

T.K. Yuldashev

Tashkent State University of Economics, Tashkent, Uzbekistan  
(E-mail: t.yuldashev@tsue.uz)

## Mixed inverse problem for a Benney–Luke type integro-differential equation with two redefinition functions and parameters

In this paper, we consider a linear Benney–Luke type partial integro-differential equation of higher order with degenerate kernel and two redefinition functions given at the endpoint of the segment and two parameters. To find these redefinition functions we use two intermediate data. Dirichlet boundary value conditions are used with respect to spatial variable. The Fourier series method of variables separation is applied. The countable system of functional-integral equations is obtained. Theorem on a unique solvability of countable system for functional-integral equations is proved. The method of successive approximations is used in combination with the method of contraction mapping. The triple of solutions of the inverse problem is obtained in the form of Fourier series. Absolutely and uniformly convergences of Fourier series are proved.

*Keywords:* Inverse problem, two redefinition functions, final conditions, intermediate functions, Fourier method, unique value solvability.

### Introduction

Historically, differential equations arose in solving applied problems. Therefore, the development of differential equations at the initial stage was carried out by applied scientists. Gradually, this direction grew into an independent theory – the theory of differential equations. Therefore, it can be said many times that differential and integro-differential equations are great interest from the point of theoretical research and applications in the mathematical physics, engineering, chemistry and in other different fields [1–8]. Recent years, a number of new problems for ordinary and partial differential and integro-differential equations are studied and a large number of research papers are published. Problems with nonlocal conditions for differential and integro-differential equations were considered in [9–29]. In [30–38], integro-differential equations with a degenerate kernel were considered.

In this paper, we study the solvability of the mixed inverse problem for a Benney–Luke type partial integro-differential equation with a degenerate kernel, two parameters, and final conditions at the endpoint of the interval. This paper differs from existing papers in that it requires to find redefinition functions considering at the endpoint of the interval. This inverse problem has features in relation to the direct problem.

In the rectangular domain  $\Omega = \{0 < t < T, 0 < x < l\}$  we consider the following partial integro-differential equation of a higher order

$$\frac{\partial^2 U}{\partial t^2} + (-1)^k \frac{\partial^{2k+2} U}{\partial t^2 \partial x^2} + \omega^2 \left[ (-1)^k \frac{\partial^{2k} U}{\partial x^{2k}} + \frac{\partial^{4k} U}{\partial x^{4k}} \right] = \alpha(t) U(t, x) + \nu \int_0^T K(t, s) U(s, x) ds, \quad (1)$$

where  $k$  is a natural number,  $0 < \alpha(t) \in C[0, T]$ ,  $T, l$  are given positive numbers,  $\omega$  is a positive parameter,  $\nu$  is a nonzero real parameter,  $K(t, s) = \sum_{i=1}^m a_i(t) b_i(s)$ ,  $a_i(t), b_i(s) \in C[0, T]$ . It is assumed that the systems of functions  $\{a_i(t)\}$  and  $\{b_i(s)\}$ ,  $i = \overline{1, m}$  are linear independent.

It is known that when applying the method of separation of variables, the Dirichlet condition allows us to reduce partial differential equations to a countable system of ordinary differential equations. So, in solving partial integro-differential equation (1), we use the following Dirichlet boundary value conditions with respect to spatial variable  $x$

$$\begin{aligned} U(t, 0) = U(t, l) &= \frac{\partial^2}{\partial x^2} U(t, 0) = \frac{\partial^2}{\partial x^2} U(t, l) = \\ &= \dots = \frac{\partial^{4k-2}}{\partial x^{4k-2}} U(t, 0) = \frac{\partial^{4k-2}}{\partial x^{4k-2}} U(t, l) = 0. \end{aligned} \quad (2)$$

We use two conditions at the endpoint of the given segment with respect to time variable  $t$ :

$$U(T, x) = \varphi_1(x), \quad U_t(T, x) = \varphi_2(x), \quad 0 \leq x \leq l, \quad (3)$$

where  $\varphi_1(x)$  and  $\varphi_2(x)$  are redefinition functions and we assume that they are enough smooth on the segment  $[0, l]$ . For these functions the following conditions will be fulfilled

$$\varphi_i(0) = \varphi_i(l) = \varphi_i''(0) = \varphi_i''(l) = \dots = \varphi_i^{(4k-2)}(0) = \varphi_i^{(4k-2)}(l) = 0, \quad i = 1, 2.$$

In determining the redefinition functions, we use the following two intermediate conditions:

$$U(t_1, x) = \psi_1(x), \quad U_t(t_1, x) = \psi_2(x), \quad 0 \leq x \leq l, \quad (4)$$

where  $\psi_1(x)$  and  $\psi_2(x)$  are known functions enough smooth on the segment  $[0, l]$ ,  $0 < t_1 < T$ . For the functions  $\psi_1(x)$  and  $\psi_2(x)$  the following conditions will be fulfilled

$$\psi_i(0) = \psi_i(l) = \psi_i''(0) = \psi_i''(l) = \dots = \psi_i^{(4k-2)}(0) = \psi_i^{(4k-2)}(l) = 0, \quad i = 1, 2.$$

The choice of conditions (3) and (4) with the final and intermediate data are important in applications. Indeed, in real practice it is not always possible to determine the initial data for unknown functions. When studying the technological process of aluminum production, before the start of the production cycle, the raw material passes through firing and the state of the raw material by the beginning of the production cycle is not known. And the final expected state of the output will be unknown in reality. We find it from known intermediate conditions. Because after each technological cycle we can determine the quality of the product. So, we have an inverse problem to solve equation (1).

*Problem statement.* To find triple of functions

$$\left\{ U(t, x) \in C(\bar{\Omega}) \cap C_{t,x}^{2,4k}(\Omega) \cap C_{t,x}^{2+2k}(\Omega), \varphi_i(x) \in C[0, l], i = 1, 2 \right\},$$

the first of which satisfies partial integro-differential equation (1) and specified conditions (2)–(4), where  $\bar{\Omega} = \{0 \leq t \leq T, 0 \leq x \leq l\}$ .

Note that problem (1)–(4) is formulated such that direct problem (1)–(3) has a unique solution for all values of the parameter  $\omega$ , and inverse problem (1)–(4) has a unique solution only for certain values of this parameter  $\omega$ .

### 1 Construction of formal solution of the direct problem (1)–(3)

Note that the functions  $\vartheta_n(x) = \sqrt{\frac{2}{l}} \sin \lambda_n x$ , where  $\lambda_n \in \frac{n\pi}{l}$ ,  $n \in \mathbb{N}$ , form a complete system of orthonormal eigenfunctions in the space  $L_2[0, l]$ . Linear equation (1) always has the trivial solution.

Therefore, by virtue of the Dirichlet condition (2), we seek nontrivial solutions to the linear partial integro-differential equation (1) of the higher order in the form of a Fourier series in sines

$$U(t, x) = \sqrt{\frac{2}{l}} \sum_{n=1}^{\infty} u_n(t) \sin \lambda_n x, \tag{5}$$

where

$$u_n(t) = \sqrt{\frac{2}{l}} \int_0^l U(t, x) \sin \lambda_n x dx, \quad \lambda_n = \frac{n\pi}{l}. \tag{6}$$

Substituting the Fourier series (5) into the given integro-differential equation (1), we obtain a linear second order countable system of ordinary differential equations

$$u_n''(t) + \omega^2 \lambda_n^{2k} u_n(t) = \frac{\nu}{1 + \lambda_n^{2k}} \sum_{i=1}^m a_i(t) \tau_{ni} + \frac{1}{1 + \lambda_n^{2k}} \alpha(t) u_n(t), \tag{7}$$

where

$$\tau_{ni} = \int_0^T b_i(s) u_n(s) ds. \tag{8}$$

Solving the countable system of differential equations (7) by the variation method of arbitrary constants, we obtain the representation for its solution

$$\begin{aligned} u_n(t) &= A_{1n} \cos \lambda_n^k \omega t + A_{2n} \sin \lambda_n^k \omega t + \\ &+ \frac{\nu}{\lambda_n^k (1 + \lambda_n^{2k}) \omega} \sum_{i=1}^m \tau_{ni} \int_0^t \sin \lambda_n^k \omega (t-s) a_i(s) ds + \\ &+ \frac{1}{\lambda_n^k (1 + \lambda_n^{2k}) \omega} \int_0^t \sin \lambda_n^k \omega (t-s) \alpha(s) u_n(s) ds, \end{aligned} \tag{9}$$

where  $A_{1n}$  and  $A_{2n}$  are arbitrary coefficients, which will be determined by the final conditions (3). By differentiating (9) one times on  $t$ , we obtain

$$\begin{aligned} u_n'(t) &= -\lambda_n^k \omega A_{1n} \sin \lambda_n^k \omega t + \lambda_n^k \omega A_{2n} \cos \lambda_n^k \omega t + \\ &+ \frac{\nu}{1 + \lambda_n^{2k}} \sum_{i=1}^m \tau_{ni} \int_0^t \cos \lambda_n^k \omega (t-s) a_i(s) ds + \\ &+ \frac{1}{1 + \lambda_n^{2k}} \int_0^t \cos \lambda_n^k \omega (t-s) \alpha(s) u_n(s) ds. \end{aligned} \tag{10}$$

Now, supposing that the redefinition functions  $\varphi_1(x)$  and  $\varphi_2(x)$  were expanded into a Fourier series, and using Fourier coefficients (6), from conditions (3) we obtain

$$u_n(T) = \sqrt{\frac{2}{l}} \int_0^l U(T, x) \sin \lambda_n x dx = \sqrt{\frac{2}{l}} \int_0^l \varphi_1(x) \sin \lambda_n x dx = \varphi_{1n}, \tag{11}$$

$$u'_n(T) = \sqrt{\frac{2}{l}} \int_0^l U_t(T, x) \sin \lambda_n x dx = \sqrt{\frac{2}{l}} \int_0^l \varphi_2(x) \sin \lambda_n x dx = \varphi_{2n}. \quad (12)$$

To find the unknown coefficients  $A_{1n}$  and  $A_{2n}$  in presentations (9) and (10), we use final conditions (11) and (12). Then we arrive at a system of algebraic equations (SAE)

$$\begin{cases} A_{1n} \cos \lambda_n^k \omega T + A_{2n} \sin \lambda_n^k \omega T = \gamma_{1n}, \\ -A_{1n} \sin \lambda_n^k \omega T + A_{2n} \cos \lambda_n^k \omega T = \gamma_{2n}, \end{cases} \quad (13)$$

where

$$\begin{aligned} \gamma_{1n} &= \varphi_{1n} - \frac{\nu}{\lambda_n^k (1 + \lambda_n^{2k}) \omega} \sum_{i=1}^m \tau_{in} \int_0^T \sin \lambda_n^k \omega (T-s) a_i(s) ds - \\ &\quad - \frac{1}{\lambda_n^k (1 + \lambda_n^{2k}) \omega} \int_0^T \sin \lambda_n^k \omega (T-s) \alpha(s) u_n(s) ds, \\ \gamma_{2n} &= \varphi_{2n} - \frac{\nu}{1 + \lambda_n^{2k}} \sum_{i=1}^m \tau_{in} \int_0^T \cos \lambda_n^k \omega (T-s) a_i(s) ds - \\ &\quad - \frac{1}{1 + \lambda_n^{2k}} \int_0^T \cos \lambda_n^k \omega (T-s) \alpha(s) u_n(s) ds. \end{aligned}$$

For uniquely solvability of SAE (13), the following condition

$$\delta_{0n} = \begin{vmatrix} \cos \lambda_n^k \omega T & \sin \lambda_n^k \omega T \\ -\sin \lambda_n^k \omega T & \cos \lambda_n^k \omega T \end{vmatrix} \neq 0$$

must be fulfilled. Since  $\delta_{0n} = 1$ , this condition are fulfilled for all values of the parameter  $\omega$ . Consequently, SAE (13) has a unique pair of solutions

$$\begin{aligned} A_{1n} = \delta_{1n} &= \begin{vmatrix} \gamma_{1n} \sin \lambda_n^k \omega T \\ \gamma_{2n} \cos \lambda_n^k \omega T \end{vmatrix} = \varphi_{1n} \cos \lambda_n^k \omega T - \varphi_{2n} \sin \lambda_n^k \omega T + \\ &+ \frac{\nu}{\lambda_n^k (1 + \lambda_n^{2k}) \omega} \sum_{i=1}^m \tau_{in} \int_0^T \sin \lambda_n^k \omega s a_i(s) ds + \frac{1}{\lambda_n^k (1 + \lambda_n^{2k}) \omega} \int_0^T \sin \lambda_n^k \omega s \alpha(s) u_n(s) ds, \end{aligned} \quad (14) \end{aligned}$$

$$\begin{aligned} A_{2n} = \delta_{2n} &= \begin{vmatrix} \cos \lambda_n^k \omega T & \gamma_{1n} \\ -\sin \lambda_n^k \omega T & \gamma_{2n} \end{vmatrix} = \varphi_{1n} \sin \lambda_n^k \omega T + \varphi_{2n} \cos \lambda_n^k \omega T + \\ &+ \frac{\nu}{1 + \lambda_n^{2k}} \sum_{i=1}^m \tau_{in} \int_0^T \cos \lambda_n^k \omega s a_i(s) ds + \frac{1}{1 + \lambda_n^{2k}} \int_0^T \cos \lambda_n^k \omega s \alpha(s) u_n(s) ds. \end{aligned} \quad (15)$$

Substituting these values of (14) and (15) into presentation (9), we obtain

$$u_n(t, \nu, \omega) = \varphi_{1n} \chi_{1n}(t, \omega) + \varphi_{2n} \chi_{2n}(t, \omega) + \frac{\nu}{\lambda_n^k (1 + \lambda_n^{2k}) \omega} \sum_{i=1}^m \tau_{in} \chi_{3in}(t, \omega) +$$

$$+ \frac{1}{\lambda_n^k (1 + \lambda_n^{2k}) \omega} \int_0^T H_n(t, s, \omega) \alpha(s) u_n(s, \nu, \omega) ds, \tag{16}$$

where

$$\chi_{1n}(t, \omega) = \cos \lambda_n^k \omega (T - t) - \sin \lambda_n^k \omega (T - t),$$

$$\chi_{2n}(t, \omega) = \cos \lambda_n^k \omega (T + t) - \sin \lambda_n^k \omega (T - t),$$

$$\chi_{3in}(t, \omega) = \int_0^T H_n(t, s, \omega) a_i(s) ds,$$

$$H_n(t, s, \omega) = \begin{cases} \sin z(t + s), & z = \lambda_n^k \omega, \quad t < s \leq T, \\ \sin z(t - s) + \cos z t \sin z s + z \sin z t \sin z s, & 0 \leq s < t. \end{cases}$$

Although functions (16) are Fourier coefficients of the solution to direct problem (1)–(3), it contains extra quantities  $\tau_{in}$  that are still unknown. To find these quantities, we substitute representation (16) into designation (8) and arrive at a new SAE:

$$\tau_{in} - \frac{\nu}{\bar{\lambda}} \sum_{j=1}^m \tau_{jn} \sigma_{3ijn}(t) = \varphi_{1n} \sigma_{1in} + \varphi_{2n} \sigma_{2in} + \sigma_{4in}(u_n), \tag{17}$$

where

$$\sigma_{1in} = \int_0^T b_i(s) \chi_{1n}(s, \omega) ds, \quad \sigma_{2in} = \int_0^T b_i(s) \chi_{2n}(s, \omega) ds, \quad \bar{\lambda} = \lambda_n^k (1 + \lambda_n^{2k}) \omega,$$

$$\sigma_{3ijn} = \int_0^T b_i(s) \int_0^T H_n(s, \theta, \omega) a_j(\theta) d\theta ds,$$

$$\sigma_{4in}(u_n) = \frac{1}{\lambda} \int_0^T b_i(s) \int_0^T H_n(s, \theta, \omega) \alpha(\theta) u_n(\theta) d\theta ds.$$

To establish the unique solvability of SAE (17), we introduce the following matrix

$$\Theta_{0n}(\nu, \omega) = \begin{pmatrix} 1 - \frac{\nu}{\bar{\lambda}} \sigma_{311n} & \frac{\nu}{\bar{\lambda}} \sigma_{312n} & \dots & \frac{\nu}{\bar{\lambda}} \sigma_{31mn} \\ \frac{\nu}{\bar{\lambda}} \sigma_{321n} & 1 - \frac{\nu}{\bar{\lambda}} \sigma_{322n} & \dots & \frac{\nu}{\bar{\lambda}} \sigma_{32mn} \\ \dots & \dots & \dots & \dots \\ \frac{\nu}{\bar{\lambda}} \sigma_{3m1n} & \frac{\nu}{\bar{\lambda}} \sigma_{3m2n} & \dots & 1 - \frac{\nu}{\bar{\lambda}} \sigma_{3mmn} \end{pmatrix}$$

and consider the values of the parameter  $\nu$ , for which the Fredholm determinant is not zero:

$$\Delta_{0n}(\nu, \omega) = \det \Theta_{0n}(\nu, \omega) \neq 0. \tag{18}$$

Determinant  $\Delta_{0n}(\nu, \omega)$  in (18) is a polynomial with respect to  $\frac{\nu}{\bar{\lambda}}$  of the degree not higher than  $m$ . The countable system of algebraic equations  $\Delta_{0n}(\nu, \omega) = 0$  has no more than  $m$  different real roots for every value of  $n$ . We denote them by  $\mu_l (l = \overline{1, p}, 1 \leq p \leq m)$ . Then  $\nu_n = \nu_{ln} = \bar{\lambda} \mu_l = \lambda_n^k (1 + \lambda_n^{2k}) \omega \mu_l$  are called the characteristic (irregular) values of the kernel for integro-differential equation (1). So, we introduce the following two designations

$$\Lambda_1 = \left\{ (\nu_n, \omega) : \nu_n = \lambda_n^k (1 + \lambda_n^{2k}) \omega \mu_l, \quad \omega \in (0, \infty) \right\},$$

$$\Lambda_2 = \left\{ (\nu_n, \omega) : |\Delta_{0n}(\nu, \omega)| > 0, \nu_n \neq \lambda_n^k (1 + \lambda_n^{2k}) \omega \mu_l, \omega \in (0, \infty) \right\}.$$

On the number set  $\Lambda_2$  we consider a matrix

$$\Theta_{ijn}(\nu, \omega) = \begin{pmatrix} 1 - \frac{\nu}{\lambda} \sigma_{311n} & \cdots & \frac{\nu}{\lambda} \sigma_{31(i-1)n} & \sigma_{j1n} & \frac{\nu}{\lambda} \sigma_{31(i+1)n} & \cdots & \frac{\nu}{\lambda} \sigma_{31mn} \\ \frac{\nu}{\lambda} \sigma_{321n} & \cdots & \frac{\nu}{\lambda} \sigma_{32(i-1)n} & \sigma_{j2n} & \frac{\nu}{\lambda} \sigma_{32(i+1)n} & \cdots & \frac{\nu}{\lambda} \sigma_{32mn} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\nu}{\lambda} \sigma_{3m1n} & \cdots & \frac{\nu}{\lambda} \sigma_{3m(i-1)n} & \sigma_{jmn} & \frac{\nu}{\lambda} \sigma_{3m(i+1)n} & \cdots & 1 - \frac{\nu}{\lambda} \sigma_{3mmm} \end{pmatrix},$$

$j = 1, 2, 4$ . Taking into account the known properties of the matrix  $\Theta_{ijn}(\nu, \omega)$ , we modified the Cramer method on the set  $\Lambda_2$  and obtain solutions of SAE (17) in the form

$$\tau_{in} = \varphi_{1n} \frac{\Delta_{1i}(\nu, \omega)}{\Delta_{0n}(\nu, \omega)} + \varphi_{2n} \frac{\Delta_{2in}(\nu, \omega)}{\Delta_{0n}(\nu, \omega)} + \frac{\Delta_{4in}(\nu, \omega, u_n)}{\Delta_{0n}(\nu, \omega)}, \quad i = \overline{1, m}, \quad (\nu, \omega) \in \Lambda_2, \quad (19)$$

where  $\Delta_{ijn}(\nu, \omega) = \det \Theta_{ijn}(\nu, \omega)$ ,  $j = 1, 2, 4$ .

Substituting solutions (19) into function (16), we obtain

$$u_n(t, \nu, \omega) = \varphi_{1n} h_{1n}(t, \nu, \omega) + \varphi_{2n} h_{2n}(t, \nu, \omega) + \frac{\nu}{\lambda_n^k (1 + \lambda_n^{2k}) \omega} \sum_{i=1}^m \frac{\Delta_{4in}(\nu, \omega, u_n)}{\Delta_{0n}(\nu, \omega)} \chi_{3in}(t) + \\ + \frac{1}{\lambda_n^k (1 + \lambda_n^{2k}) \omega} \int_0^T H_n(t, s, \omega) u_n(s, \nu, \omega) ds, \quad (\nu, \omega) \in \Lambda_2, \quad (20)$$

where

$$h_{jn}(t, \nu, \omega) = \chi_{jn}(t, \omega) + \frac{\nu}{\lambda_n^k (1 + \lambda_n^{2k}) \omega} \sum_{i=1}^m \frac{\Delta_{jn}(\nu, \omega)}{\Delta_{0n}(\nu, \omega)} \chi_{3in}(t, \omega), \quad j = 1, 2,$$

$$\chi_{1n}(t, \omega) = \cos \lambda_n^k \omega (T - t) - \sin \lambda_n^k \omega (T - t), \quad \chi_{2n}(t, \omega) = \cos \lambda_n^k \omega (T + t) - \sin \lambda_n^k \omega (T - t),$$

$$\chi_{3in}(t, \omega) = \int_0^T H_n(t, s, \omega) a_i(s) ds,$$

$$H_n(t, s, \omega) = \begin{cases} \sin z(t+s), & z = \lambda_n^k \omega, \quad t < s \leq T, \\ \sin z(t-s) + \cos zt \sin zs + z \sin zt \sin zs, & 0 \leq s < t. \end{cases}$$

Representation (20) is a countable system of functional-integral equations. Substituting representation (20) into the Fourier series (5), we obtain a formal solution of direct problem (1)–(3) on the domain  $\Omega$

$$U(t, x) = \sqrt{\frac{2}{l}} \sum_{n=1}^{\infty} \sin \lambda_n x \times \\ \times \left[ \varphi_{1n} h_{1n}(t, \nu, \omega) + \varphi_{2n} h_{2n}(t, \nu, \omega) + \frac{\nu}{\lambda_n^k (1 + \lambda_n^{2k}) \omega} \sum_{i=1}^m \frac{\Delta_{4in}(\nu, \omega, u_n)}{\Delta_{0n}(\nu, \omega)} \chi_{3im}(t) + \right. \\ \left. + \frac{1}{\lambda_n^k (1 + \lambda_n^{2k}) \omega} \int_0^T H_n(t, s, \omega) u_n(s, \nu, \omega) ds \right], \quad (\nu, \omega) \in \Lambda_2. \quad (21)$$

But, there are two unknown quantities  $\varphi_{1n}$  and  $\varphi_{2n}$  in (21).

2 Formal solution of the inverse problem (1)–(4)

We will now formally define the redefinition functions  $\varphi_1(x)$  and  $\varphi_2(x)$ . We subordinate function (20) to intermediate conditions (4). For this purpose, we differentiate (21) one times on the time-variable  $t$ :

$$U_t(t, x) = \sqrt{\frac{2}{l}} \sum_{n=1}^{\infty} \sin \lambda_n x [\varphi_{1n} h'_{1n}(t, \nu, \omega) + \varphi_{2n} h'_{2n}(t, \nu, \omega) + \frac{\nu}{\lambda_n^k (1 + \lambda_n^{2k}) \omega} \sum_{i=1}^m \frac{\Delta_{4in}(\nu, \omega, u_n)}{\Delta_{0n}(\nu, \omega)} \chi'_{3in}(t) + \frac{1}{\lambda_n^k (1 + \lambda_n^{2k}) \omega} \int_0^T H'_n(t, s, \omega) u_n(s, \nu, \omega) ds], \quad (22)$$

where

$$h'_{jn}(t, \nu, \omega) = \chi'_{jn}(t, \omega) + \frac{\nu}{\lambda_n^k (1 + \lambda_n^{2k}) \omega} \sum_{i=1}^m \frac{\Delta_{jn}(\nu, \omega)}{\Delta_{0n}(\nu, \omega)} \chi'_{3in}(t, \omega), \quad j = 1, 2,$$

$$\chi'_{1n}(t, \omega) = \lambda_n^k \omega (\sin \lambda_n^k \omega (T - t) + \cos \lambda_n^k \omega (T - t)),$$

$$\chi'_{2n}(t, \omega) = -\lambda_n^k \omega (\sin \lambda_n^k \omega (T + t) + \cos \lambda_n^k \omega (T - t)),$$

$$\chi'_{3in}(t, \omega) = \int_0^T H'_n(t, s, \omega) u_i(s) ds,$$

$$H'_n(t, s, \omega) = \begin{cases} z \cos z (t + s), & z = \lambda_n^k \omega, \quad t < s \leq T, \\ z \cos z (t - s) - z \sin z t \sin z s + z^2 \cos z t \sin z s, & 0 \leq s < t. \end{cases}$$

Then, applying intermediate conditions (4) to functions (21) and (22), we arrive at the solution of the following SAE:

$$\begin{cases} \varphi_{1n} [\chi_{1n}(t_1, \omega) + \varepsilon_{11n}] + \varphi_{2n} [\chi_{2n}(t_1, \omega) + \varepsilon_{12n}] = \bar{\psi}_{1n}, \\ \varphi_{1n} [\chi'_{1n}(t_1, \omega) + \varepsilon_{21n}] + \varphi_{2n} [\chi'_{2n}(t_1, \omega) + \varepsilon_{22n}] = \bar{\psi}_{2n}, \end{cases} \quad (23)$$

where

$$\varepsilon_{1jn} = \frac{\nu}{\bar{\lambda}} \sum_{i=1}^m \frac{\Delta_{jn}(\nu, \omega)}{\Delta_{0n}(\nu, \omega)} \chi_{3in}(t_1, \omega), \quad \varepsilon_{2jn} = \frac{\nu}{\bar{\lambda}} \sum_{i=1}^m \frac{\Delta_{jn}(\nu, \omega)}{\Delta_{0n}(\nu, \omega)} \chi'_{3in}(t_1, \omega), \quad j = 1, 2,$$

$$\bar{\psi}_{1n} = \psi_{1n} - \frac{\nu}{\bar{\lambda}} \sum_{i=1}^m \frac{\Delta_{4in}(\nu, \omega, u_n)}{\Delta_{0n}(\nu, \omega)} \chi_{3in}(t_1, \omega) + \frac{1}{\bar{\lambda}} \int_0^T H_n(t_1, s, \omega) u_n(s, \nu, \omega) ds, \quad (24)$$

$$\bar{\psi}_{2n} = \psi_{2n} - \frac{\nu}{\bar{\lambda}} \sum_{i=1}^m \frac{\Delta_{4in}(\nu, \omega, u_n)}{\Delta_{0n}(\nu, \omega)} \chi'_{3in}(t_1, \omega) + \frac{1}{\bar{\lambda}} \int_0^T H'_n(t_1, s, \omega) u_n(s, \nu, \omega) ds, \quad (25)$$

$$\bar{\lambda} = \lambda_n^k (1 + \lambda_n^{2k}) \omega.$$

The fulfillment of the following condition ensures the unique solvability of SAE (23):

$$V_{0n}(\omega) = \begin{vmatrix} \chi_{1n}(t_1, \omega) + \varepsilon_{11n} & \chi_{2n}(t_1, \omega) + \varepsilon_{12n} \\ \chi'_{1n}(t_1, \omega) + \varepsilon_{21n} & \chi'_{2n}(t_1, \omega) + \varepsilon_{22n} \end{vmatrix} = \\ = -z \sin 2zT - z \cos 2zT + 2z \sin z(T - t_1) \cos z(T - t_1) - z \cos 2z(T - t_1) - \\ - z \varepsilon_{11n} [\sin z(T + t_1) + \cos z(T - t_1)] - z \varepsilon_{12n} [\sin z(T - t_1) + \cos z(T - t_1)] -$$

$$\begin{aligned}
& -\varepsilon_{21n}[\cos z(T+t_1) - z \sin z(T-t_1)] - \varepsilon_{22n}[\sin z(T-t_1) - z \cos z(T-t_1)] + \\
& + \varepsilon_{11n}\varepsilon_{22n} - \varepsilon_{21n}\varepsilon_{12n} \neq 0.
\end{aligned} \tag{26}$$

Before proceeding to find the solution of SAE (23), we consider nonzero condition (26). To do this, we suppose the opposite:

$$\begin{aligned}
& -z \sin 2zT - z \cos 2zT + 2z \sin z(T-t_1) \cos z(T-t_1) - z \cos 2z(T-t_1) - \\
& -z \varepsilon_{11n}[\sin z(T+t_1) + \cos z(T-t_1)] - z \varepsilon_{12n}[\sin z(T-t_1) + \cos z(T-t_1)] - \\
& -\varepsilon_{21n}[\cos z(T+t_1) - z \sin z(T-t_1)] - \varepsilon_{22n}[\sin z(T-t_1) - z \cos z(T-t_1)] + \\
& + \varepsilon_{11n}\varepsilon_{22n} - \varepsilon_{21n}\varepsilon_{12n} = 0, \quad z = \lambda_n^k \omega.
\end{aligned} \tag{27}$$

Condition (27) is a transcendental equation, and the set of its solutions with respect to  $\omega$  is denoted by  $\mathfrak{S}$ . So, on the set

$$\Lambda_3 = \left\{ (\nu_n, \omega) : |\Delta_{0n}(\nu, \omega)| > 0, \nu_n \neq \lambda_n^k (1 + \lambda_n^{2k}) \omega \mu_l, \omega \in \mathfrak{S} \right\}$$

SAE (23) is not uniquely solvable. But, on the other set

$$\Lambda_4 = \left\{ (\nu_n, \omega) : |\Delta_{0n}(\nu, \omega)| > 0, |V_{0n}(\omega)| > 0, \nu_n \neq \lambda_n^k (1 + \lambda_n^{2k}) \omega \mu_l, \omega \in (0, \infty) \setminus \mathfrak{S} \right\}$$

SAE (23) is uniquely solvable. So, taking into account notations (24) and (25), we obtain

$$\begin{aligned}
\varphi_{jn} &= \psi_{1n} w_{j1n}(\omega) + \psi_{2n} w_{j2n}(\omega) + \frac{\nu}{\lambda_n^k (1 + \lambda_n^{2k}) \omega} \sum_{i=1}^m \frac{\Delta_{4in}(\nu, \omega, u_n)}{\Delta_{0n}(\nu, \omega)} w_{j3in}(\omega) + \\
& + \frac{1}{\lambda_n^k (1 + \lambda_n^{2k}) \omega} \int_0^T W_{jn}(s, \omega) u_n(s, \nu, \omega) ds, \quad j = 1, 2, (\nu, \omega) \in \Lambda_4,
\end{aligned} \tag{28}$$

where

$$\begin{aligned}
w_{11n}(\omega) &= V_{0n}^{-1} (\chi'_{2n}(t_1, \omega) + \varepsilon_{22n}(\omega)), \quad w_{12n}(\omega) = V_{0n}^{-1} (-\chi_{2n}(t_1, \omega) + \varepsilon_{12n}(\omega)), \\
w_{21n}(\omega) &= V_{0n}^{-1} (\chi'_{1n}(t_1, \omega) + \varepsilon_{21n}(\omega)), \quad w_{22n}(\omega) = V_{0n}^{-1} (\chi_{1n}(t_1, \omega) + \varepsilon_{11n}(\omega)), \\
w_{13n}(\omega) &= -[\chi_{3in}(t_1, \omega) w_{11n}(\omega) + \chi'_{3in}(t_1, \omega) w_{12n}(\omega)], \\
w_{23n}(\omega) &= -[\chi_{3in}(t_1, \omega) w_{21n}(\omega) + \chi'_{3in}(t_1, \omega) w_{22n}(\omega)], \\
W_{1n}(s, \omega) &= H_n(t_1, s) w_{11n}(\omega) + H'_n(t_1, s) w_{12n}(\omega), \\
W_{2n}(s, \omega) &= H_n(t_1, s) w_{21n}(\omega) + H'_n(t_1, s) w_{22n}(\omega).
\end{aligned}$$

Since  $\varphi_{1n}$  and  $\varphi_{2n}$  are Fourier coefficients, from presentations (28) we obtain the following Fourier series

$$\begin{aligned}
\varphi_j(x) &= \sqrt{\frac{2}{l}} \sum_{n=1}^{\infty} \sin \lambda_n x \left[ \psi_{1n} w_{j1n} + \psi_{2n} w_{j2n} + \frac{\nu}{\lambda_n^k (1 + \lambda_n^{2k}) \omega} \sum_{i=1}^m \frac{\Delta_{4in}(\nu, \omega, u_n)}{\Delta_{0n}(\nu, \omega)} w_{j3in} + \right. \\
& \left. + \frac{1}{\lambda_n^k (1 + \lambda_n^{2k}) \omega} \int_0^T W_{jn}(s, \omega) u_n(s, \nu, \omega) ds \right], \quad (\nu, \omega) \in \Lambda_4.
\end{aligned} \tag{29}$$

The functions  $u_n(t, \nu, \omega)$  in series (29) are Fourier coefficients of the unknown function  $U(t, x, \nu, \omega)$ . Therefore, we need to define the Fourier coefficients  $u_n(t, \nu, \omega)$  uniquely. Substituting representation (28) into equations (20), we obtain the following countable system of functional-integral equations in the final form

$$\begin{aligned}
 u_n(t, \nu, \omega) = & S(t, \nu, \omega; u_n) \equiv \psi_{1n} g_{1n}(t, \nu, \omega) + \psi_{2n} g_{2n}(t, \nu, \omega) + \\
 & + \frac{\nu}{\lambda_n^k (1 + \lambda_n^{2k}) \omega} \sum_{i=1}^m \frac{\Delta_{4in}(\nu, \omega, u_n)}{\Delta_{0n}(\nu, \omega)} g_{3in}(t, \omega) + \\
 & + \frac{1}{\lambda_n^k (1 + \lambda_n^{2k}) \omega} \int_0^T G_n(t, s, \nu, \omega) u_n(s, \nu, \omega) ds, \quad (\nu, \omega) \in \Lambda_4, \quad (30)
 \end{aligned}$$

where

$$\begin{aligned}
 g_{1n}(t, \nu, \omega) &= w_{11n}(\omega) h_{1n}(t, \nu, \omega) + w_{21n}(\omega) h_{2n}(t, \nu, \omega), \\
 g_{2n}(t, \nu, \omega) &= w_{12n}(\omega) h_{1n}(t, \nu, \omega) + w_{22n}(\omega) h_{2n}(t, \nu, \omega), \\
 g_{3in}(t, \omega) &= g_{1n}(t, \nu, \omega) \chi_{3in}(t_1, \omega) + g_{2n}(t, \nu, \omega) \chi'_{3in}(t_1, \omega) + \chi_{3in}(t, \omega), \\
 G_n(t, s, \nu, \omega) &= g_{1n}(t, \nu, \omega) H_n(t_1, s, \omega) + g_{2n}(t, \nu, \omega) H'(t_1, s, \omega) + H_n(t, s, \omega).
 \end{aligned}$$

Note that this functional-integral equation (30) makes sense only for values of parameters  $\nu, \omega$  from the set  $\Lambda_4$ . In addition, in the countable system of functional-integral equations (30), the unknown function  $u_n(t, \nu, \omega)$  is under the sign of the determinant and under the sign of the integral.

### 3 Solvable of the countable system of functional-integral equations (30)

Let us investigate the system of equations (30) in the sense of the unique solvability. To this, we consider the following well-known Banach spaces, in which we need in our further actions [26, 32, 33, 36]. We consider the space  $B_2$  of function sequences  $\{u_n(t)\}_{n=1}^\infty$  on the segment  $[0, T]$  with the norm

$$\|u(t)\|_{B_2} = \sqrt{\sum_{n=1}^\infty \left( \max_{t \in [0, T]} |u_n(t)| \right)^2} < \infty;$$

the space  $\ell_2$  of number sequences  $\{\varphi_n\}_{n=1}^\infty$  with the norm

$$\|\varphi\|_{\ell_2} = \sqrt{\sum_{n=1}^\infty |\varphi_n|^2} < \infty;$$

the space  $L_2[0, l]$  of square-integrable functions on an interval  $[0, l]$  with norm

$$\|\vartheta(x)\|_{L_2[0, l]} = \sqrt{\int_0^l |\vartheta(x)|^2 dx} < \infty.$$

*Smoothness conditions.* Let on the segments  $[0, l]$  there exist peace-wise continuous derivatives with respect to  $x$  up  $(4k + 2)$ -th order for the functions  $\psi_i(x) \in C^{4k+1}[0, l]$ ,  $i = 1, 2$ . Then, after integration the integrand functions  $\psi_{in} = \sqrt{\frac{2}{l}} \int_0^l \psi_i(x) \sin \lambda_n x dx$ ,  $i = 1, 2$  by part  $(4k + 2)$  times on the variable  $x$ , we obtain the following relation

$$|\psi_{i,n}| = \left(\frac{l}{\pi}\right)^{4k+2} \frac{|\psi_{i,n}^{(4k+2)}|}{n^{4k+2}}, \quad i = 1, 2, \quad (31)$$

where

$$\psi_{i,n}^{(4k+2)} = \int_0^l \frac{\partial^{4k+2} \psi_i(x)}{\partial x^{4k+2}} \vartheta_n(x) dx, \quad i = 1, 2.$$

Here we note that the Bessel inequality is true

$$\sum_{n=1}^{\infty} [\psi_{i,n}^{(4k+2)}]^2 \leq \left(\frac{2}{l}\right)^{4k+2} \int_0^l \left[ \frac{\partial^{4k+2} \psi_i(x)}{\partial x^{4k+2}} \right]^2 dx, \quad i = 1, 2. \quad (32)$$

*Theorem 1.* Let the smoothness conditions and the following conditions be fulfilled:

$$\max_{t \in [0, T]} [|g_{1n}(t, \nu, \omega)|; |g_{2n}(t, \nu, \omega)|] = \delta_{1n} \leq \delta_1 < \infty, \quad (33)$$

$$\rho = |\nu| \delta_2 \left\| \sum_{i=1}^m \left| \frac{\bar{\Delta}_{4in}(\nu, \omega)}{\Delta_{0n}(\nu, \omega)} \right| \delta_{0i} \right\|_{\ell_2} + \delta_3 < 1, \quad (34)$$

where  $\delta_2$ ,  $\delta_3$  and  $\delta_{0i}$  will be defined from (38) and (39), while  $\bar{\Delta}_{4in}(\nu, \omega)$  is defined from (41). Then the countable systems of functional-integral equations (30) is uniquely solvable in the space  $B_2$ . The desired solution can be founded from the following iterative process:

$$\begin{cases} u_n^0(t, \nu, \omega) = \psi_{1n} g_{1n}(t, \nu, \omega) + \psi_{2n} g_{2n}(t, \nu, \omega), \\ u_n^{r+1}(t, \nu, \omega) = S(t, \nu, \omega; u_n^r), \quad r = 0, 1, 2, \dots \end{cases} \quad (35)$$

*Proof.* We use the method of contraction maps in combination with the method of successive approximations in the space  $B_2$ . Then, by virtue of smoothness condition (31) and estimate (33), applying the Cauchy–Schwartz inequality and Bessel inequality (32), from approximations (35) we obtain that the following estimate is valid:

$$\begin{aligned} \sum_{n=1}^{\infty} \max_{t \in [0, T]} |u_n^0(t)| &\leq \sum_{n=1}^{\infty} \max_{t \in [0, T]} [| \psi_{1n} | \cdot |g_{1n}(t, \nu, \omega)| + | \psi_{2n} | \cdot |g_{2n}(t, \nu, \omega)|] \leq \\ &\leq \delta_1 \left(\frac{l}{\pi}\right)^{4k+2} \left[ \sum_{n=1}^{\infty} \frac{|\psi_{1,n}^{(4k+2)}|}{n^{4k+2}} + \sum_{n=1}^{\infty} \frac{|\psi_{2,n}^{(4k+2)}|}{n^{4k+2}} \right] \leq \\ &\leq \delta_1 \left(\frac{\sqrt{2}l}{\pi}\right)^{4k+2} \sqrt{\sum_{n=1}^{\infty} \frac{1}{n^{8k+4}} \left[ \left\| \frac{\partial^{4k+2} \psi_1(x)}{\partial x^{4k+2}} \right\|_{L_2[0, l]}^2 + \left\| \frac{\partial^{4k+2} \psi_2(x)}{\partial x^{4k+2}} \right\|_{L_2[0, l]}^2 \right]} = \delta_0 < \infty. \end{aligned} \quad (36)$$

Taking into account estimate (36), applying the Cauchy–Schwartz inequality, for the first difference of approximations (35) we obtain:

$$\begin{aligned} \sum_{n=1}^{\infty} \max_{t \in [0, T]} |u_n^1(t) - u_n^0(t)| &\leq |\nu| \sum_{n=1}^{\infty} \frac{1}{\lambda_n^{3k} \omega} \sum_{i=1}^m \left| \frac{\Delta_{4in}(\nu, \omega, u_n^0)}{\Delta_{0n}(\nu, \omega)} \right| \max_{t \in [0, T]} |g_{3in}(t, \omega)| + \\ &+ \sum_{n=1}^{\infty} \frac{1}{\lambda_n^{3k} \omega} \max_{t \in [0, T]} \left| \int_0^T G_n(t, s, \nu, \omega) u_n^0(s, \nu, \omega) ds \right| \leq \end{aligned}$$

$$\leq |\nu| \delta_2 \sqrt{\sum_{n=1}^{\infty} \left[ \sum_{i=1}^m \left| \frac{\Delta_{4in}(\nu, \omega, u_n^0)}{\Delta_{0n}(\nu, \omega)} \right| \delta_{0i} \right]^2} + \delta_3 \delta_0 < \infty, \tag{37}$$

where

$$\delta_{0i} \geq \delta_{0in} = \max_{t \in [0, T]} |g_{3in}(t, \omega)|, \quad \delta_2 = \sqrt{\sum_{n=1}^{\infty} \frac{1}{\lambda_n^{6k} \omega^2}}, \tag{38}$$

$$\delta_3 = \sqrt{\sum_{n=1}^{\infty} \max_{t \in [0, T]} \left[ \frac{1}{\lambda_n^{3k} \omega} \int_0^T |G_n(t, s, \nu, \omega)| ds \right]^2}. \tag{39}$$

Continuing this process, similarly to estimate (37) we obtain

$$\begin{aligned} & \sum_{n=1}^{\infty} \max_{t \in [0, T]} |u_n^{r+1}(t) - u_n^r(t)| \leq \\ & \leq |\nu| \delta_2 \sqrt{\sum_{n=1}^{\infty} \left[ \sum_{i=1}^m \left| \frac{\Delta_{4in}(\nu, \omega, u_n^r) - \Delta_{4in}(\nu, \omega, u_n^{r-1})}{\Delta_{0n}(\nu, \omega)} \right| \delta_{0i} \right]^2} + \\ & + \delta_3 \sqrt{\sum_{n=1}^{\infty} \max_{t \in [0, T]} |u_n^r(t, \nu, \omega) - u_n^{r-1}(t, \nu, \omega)|^2} \leq \\ & \leq |\nu| \delta_2 \sqrt{\sum_{n=1}^{\infty} \left[ \sum_{i=1}^m \left| \frac{\bar{\Delta}_{4in}(\nu, \omega)}{\Delta_{0n}(\nu, \omega)} \right| \delta_{0i} \right]^2} \|u^r(t, \nu, \omega) - u^{r-1}(t, \nu, \omega)\|_{B_2} + \\ & + \delta_3 \|u^r(t, \nu, \omega) - u^{r-1}(t, \nu, \omega)\|_{B_2} \leq \rho \cdot \|u^r(t, \nu, \omega) - u^{r-1}(t, \nu, \omega)\|_{B_2}, \end{aligned} \tag{40}$$

where

$$\rho = |\nu| \delta_2 \left\| \sum_{i=1}^m \left| \frac{\bar{\Delta}_{4in}(\nu, \omega)}{\Delta_{0n}(\nu, \omega)} \right| \delta_{0i} \right\|_{\ell_2} + \delta_3, \tag{41}$$

$$\bar{\Delta}_{4in}(\nu, \omega) = \begin{vmatrix} 1 - \frac{\nu}{\lambda} \sigma_{311n} & \dots & \frac{\nu}{\lambda} \sigma_{31(i-1)n} & \bar{\sigma}_{41n} & \frac{\nu}{\lambda} \sigma_{31(i+1)n} & \dots & \frac{\nu}{\lambda} \sigma_{31mn} \\ \frac{\nu}{\lambda} \sigma_{321n} & \dots & \frac{\nu}{\lambda} \sigma_{32(i-1)n} & \bar{\sigma}_{42n} & \frac{\nu}{\lambda} \sigma_{32(i+1)n} & \dots & \frac{\nu}{\lambda} \sigma_{32mn} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \frac{\nu}{\lambda} \sigma_{3m1n} & \dots & \frac{\nu}{\lambda} \sigma_{3m(i-1)n} & \bar{\sigma}_{4mn} & \frac{\nu}{\lambda} \sigma_{3m(i+1)n} & \dots & 1 - \frac{\nu}{\lambda} \sigma_{3mmn} \end{vmatrix},$$

$$\bar{\sigma}_{4in} = \frac{1}{\lambda} \int_0^T |b_i(s)| \int_0^T |H_n(s, \theta, \omega) \alpha(\theta)| d\theta ds.$$

According to the last condition (34) of the theorem, we have  $\rho < 1$ . Consequently, it follows from estimate (40) that the operator on the right-hand sides of the countable system of functional-integral equations (30) is contracting. It follows from estimates (36), (37) and (40) that there is a unique fixed point, which is a solution to the countable system of functional-integral equations (30) in the space  $B_2$ . Theorem 1 is proved.

## 4 Uniformly convergence of Fourier series

*Theorem 2.* Let the conditions of Theorem 1 are fulfilled. Then the series in (29) are convergence in the segment  $[0, l]$ .

*Proof.* Let  $u_n(t, \nu, \omega) \in B_2$  be a solution of system (30). As in the case of estimates (36) and (40), we obtain

$$|\varphi_j(x)| \leq \sqrt{\frac{2}{l}} \left( \frac{\sqrt{2l}}{\pi} \right)^{4k+2} \delta_1 \delta_2 \left[ \left\| \frac{\partial^{4k+2} \psi_1(x)}{\partial x^{4k+2}} \right\|_{L_2(\Omega_l)} + \left\| \frac{\partial^{4k+2} \psi_2(x)}{\partial x^{4k+2}} \right\|_{L_2(\Omega_l)} \right] +$$

$$+ |\nu| \delta_2 \sqrt{\sum_{n=1}^{\infty} \left[ \sum_{i=1}^m \left| \frac{\Delta_{4in}(\nu, \omega, u_n)}{\Delta_{0n}(\nu, \omega)} \right| \delta_{0i} \right]^2} + \delta_3 \|u(t, \nu, \omega)\|_{B_2} < \infty, \quad j=1, 2. \quad (42)$$

Absolutely and uniformly convergence of the series (29) implies from estimate (42).

Substituting system (30) into Fourier series (5), we obtain

$$U(t, x, \nu, \omega) = \sqrt{\frac{2}{l}} \sum_{n=1}^{\infty} \sin \lambda_n x [\psi_{1n} g_{1n}(t, \nu, \omega) + \psi_{2n} g_{2n}(t, \nu, \omega) +$$

$$+ \frac{\nu}{\lambda_n^k (1 + \lambda_n^{2k}) \omega} \sum_{i=1}^m \frac{\Delta_{4in}(\nu, \omega, u_n)}{\Delta_{0n}(\nu, \omega)} g_{3in}(t, \omega) +$$

$$+ \frac{1}{\lambda_n^k (1 + \lambda_n^{2k}) \omega} \int_0^T G_n(t, s, \nu, \omega) u_n(s, \nu, \omega) ds], \quad (\nu, \omega) \in \Lambda_4. \quad (43)$$

*Theorem 3.* Let the conditions of Theorem 1 are fulfilled. Then the main unknown function  $U(t, x, \nu, \omega)$  of inverse problem (1)–(4) is defined by Fourier series (43) and this series (43) converges absolutely and uniformly in the domain  $\Omega$  for all  $(\nu, \omega) \in \Lambda_4$ . Moreover, function (43) belongs to the class  $C(\bar{\Omega}) \cap C_{t,x}^{2,4k}(\Omega) \cap C_{t,x}^{2+2k}(\Omega)$ .

The *proof* of Theorem 3 is similar to the proof of Theorem 2.

## Conclusion

In the rectangular domain  $\Omega = \{0 < t < T, 0 < x < l\}$  we consider a linear Benney–Luke type partial integro-differential equation (1) of a higher order with degenerate kernel and two redefinition functions (3) given at the endpoint of the segment  $[0, T]$ . With respect to spatial variable  $x$  Dirichlet boundary value conditions (2) is used. To find these redefinition functions intermediate data (4) are used. The Fourier series method of variables separation is applied. The countable system of functional-integral equations (30) is obtained. Theorem 1 on a unique solvability of countable system of functional-integral equations (30) is proved. The method of successive approximations is used in combination with the method of contraction mappings. The triple of solutions for the inverse problem is obtained in the form of Fourier series (29) and (43). The absolutely and uniformly convergence of Fourier series is proved (Theorem 2 and 3).

## References

- 1 Cavalcanti M.M. Existence and uniform decay for a nonlinear viscoelastic equation with strong damping / M.M. Cavalcanti, D.N. Cavalcanti, J. Ferreira // *Math. Methods in the Appl. Sciences.* — 2001. — 24. — P. 1043–1053.

- 2 Быков Я.В. Некоторые задачи теории интегро-дифференциальных уравнений / Я.В. Быков. — Фрунзе: Изд-во Киргиз. гос. ун-та, 1957. — 327 с.
- 3 Джумабаев Д.С. Признаки корректной разрешимости линейной двухточечной краевой задачи для систем интегро-дифференциальных уравнений / Д.С. Джумабаев, Э.А. Бакирова // Дифф. уравнения. — 2010. — 46. — №. 4. — С. 550–564.
- 4 Dzhumabaev D.S. New general solution to a nonlinear Fredholm integro-differential equation / D.S. Dzhumabaev, S.T. Mynbayeva // Eurasian Math. Journal. — 2019. — 10. — No. 4. — P. 24–33. <https://doi.org/10.32523/2077-9879-2019-10-4-24-33>
- 5 Вайнберг М.М. Интегро-дифференциальные уравнения / М.М. Вайнберг // Итоги науки. — 1962. — М.: ВИНТИ, 1964. — С. 5–37.
- 6 Фалалаев М.В. Интегро-дифференциальные уравнения с фредгольмовым оператором при старшей производной в банаховых пространствах и их приложения / М.В. Фалалаев // Изв. Иркут. гос. ун-та. Математика. — 2012. — 5. — № 2. — С. 90–102. <https://www.mathnet.ru/rus/iigum/v5/i2/p90>
- 7 Сидоров Н.А. Решения задачи Коши для класса интегро-дифференциальных уравнений с аналитическими нелинейностями / Н.А. Сидоров // Дифф. уравнения. — 1968. — 4. — №. 7. — С. 1309–1316. <https://www.mathnet.ru/rus/de/v4/i7/p1309>
- 8 Ушаков Е.И. Статистическая устойчивость электрических систем / Е.И. Ушаков. — Новосибирск: Наука, 1988. — 273 с.
- 9 Abildayeva A.T. To a unique solvability of a problem with integral condition for integro-differential equation / A.T. Abildayeva, R.M. Kaparova, A.T. Assanova // Lobachevskii Journal of Mathematics. — 2021. — 42. — No. 12. — P. 2697–2706. <https://doi.org/10.1134/S1995080221120039>
- 10 Asanova A.T. Correct solvability of a nonlocal boundary value problem for systems of hyperbolic equations / A.T. Asanova, D.S. Dzhumabaev // Doklady Mathematics. — 2003. — 68. — No. 1. — P. 46–49. <https://www.mathnet.ru/eng/dan1763>
- 11 Assanova A.T. On the solvability of nonlocal problem for the system of Sobolev-type differential equations with integral condition / A.T. Assanova // Georgian Mathematical Journal. — 2021. — 28. — No. 1. — P. 49–57. <https://doi.org/10.1515/gmj-2019-2011>
- 12 Asanova A.T. Well-posedness of nonlocal boundary value problems with integral condition for the system of hyperbolic equations / A.T. Asanova, D.S. Dzhumabaev // Journal of Mathematical Analysis and Applications. — 2013. — 402. — No. 1. — P. 167–178. <https://doi.org/10.1016/j.jmaa.2013.01.012>
- 13 Assanova A.T. A nonlocal problem for loaded partial differential equations of fourth order / A.T. Assanova, A.E. Imanchiyev, Zh.M. Kadirbayeva // Bulletin of the Karaganda University. Mathematics Series. — 2020. — No. 1(97). — P. 6–16. <https://doi.org/10.31489/2020M1/6-16>
- 14 Assanova A.T. A nonlocal multipoint problem for a system of fourth-order partial differential equations / A.T. Assanova, Z.S. Tokmurzin // Eurasian Math. Journal. — 2020. — 11. — No. 3. — P. 8–20. <https://doi.org/10.32523/2077-9879-2020-11-3-08-20>
- 15 Ashurov R.R. On the nonlocal problems in time for subdiffusion equations with the Riemann–Liouville derivatives / R.R. Ashurov, Yu.E. Fayziev // Bulletin of the Karaganda University. Mathematics Series. — 2022. — No. 2(106). — P. 18–37. <https://doi.org/10.31489/2022M2/18-37>
- 16 Гордезиани Д.Г. Решения нелокальных задач для одномерных колебаний среды / Д.Г. Гордезиани, Г.А. Авалишвили // Математическое моделирование. — 2000. — 12. — № 1. — С. 94–103. <https://www.mathnet.ru/rus/mm/v12/i1/p94>
- 17 Dzhumabaev D.S. Well-posedness of nonlocal boundary value problem for a system of loaded hyperbolic equations and an algorithm for finding its solution / D.S. Dzhumabaev // Journal of

- Mathematical Analysis and Applications. — 2018. — 461. — No. 1. — P. 1439–1462. <https://doi.org/10.1016/j.jmaa.2017.12.005>
- 18 Isgenderov N.Sh. On solvability of an inverse boundary value problem for the Boussinesq-Love equation with periodic and integral condition / N.Sh. Isgenderov, S.I. Allahverdiyeva // Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. Mathematics. — 2021. — 41. — No. 1. — P. 118–132.
  - 19 Иванчов Н.И. Краевые задачи для параболического уравнения с интегральными условиями / Н.И. Иванчов // Дифф. уравнения. — 2004. — 40. — № 4. — С. 591–609. <https://doi.org/10.1023/B:DIEQ.0000035796.56467.44>
  - 20 Zunnunov R.T. A problem with shift for a mixed-type model equation of the second kind in an unbounded domain / R.T. Zunnunov, A.A. Ergashev // Bulletin of the Karaganda University. Mathematics Series. — 2022. — No. 2(106). — P. 202–207. <https://doi.org/10.31489/2022M2/202-207>
  - 21 Mammedzadeh G.S. On a boundary value problem with spectral parameter quadratically contained in the boundary condition / G.S. Mammedzadeh // Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. Mathematics. — 2022. — 42. — No. 1. — P. 141–150.
  - 22 Ochilova N.K. On a nonlocal boundary value problem for a degenerate parabolichyperbolic equation with fractional derivative / N.K. Ochilova, T.K. Yuldashev // Lobachevskii Journal of Mathematics. — 2022. — 43. — No. 1. — P. 229–236. <https://doi.org/10.1134/S1995080222040175>
  - 23 Тихонов И.В. Теоремы единственности в линейных нелокальных задачах для абстрактных дифференциальных уравнений / И.В. Тихонов // Изв. РАН. Сер. мат. — 2003. — 67. — № 2. — С. 133–166. <https://doi.org/10.4213/im429>
  - 24 Юрко В.А. Обратные задачи для интегро-дифференциальных операторов первого порядка / В.А. Юрко // Мат. заметки. — 2016. — 100. — № 6. — С. 939–946. <https://doi.org/10.4213/mzm11112>
  - 25 Yuldashev T.K. Determination of the coefficient and boundary regime in boundary value problem for integro-differential equation with degenerate kernel / T.K. Yuldashev // Lobachevskii Journal of Mathematics. — 2017. — 38. — No. 3. — P. 547–553. <https://doi.org/10.1134/S199508021703026X>
  - 26 Yuldashev T.K. Nonlocal problem for a nonlinear fractional mixed type integrodifferential equation with spectral parameters / T.K. Yuldashev, F.D. Rakhmonov // AIP Conference Proceedings. — 2021. — No. 2365. ID 060003. — 20 p. <https://doi.org/10.1063/5.0057147>
  - 27 Зарипов С.К. Построение аналога теоремы Фредгольма для одного класса модельных интегродифференциальных уравнений первого порядка с особой точкой в ядре / С.К. Зарипов // Вестн. Том. гос. ун-та. Математика и механика. — 2017. — 46. — С. 24–35. <https://doi.org/10.17223/19988621/46/4>
  - 28 Зарипов С.К. Построение аналога теорем Фредгольма для одного класса модельных интегродифференциальных уравнений первого порядка с логарифмической особенностью в ядре / С.К. Зарипов // Вестн. Самар. гос. техн. ун-та. Физ.-мат. науки. — 2017. — 21. — № 2. — С. 236–248. <https://doi.org/10.14498/vsgtu1515>
  - 29 Зарипов С.К. О новом методе решения одного класса модельных интегро-дифференциальных уравнений первого порядка с особенностью в ядре / С.К. Зарипов // Матем. физ. и комп. моделирование. — 2017. — 20. — № 4. — С. 68–75.
  - 30 Юлдашев Т.К. Об одном интегро-дифференциальном уравнении Фредгольма в частных производных третьего порядка / Т.К. Юлдашев // Изв. вузов. Математика. — 2015. — № 9. — С. 74–79.
  - 31 Юлдашев Т.К. Об одной краевой задаче для интегро-дифференциального уравнения в част-

- ных производных четвертого порядка с вырожденным ядром / Т.К. Юлдашев // Итоги науки и техн. Сер. Сов. мат. и ее прилож. Тем. обз. — 145. — М.: ВИНТИ РАН, 2018. — С. 95–109. <https://www.mathnet.ru/rus/into/v145/p95>
- 32 Юлдашев Т.К. Определение коэффициента и классическая разрешимость нелокальной краевой задачи для интегро-дифференциального уравнения Бенни-Люка с вырожденным ядром / Т.К. Юлдашев // Итоги науки и техн. Сер. Совр. мат. и ее прилож. Тем. обз. — 156. — М.: ВИНТИ РАН, 2018. — С. 89–102. <https://www.mathnet.ru/rus/into/v156/p89>
- 33 Юлдашев Т.К. Обратная краевая задача для интегро-дифференциального уравнения типа Буссинеска с вырожденным ядром / Т.К. Юлдашев // Итоги науки и техн. Сер. Совр. мат. и ее прилож. Тем. обз. — 149. — М.: ВИНТИ РАН, 2018. — С. 129–140. <https://www.mathnet.ru/rus/into/v149/p129>
- 34 Yuldashev T.K. Boundary value problem for third order partial integro-differential equation with a degenerate kernel / T.K. Yuldashev, Yu.P. Apakov, A.Kh. Zhuraev // Lobachevskii Journal of Mathematics. — 2021. — 42. — No. 6. — P. 1317–1327. <https://doi.org/10.1134/S1995080221060329>
- 35 Yuldashev T.K. On a boundary value problem for a fifth order partial integro-differential equation / T.K. Yuldashev // Azerbaijan Journal of Mathematics. — 2022. — 12. — No. 2. — P. 154–172.
- 36 Yuldashev T.K. On a nonlocal boundary value problem for a partial integro-differential equations with degenerate kernel / T.K. Yuldashev // Vladikavkaz. Mathematical Journal. — 2022. — 24. — No. 2. — P. 130–141. <https://doi.org/10.46698/h5012-2008-4560-g>
- 37 Pskhu A.V. Boundary value problem for fractional diffusion equation in a curvilinear angle domain / A.V. Pskhu, M.I. Ramazanov, N.K. Gulmanov, S.A. Iskakov // Bulletin of the Karaganda University. Mathematics Series. — 2022. — No. 1(105). — P. 83–95. <https://doi.org/10.31489/2022M1/83-95>
- 38 Jenaliyev M.T. To the solution of the Solonnikov-Fasano problem with boundary moving on arbitrary law  $x = \gamma(t)$  / M.T. Jenaliyev, M.I. Ramazanov, A.O. Tanin // Bulletin of the Karaganda University. Mathematics Series. — 2021. — No. 1(101). — P. 37–49. <https://doi.org/10.31489/2021M1/37-49>

Т.К. Юлдашев

*Ташкент мемлекеттік экономикалық университети, Ташкент, Өзбекстан*

## Екі қайта анықтау функциясы мен параметрлері бар Бенни-Люк типті интегралдық-дифференциалдық теңдеу үшін аралас кері есеп

Мақалада сегменттің шеттерінде берілген екі қайта анықтау функциясы мен өзгешеленетін ядросы бар Бенни-Люк типті жоғары ретті сызықтық интегралдық-дифференциалды дербес туындылы дифференциалдық теңдеуі қарастырылған. Қайта анықтау функцияларын табу үшін аралық берілгендер пайдаланылған. Кеңістіктік айнымалыға қатысты Дирихле типінің шекаралық шарттары қолданылған. Айнымалыны бөліктеу үшін Фурье әдісі пайдаланылды. Функционалдық интегралдық теңдеулердің есептелетін жүйесі алынды. Функционалдық интегралдық теңдеулердің санаулы жүйесінің бірімәнді шешілетіндігі туралы теорема дәлелденді. Бұл жағдайда біртіндеп жуықтау әдісі сығылған бейнелеу әдісімен бірге қолданылады. Кері есептің шешімі Фурье қатары түрінде құрылады. Алынған Фурье қатарының абсолютті және бірқалыпты жинақтылығы нақтыланды.

*Кілт сөздер:* кері есеп, екі қайта анықтау функциясы, кейінгі шарттар, аралық функциялар, Фурье әдісі, бірімәнді шешім.

Т. К. Юлдашев

*Ташкентский государственный экономический университет, Ташкент, Узбекистан*

## Смешанная обратная задача для интегро-дифференциального уравнения типа Бенни–Люка с двумя функциями переопределения и параметрами

В статье рассмотрено линейное интегро-дифференциальное уравнение в частных производных типа Бенни–Люка высокого порядка с вырожденным ядром и двумя функциями переопределения, заданными в конце отрезка. Для нахождения этих функций переопределения использованы промежуточные данные. По отношению к пространственной переменной применены краевые условия типа Дирихле. Применяется метод разделения переменных Фурье. Получена счетная система функционально-интегральных уравнений. Доказана теорема об однозначной разрешимости счетной системы функционально-интегральных уравнений. При этом используется метод последовательных приближений в сочетании с методом сжатых отображений. Решение обратной задачи строится в виде ряда Фурье. Доказана абсолютная и равномерная сходимость полученных рядов Фурье.

*Ключевые слова:* обратная задача, две функции переопределения, финальные условия, промежуточные функции, метод Фурье, однозначная разрешимость.

### References

- 1 Cavalcanti, M.M., Cavalcanti, V.D., & Ferreira, J. (2001). Existence and uniform decay for a nonlinear viscoelastic equation with strong damping. *Math. Methods in the Appl. Sciences*, 24, 1043–1053. <https://doi.org/10.1002/mma.250>
- 2 Выков, Ya.V. (1957). *Nekotorye zadachi teorii integro-differentsialnykh uravnenii [On some problems in the theory of integro-differential equations]*. Frunze: Izdatelstvo Kirgizskogo gosudarstvennogo universiteta, 327 [in Russian].
- 3 Dzhumabaev, D.S., & Bakirova, E.A. (2010). Criteria for the well-posedness of a linear two-point boundary value problem for systems of integro-differential equations. *Differential Equations*, 46(4), 553–567. <https://doi.org/10.1134/S0012266110040117>
- 4 Dzhumabaev, D.S., & Mynbayeva, S.T. (2019). New general solution to a nonlinear Fredholm integro-differential equation. *Eurasian Math. Journal*, 10(4), 24–33. <https://doi.org/10.32523/2077-9879-2019-10-4-24-33>
- 5 Vainberg, M.M. (1964). Integro-differentsialnye uravneniia [Integro-differential equations]. *Itogi nauki – Results of Science*, 1962. Moscow: VINITI, 5–37 [in Russian].
- 6 Falaleev, M.V. (2012). Integro-differentsialnye uravneniia s operatorom Fredgolma pri starshei proizvodnoi v banakhovykh prostranstvakh i ikh prilozheniia [Integro-differential equations with a Fredholm operator at the highest derivative in Banach spaces and their applications]. *Izvestiia Irkutskogo gosudarstvennogo universiteta. Matematika – News of the Irkutsk State University. Series «Mathematics»*, 5(2), 90–102 [in Russian]. <https://www.mathnet.ru/rus/iigum/v5/i2/p90>
- 7 Sidorov, N.A. (1968). Reshenie zadachi Koshi dlia odnogo klassa integro-differentsialnykh uravnenii s analiticheskimi nelineinostiami [Solution of the Cauchy problem for a class of integro-differential equations with analytic nonlinearities]. *Differentsialnye uravneniia – Differential Equations*, 4(7), 1309–1316 [in Russian].
- 8 Ushakov, E.I. (1988). *Staticheskaiia ustoiчивost elektricheskikh sistem [Static stability of electrical systems]*. Novosibirsk: Nauka [in Russian].

- 9 Abildayeva, A.T., Kaparova, R.M., & Assanova, A.T. (2021). To a unique solvability of a problem with integral condition for integro-differential equation. *Lobachevskii Journal of Mathematics*, 42(12), 2697–2706. <https://doi.org/10.1134/S1995080221120039>
- 10 Asanova, A.T., & Dzhumabaev, D.S. (2003). Correct solvability of a nonlocal boundary value problem for systems of hyperbolic equations. *Doklady Mathematics*, 68(1), 46–49. <https://www.mathnet.ru/eng/dan1763>
- 11 Assanova, A.T. (2021). On the solvability of nonlocal problem for the system of Sobolev-type differential equations with integral condition. *Georgian Mathematical Journal*, 28(1), 49–57. <https://doi.org/10.1515/gmj-2019-2011>
- 12 Asanova, A.T., & Dzhumabaev, D.S. (2013). Well-posedness of nonlocal boundary value problems with integral condition for the system of hyperbolic equations. *Journal of Mathematical Analysis and Applications*, 402(1), 167–178. <https://doi.org/10.1016/j.jmaa.2013.01.012>
- 13 Assanova, A.T., Imanchiyev, A.E., & Kadirbayeva, Zh.M. (2020). A nonlocal problem for loaded partial differential equations of fourth order. *Bulletin of the Karaganda University. Mathematics Series*, 1(97), 6–16. <https://doi.org/10.31489/2020M1/6-16>
- 14 Assanova, A.T., & Tokmurzin, Z.S. (2020). A nonlocal multipoint problem for a system of fourth-order partial differential equations. *Eurasian Math. Journal*, 11(3), 8–20. <https://doi.org/10.32523/2077-9879-2020-11-3-08-20>
- 15 Ashurov, R.R., & Fayziev, Yu.E. (2022). On the nonlocal problems in time for subdiffusion equations with the Riemann-Liouville derivatives. *Bulletin of the Karaganda University. Mathematics Series*, 2(106), 18–37. <https://doi.org/10.31489/2022M2/18-37>
- 16 Gordeziani, D.G., & Avalishvili, G.A. (2000). Resheniia nelokalnykh zadach dlia odnomernykh kolebaniy sredy [Solutions of nonlocal problems for one-dimensional vibrations of a medium]. *Matematicheskoe modelirovanie – Mathematical Models and Computer Simulations*, 12(1), 94–103 [in Russian].
- 17 Dzhumabaev, D.S. (2018). Well-posedness of nonlocal boundary value problem for a system of loaded hyperbolic equations and an algorithm for finding its solution. *Journal of Mathematical Analysis and Applications*, 461(1), 1439–1462. <https://doi.org/10.1016/j.jmaa.2017.12.005>
- 18 Isgenderov, N.Sh., & Allahverdiyeva, S.I. (2021). On solvability of an inverse boundary value problem for the Boussinesq-Love equation with periodic and integral condition. *Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. Mathematics*, 41(1), 118–132.
- 19 Ivanchov, N.I. (2004). Boundary value problems for a parabolic equation with integral conditions. *Differential Equations*, 40(4), 591–609. <https://doi.org/10.1023/B:DIEQ.0000035796.56467.44>
- 20 Zunnunov, R.T., & Ergashev, A.A. (2022). A problem with shift for a mixed-type model equation of the second kind in an unbounded domain. *Bulletin of the Karaganda University. Mathematics Series*, 2(106), 202–207. <https://doi.org/10.31489/2022M2/202-207>
- 21 Mammedzadeh, G.S. (2022). On a boundary value problem with spectral parameter quadratically contained in the boundary condition. *Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. Mathematics*, 42(1), 141–150.
- 22 Ochilova, N.K., & Yuldashev, T.K. (2022). On a nonlocal boundary value problem for a degenerate parabolichyperbolic equation with fractional derivative. *Lobachevskii Journal of Mathematics*, 43(1), 229–236. <https://doi.org/10.1134/S1995080222040175>
- 23 Tikhonov, I.V. (2003). Theorems on uniqueness in linear nonlocal problems for abstract differential equations. *Izvestiya: Mathematics*, 67(2), 333–363. <https://doi.org/10.1070/IM2003v067n02ABEH000429>
- 24 Yurko, V.A. (2016). Inverse problems for first-order integro-differential operators. *Math. Notes*,

- 100(6), 876–882. <https://doi.org/10.1134/S0001434616110286>
- 25 Yuldashev, T.K. (2017). Determination of the coefficient and boundary regime in boundary value problem for integro-differential equation with degenerate kernel. *Lobachevskii Journal of Mathematics*, 38(3), 547–553. <https://doi.org/10.1134/S199508021703026X>
- 26 Yuldashev, T.K., & Rakhmonov, F.D. (2021). Nonlocal problem for a nonlinear fractional mixed type integro-differential equation with spectral parameters. *AIP Conference Proceedings*, 2365 (060003), 1–20. <https://doi.org/10.1063/5.0057147>
- 27 Zaripov, S.K. (2017). Postroenie analoga teoremy Fredgolma dlia odnogo klassa modelnykh integro-differentsialnykh uravnenii pervogo poriadka s osoboi tochkoii v yadre [Construction of an analog of the Fredholm theorem for a class of model first order integrodifferential equations with a singular point in the kernel]. *Vestnik Tomskogo gosudarstvennogo universiteta. Matematika i mekhanika — Vestnik of Tomsk State University. Mathematics and Mechanics*, 46, 24–35 [in Russian]. <https://doi.org/10.17223/19988621/46/4>
- 28 Zaripov, S.K. (2017). Postroenie analoga teorem Fredgolma dlia odnogo klassa modelnykh integro-differentsialnykh uravnenii pervogo poriadka s logarifmicheskoi osobennostiu v yadre [A construction of analog of Fredholm theorems for one class of first order model integrodifferential equation with logarithmic singularity in the kernel]. *Vestnik Samarskogo gosudarstvennogo tekhnicheskogo universiteta. Fiziko-matematicheskie nauki — Bulletin of the Samara State Technical University. Series: Physical and Mathematical Sciences*, 21(2), 236–248 [in Russian]. <https://doi.org/10.14498/vsgtu1515>
- 29 Zaripov, S.K. (2017). O novom metode resheniia odnogo klassa modelnykh integro-differentsialnykh uravnenii pervogo poriadka s osobennostiu v yadre [On a new method of solving of one class of model first-order integro-differential equations with singularity in the kernel]. *Matemematischekaia fizika i kompiuternoe modelirovanie — Mathematical Physics and Computer Modeling*, 20(4), 68–75 [in Russian].
- 30 Yuldashev, T.K. (2015). A Certian Fredholm Partial Integro-Differential Equation of the Third Order. *Russian Mathematics (Iz VUZ)*, 59(9), 62–66. <https://doi.org/10.3103/S1066369X15090091>
- 31 Yuldashev, T.K. (2020). On a boundary-value problem for a fourth-order partial integro-differential equation with degenerate kernel. *Journal of Mathematical Sciences*, 245(4), 508–523. <https://doi.org/10.1007/s10958-020-04707-2>
- 32 Yuldashev, T.K. (2021). Determining of coefficients and the classical solvability of a nonlocal boundary-value problem for the Benney–Luke integro-differential equation with degenerate kernel. *Journal of Mathematical Sciences*, 254(6), 793–807. <https://doi.org/10.1007/s10958-021-05341-2>
- 33 Yuldashev, T.K. (2020). Inverse boundary-value problem for an integro-differential Boussinesq-type equation with degenerate kernel. *Journal of Mathematical Sciences*, 250(5), 847–858. <https://doi.org/10.1007/s10958-020-05050-2>
- 34 Yuldashev, T.K., Apakov, Yu.P., & Zhuraev, A.Kh. (2021). Boundary value problem for third order partial integro-differential equation with a degenerate kernel. *Lobachevskii Journal of Mathematics*, 42(6), 1317–1327. <https://doi.org/10.1134/S1995080221060329>
- 35 Yuldashev, T.K. (2022). On a boundary value problem for a fifth order partial integro-differential equation. *Azerbaijan Journal of Mathematics*, 12(2), 154–172.
- 36 Yuldashev, T.K. (2022). On a nonlocal boundary value problem for a partial integro-differential equations with degenerate kernel. *Vladikavkaz. Mathematical Journal*, 24(2), 130–141. <https://doi.org/10.46698/h5012-2008-4560-g>
- 37 Pskhu, A.V., Ramazanov, M.I., Gulmanov, N.K., & Iskakov, S.A. (2022). Boundary value problem

for fractional diffusion equation in a curvilinear angle domain. *Bulletin of the Karaganda University. Mathematics Series*, 1(105), 83–95. <https://doi.org/10.31489/2022M1/83-95>

- 38 Jenaliyev, M.T., Ramazanov, M.I., & Tanin, A.O. (2021). To the solution of the Solonnikov-Fasano problem with boundary moving on arbitrary law  $x = \gamma(t)$ . *Bulletin of the Karaganda University. Mathematics Series*, 1(101), 37–49. <https://doi.org/10.31489/2021M1/37-49>

Buketov University