Gamma(m_1+\gamma_1+1)\Gamma(m_2+\gamma_2+1)}==J_{\gamma_1,\gamma_2}(z_1,z_2)$$

Theorem 2.1. The Bessel function of two variables is represented as

$$J_{\gamma_1,\gamma_2}(z_1,z_2)=\left(\frac{z_1}{2}\right)^{\gamma_1}\left(\frac{z_2}{2}\right)^{\gamma_2}\sum_{m_1,m_2=0}^{\infty}\frac{(-1)^{m_1+m_2}\left(\frac{z_1}{2}\right)^{2m_1}\left(\frac{z_2}{2}\right)^{2m_2}}{m_1!m_2!\Gamma(m_1+\gamma_1+1)\Gamma(m_2+\gamma_2+1)} \tag{18}$$

Theorem 2.2. System (2.22) has four linearly independent particular solutions, one of which is the product of the function

$$W_1(z_1,z_2)=\sum_{m_1,m_2=0}^{\infty}\frac{1}{(\gamma_1,m_1)(\gamma_2,m_2)}\frac{z_1^{m_1}}{m_1!}\frac{z_2^{m_2}}{m_2!}=J(\gamma_1,z_1)\,J(\gamma_2,z_2)$$

Particular solutions of the given system (16) can be constructed by direct substitution of the generalized step series of two variables

$$W=z_1^{\rho_1}z_2^{\rho_2}\sum_{m_1,m_2=0}^{\infty}A_{m_1,m_2}z_1^{m_1}z_2^{m_2},\,A_{0,0}\neq 0$$

into the given system. Then determine the unknown $\rho_1,\rho_2,A_{m_1,m_2}\,(m_1,m_2=0,1,2,...)$ coefficients.

Theorem 2.3. System (2.20) has three linearly independent particular solutions:

$$W_1(z_1,z_2)=\sum_{m_1,m_2=0}^{\infty}\frac{1}{(\gamma)_{m_1+m_2}}\frac{z_1^{m_1}}{m_1!}\frac{z_2^{m_2}}{m_2!},$$

$$W_2(z_1,z_2)=z_2^{1-\gamma}\left\{1+z_1+\frac{1}{2-\gamma}z_2+\frac{1}{2(2-\gamma)}z_1z_2+...\right\},$$

$$W_3(z_1,z_2)=z_2^{1-\gamma}\left\{1+\frac{1}{2-\gamma}z_1+z_2+\frac{1}{2(2-\gamma)}z_1z_2+...\right\} \tag{19}$$

For the system, the compatibility condition does not hold, therefore, it has no more than three linearly independent particular solutions. Solution (19) can be written in the following form:

$$W_1(z_1,z_2)=\sum_{m_1,m_2=0}^{\infty}\frac{1}{\gamma(\gamma+1)...(\gamma+m_1+m_2+1)}\frac{(z_1+z_2)^{m_1+m_2}}{(m_1+m_2)!}=$$

$$=\sum_{m_1,m_2=0}^{\infty}\frac{1}{(\gamma)_{m_1+m_2}}\frac{z_1^{m_1}}{m_1!}\frac{z_2^{m_2}}{m_2!}=J(\gamma,z_1+z_2) \tag{20}$$

that is, we again obtained a degenerate hypergeometric series, which can be reduced to the Bessel function of two variables.

According to the general theory of such systems, the statement holds [13].

Theorem 2.4. The general solution of system (14) can be represented as the sum of three linearly independent particular solutions:

$$\overline{W}\left(z_1,z_2\right)=\sum_{i=1}^{2}W_i\left(z_1,z_2\right)=C_1J\left(\gamma,z_1+z_2\right)+C_2z_2^{1-\gamma}J_1\left(2-\gamma,z_1+z_2\right)+C_3z_1^{1-\gamma}J_2\left(2-\gamma,z_1+z_2\right)$$

In (20) we represent $(\gamma)_{m_1+m_2}$ in the form: $(\gamma)_{m_1+m_2}=(\gamma)_{m_1}\left(\gamma+m_1\right)_{m_1}$, then we get

$$\sum_{m_1,m_2=0}^{\infty}\frac{1}{(\gamma)_{m_1+m_2}}\frac{z_1^{m_1}}{m_1!}\frac{z_2^{m_2}}{m_2!}=\sum_{m_1,m_2=0}^{\infty}\frac{1}{(\gamma)_{m_1}\left(\gamma+m_1\right)_{m_1}}\frac{z_1^{m_1}}{m_1!}\frac{z_2^{m_2}}{m_2!}=J\left(\gamma,z_1\right)J\left(\gamma+m_1,z_2\right) \tag{21}$$

Using (21), the validity of the assertion is confirmed.

Theorem 2.5. The solution of system (14) can be expressed through the Bessel function:

$$J_{\gamma,\gamma+m_1}\left(z_1,z_2\right)=\left(\frac{z_1}{2}\right)^{\gamma_1}\left(\frac{z_2}{2}\right)^{\gamma_2}\sum_{m_1,m_2=0}^{\infty}\frac{(-1)^{m_1+m_2}\left(\frac{z_1}{2}\right)^{2m_1}\left(\frac{z_2}{2}\right)^{2m_2}}{m_1!m_2!\Gamma\left(m_1+\gamma_1+1\right)\Gamma\left(m_2+\gamma_2+1\right)}$$

For the proof, the following identity is used:

$$J_\gamma\left(z_1\right)J_{\gamma+m_1}\left(z_2\right)=\left(\frac{z_1}{2}\right)^{\gamma}\frac{1}{\Gamma\left(\gamma+1\right)}J\left(\gamma+1;-\frac{z_1^2}{2^2}\right)$$
$$\left(\frac{z_2}{2}\right)^{\gamma+m_1}\frac{1}{\Gamma\left(\gamma+m_1+1\right)}J\left(\gamma+m_1+1;-\frac{z_2^2}{2^2}\right) \tag{22}$$

The presented theorems can be generalized to the case of three or more variables.

Theorem 2.6. The system consisting of three equations:

$$\sum_{j=1}^{3}z_j\frac{\partial^2W}{\partial z_j\partial z_i}+\gamma\frac{\partial W}{\partial z_i}-W=0, i=1,2,3$$

has no more than 2^3-1linearly independent particular solutions, one of which is a function reducible to a Bessel function of three variables:

$$W_1(z_1,z_2,z_3)=\sum_{m_1,m_2,m_3=0}^{\infty}\frac{1}{\gamma_{m_1+m_2+m_3}}\frac{\left(z_1+z_2+z_3\right)^{m_1+m_2+m_3}}{(m_1+m_2+m_3)!}$$
$$=\sum_{m_1,m_2,m_3=0}^{\infty}\frac{1}{\gamma_{m_1+m_2+m_3}}\frac{z_1^{m_1}}{m_1!}\frac{z_2^{m_2}}{m_2!}\frac{z_3^{m_3}}{m_3!}$$

Reasoning as in (21) and (22), we obtain a proof of Theorem 2.5 and a generalization of Theorem 2.4 to the case of a Bessel function of three or more variables.

In the general case, a particular solution that reduces to a Bessel function of n variables is represented as

$$W_1(z_1,z_2,...z_B)=\sum_{m_1+...+m_l=0}^{\infty}\frac{1}{\gamma_{m_1+...+m_l}}\frac{z_1^{m_1}}{m_1!}\frac{z_2^{m_2}}{m_2!}...\frac{z_l^{m_l}}{m_l!}$$

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EXISTENCE OF SOLUTIONS IN BOUNDARY VALUE PROBLEMS FOR NONLINEAR EQUATIONS INVOLVING Q-ANALOGS OF FRACTIONAL DERIVATIVES

Tolegen Bektas Kuanyshuly[1], **Tokmagambetov Nariman Sarsenovich**[2]