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On the well-posedness of periodic problems for the system of hyperbolic equations with finite time delay

Anar T. Assanova[1] | **Narkesh B. Iskakova**[1,2] | **Nurgul T. Orumbayeva**[3]

[1]Department of Differential Equations, Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan

[2]Abai Kazakh National Pedagogical University, Almaty, Kazakhstan

[3]E.A. Buketov Karaganda State University, Karagandy, Kazakhstan

Correspondence
Anar T. Assanova, Department of Differential Equations, Institute of Mathematics and Mathematical Modeling, Pushkin str, 125, Almaty 050010, Kazakhstan.
Email: anartasan@gmail.com

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Abstract

A periodic problem for the system of hyperbolic equations with finite time delay is investigated. The investigated problem is reduced to an equivalent problem, consisting the family of periodic problems for a system of ordinary differential equations with finite delay and integral equations using the method of a new functions introduction. Relationship of periodic problem for the system of hyperbolic equations with finite time delay and the family of periodic problems for the system of ordinary differential equations with finite delay is established. Algorithms for finding approximate solutions of the equivalent problem are constructed, and their convergence is proved. Criteria of well-posedness of periodic problem for the system of hyperbolic equations with finite time delay are obtained.

KEYWORDS

algorithm, family of periodic problems for system of differential equations with finite delay, periodic problem, system of hyperbolic equations with delayed argument, unique solvability

MSC CLASSIFICATION

34K06; 34K13; 35L20; 35L55

1 | INTRODUCTION AND STATEMENT OF PROBLEM

Various problems of population dynamics, mathematical biology, ecology, management of technical systems, the problem of physics, and etc, variational problems related to the regulatory process, the optimal control problem with delay systems lead to boundary value problems for differential equations with time delay.[1-24] Periodic and nonlocal problems for the hyperbolic equations with time delay arise of mathematical modeling of the numerous processes in biology, physics, chemistry, and mechanics.[9,18,25-30] To investigate the questions of solvability of these problems' classes, the methods of the qualitative theory of differential equations and the theory of oscillations, Riemann's method, numerical-analytical method, the method of monotone iteration, asymptotic methods, the method of upper and lower solutions and others have been applied. Based on their results, there have been obtained the solvability conditions for the considered problems and suggested the ways of finding solutions. Study of qualitative properties of periodic and nonlocal problems for the equations of hyperbolic type with time delay, as well as the conditions of solvability and finding solutions is associated with many problems, such as the complexity of considered objects, the impossibility of constructing the analytical solution, the lack of universal methods of solving, and difficulties with adaptation of known methods. Note that the periodic problems for system of hyperbolic equations with finite and infinite time delay are widely applied in various fields. Nevertheless, the problem of finding effective features of unique solvability of periodic problems for system of hyperbolic equations with time delay still holds relevant today.

In this paper, we study the questions of existence and uniqueness of periodic solution to the system of hyperbolic equations with finite time delay. Periodic problems for the system of hyperbolic equations with finite time delay will be reduced to the family of periodic problems for the system of ordinary differential equations with finite time delay and the integral equations. We establish a connection between conditions of the solvability to the periodic problem for the system of hyperbolic equations with finite time delay and the solvability of the family of periodic problems for the system of ordinary differential equations with finite delay.

So, we consider the following periodic problem for the system of hyperbolic equations of the second order with time delay on the domain $\Omega_\tau = [-\tau, T] \times [0, \omega]$

$$\frac{\partial^2 u(t,x)}{\partial t \partial x} = A(t,x)\frac{\partial u(t,x)}{\partial x} + A_0(t,x)\frac{\partial u(t-\tau,x)}{\partial x} + B(t,x)\frac{\partial u(t,x)}{\partial t} + C(t,x)u(t,x) + f(t,x), \tag{1}$$

$$(t,x) \in \Omega = [0,T] \times [0,\omega],$$

$$\frac{\partial u(z,x)}{\partial x} = diag\left[\frac{\partial u(0,x)}{\partial x}\right] \cdot \varphi(z), \quad z \in [-\tau, 0], \qquad x \in [0,\omega], \tag{2}$$

$$u(0,x) = u(T,x), \qquad x \in [0,\omega], \tag{3}$$

$$u(t,0) = \psi(t), \qquad t \in [-\tau, T], \tag{4}$$

where $u(t,x) = col(u_1(t,x), u_2(t,x), \ldots, u_n(t,x))$ is unknown function, the $(n \times n)$ matrices $A(t,x)$, $A_0(t,x)$, $B(t,x)$, $C(t,x)$ and n vector-function $f(t,x)$ are continuous on Ω, the n vector-function $\varphi(t)$ is continuously differentiable and given on the initial set $[-\tau, 0]$ such that $\varphi_i(0) = 1$, $i = \overline{1,n}$, $\tau > 0$ is constant delay, the n vector-function $\psi(t)$ is continuously differentiable on $[-\tau, T]$, and the compatibility condition is valid: $\psi(0) = \psi(T)$.

Let

$C(\Omega_\tau, R^n)$ be the space of continuous on Ω_τ vector functions $u(t,x)$ with the norm

$$||u||_0 = \max_{(t,x)\in\Omega_\tau} ||u(t,x)||, \, ||u(t,x)|| = \max_{i=\overline{1,n}} |u_i(t,x)|;$$

$C([0,\omega], R^n)$ be a space of continuous on $[0,\omega]$ vector functions $\varphi(x)$ with the norm

$$||\varphi||_{0,1} = \max_{x\in[0,\omega]} ||\varphi(x)||;$$

$C^1([-\tau, T], R^n)$ be a space of continuously differentiable on $[-\tau, T]$ vector functions $\psi(t)$ with the norm

$$||\psi||_{1,0} = \max\left(\max_{t\in[-\tau,T]} ||\psi(t)||, \max_{t\in[-\tau,T]} ||\dot{\psi}(t)||\right);$$

$\Omega_0 = \{(t,x) : t = 0, 0 \le x \le \omega\}$.

The function $u(t,x) \in C(\Omega_\tau, R^n)$, that has partial derivatives $\frac{\partial u(t,x)}{\partial x} \in C(\Omega_\tau, R^n)$, $\frac{\partial u(t,x)}{\partial t} \in C(\Omega_\tau \backslash \Omega_0, R^n)$, $\frac{\partial^2 u(t,x)}{\partial t \partial x} \in C(\Omega_\tau \backslash \Omega_0, R^n)$ is called a *classical solution* to periodic problems (1) to (4) if it satisfies system (1) for all $(t,x) \in \Omega$ and the condition (2) in the initial set $[-\tau, 0]$, the boundary conditions (3), (4).

Periodic and nonlocal problems for system (1) without delayed argument were researched by numerous authors. In the monograph,[27] the numerical-analytical method is applied to the study of periodic problems for equations and systems of partial differential equations of hyperbolic type with deviating argument. In Asanova and Dzhumabaev,[31] the well-posedness of the nonlocal problem with integral condition for the system of hyperbolic equations (1) without delayed argument was investigated. In Dzhumabaev,[32] the well-posedness of the nonlocal boundary value problem for the system of loaded hyperbolic equations of the second order without delayed argument was studied. In the present paper, the proposed approach is developed to periodic problem for the system of hyperbolic equations with finite time delay (1) to (4).

2 | UNIQUE SOLVABILITY OF A FAMILY OF PERIODIC PROBLEMS FOR SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS WITH FINITE DELAY

In this section, the periodic problem is reduced to a family of periodic problems for a system of ordinary differential equations with finite delay and integral relations. Using the parametrization method,[33] the sufficient conditions of unique solvability of the family periodic problems for system of ordinary differential equations with finite delay are obtained in the terms of initial data.

We introduce a new unknown functions $v(t,x) = \frac{\partial u(t,x)}{\partial x}$ and $w(t,x) = \frac{\partial u(t,x)}{\partial t}$ and reduce problems (1) to (4) to the equivalent problem

$$\frac{\partial v(t,x)}{\partial t} = A(t,x)v(t,x) + A_0(t,x)v(t-\tau,x) + F(t,x,w(t,x),u(t,x)), \quad (t,x) \in \Omega, \tag{5}$$

$$v(z,x) = diag[v(0,x)] \cdot \varphi(z), \quad z \in [-\tau,0], \quad x \in [0,\omega], \tag{6}$$

$$v(0,x) = v(T,x), \quad x \in [0,\omega], \tag{7}$$

$$u(t,x) = \psi(t) + \int_0^x v(t,\xi)d\xi, \quad w(t,x) = \dot{\psi}(t) + \int_0^x \frac{\partial v(t,\xi)}{\partial t}d\xi, \tag{8}$$

where $F(t,x,w(t,x),u(t,x)) = B(t,x)w(t,x) + C(t,x)u(t,x) + f(t,x)$.

In problems (5) to (8), the condition $u(t,0) = \psi(t)$ is included in integral relations (8).

A triple $\{v(t,x), u(t,x), w(t,x)\}$ of functions is called a solution to problems (5) to (8) if the function $v(t,x)$ belonging to $C(\Omega_\tau, R^n)$ has a continuous derivative with respect to t on $\Omega_\tau \backslash \Omega_0$ and satisfies the one-parameter family of periodic problems for ordinary differential equations with finite delay (5) to (7), where the functions $u(t,x)$ and $w(t,x)$ are connected with $v(t,x)$ and $\frac{\partial v(t,x)}{\partial t}$ by the integral relations (8).

Let $u^*(t,x)$ be a classical solution of periodic problems (1) to (4). Then the triple $\{v^*(t,x), u^*(t,x), w^*(t,x)\}$, where $v^*(t,x) = \frac{\partial u^*(t,x)}{\partial x}$, $w^*(t,x) = \frac{\partial u^*(t,x)}{\partial t}$, is a solution to problems (5) to (8). Conversely, if a triple $\{\widetilde{v}(t,x), \widetilde{u}(t,x), \widetilde{w}(t,x)\}$ is a solution to problems (5) to (8), then $\widetilde{u}(t,x)$ is a classical solution to periodic problems (1) to (4).

For fixed $w(t,x)$, $u(t,x)$ in problems (5) to (8), it is necessary to find a solution to a one-parameter family of periodic problems for system of ordinary differential equations with finite delay.

Hereby, problems (1) to (4) reduce to an equivalent problem, consisting the family of periodic problem for system of differential equations with finite delay and integral relations.

Consider the following family of periodic problems for system of ordinary differential equations with finite delay

$$\frac{\partial v(t,x)}{\partial t} = A(t,x)v(t,x) + A_0(t,x)v(t-\tau,x) + g(t,x), \quad (t,x) \in \Omega, \quad v \in R^n, \tag{9}$$

$$v(z,x) = diag[v(0,x)] \cdot \varphi(z), \quad z \in [-\tau,0], \qquad x \in [0,\omega], \tag{10}$$

$$v(0,x) = v(T,x), \qquad x \in [0,\omega], \tag{11}$$

where the n vector function $g(t,x)$ is continuous on Ω_τ.

Continuous function $v : \Omega_\tau \to R^n$ that has a continuous derivative with respect to t on $\Omega_\tau \backslash \Omega_0$ is called a solution to the family periodic problems with finite delay (9) to (11) if it satisfies system (9) for all $(t,x) \in \Omega$ and has the values $v(0,x)$, $v(T,x)$ on the lines $t = 0$, $t = T$ and the equalities (10), (11) are valid for all $x \in [0,\omega]$, respectively.

For fixed $x \in [0,\omega]$ problems (9) to (11) are a linear periodic problem for the system of ordinary differential equations with finite delay.[7,9,17,18,20] Suppose a variable x is changed on $[0,\omega]$; then we obtain a family of periodic problems for ordinary differential equations with finite delay.

Let $\|A(t,x)\| = \max_{i=\overline{1,n}} \sum_{j=1}^{n} \|a_{ij}(t,x)\| \le \alpha(x)$, $\|A_0(t,x)\| = \max_{i=\overline{1,n}} \sum_{j=1}^{n} \|a_{ij}^0(t,x)\| \le \alpha_0(x)$, where the functions $\alpha(x)$ and $\alpha_0(x)$ are positive and continuous on $[0,\omega]$.

Further, we apply the parametrization method[33] to the family of periodic problems for system of ordinary differential equations with finite delay (9) to (11).

The scheme of parametrization method. Take the step $h = \frac{\tau}{l}$: $N\tau = T$, $l \in \mathbb{N}$, and we make the partition of domain Ω_τ on the following form

$$[-\tau,0)\times[0,\omega] \cup [0,T)\times[0,\omega] = \bigcup_{s=l}^{1}[-t_s,-t_{s-1})\times[0,\omega] \bigcup \bigcup_{r=1}^{lN}[t_{r-1},t_r)\times[0,\omega],$$

where $t_0 = 0, -t_s = -sh, s = \overline{1,l}$, $t_r = rh$, $r = \overline{1,lN}$.

Let $C(\Omega, t_r, R^{nlN})$ be the space of systems functions $v([t],x) = (v_1(t,x), \dots, v_{lN}(t,x))'$, where the function $v_r : [t_{r-1},t_r)\times[0,\omega] \to R^n$ is continuous and uniformly for $x \in [0,\omega]$ has a finite left-hand side limit $\lim_{t\to t_r-0} v_r(t,x)$, and $r = \overline{1,lN}$ with the norm

$$\|v([\cdot],x)\|_2 = \max_{r=\overline{1,lN}} \sup_{t\in[t_{r-1},t_r)} \|v_r(t,x)\|.$$

Let the function $v(t,x)$ be a solution to problems (9) to (11). By v_r denote the restriction of v to the domain $\Omega_r = [t_{r-1},t_r)\times[0,\omega]$ such that $v_r : \Omega_r \to R^n$ and $v_r(t,x) = v(t,x)$ for all $(t,x)\in\Omega_r$ and $r = \overline{1,lN}$. By $\varphi_s(t)$, $s = 1,2,\dots,l$ denote the restriction of initial function $\varphi(t)$ to the sth interval $[-t_{l-(s-1)},-t_{l-s})$. Then family of periodic problems (9) to (11) reduces to an equivalent family of multi-point problems

$$\frac{\partial v_r(t,x)}{\partial t} = A(t,x)v_r(t,x) + A_0(t,x)v_r(t-\tau,x) + g(t,x),\quad (t,x)\in\Omega_r,\quad r=\overline{1,l}, \tag{12}$$

$$\frac{\partial v_r(t,x)}{\partial t} = A(t,x)v_r(t,x) + A_0(t,x)v_{r-l}(t-\tau,x) + g(t,x),\quad (t,x)\in\Omega_r,\quad r=\overline{l+1,lN}, \tag{13}$$

$$v_s(z,x) = diag[v_1(0,x)]\cdot\varphi_s(z),\quad z\in[-t_{l-(s-1)},-t_{l-s}),\quad x\in[0,\omega],\quad s=\overline{1,l}, \tag{14}$$

$$v_1(0,x) = \lim_{t\to T-0} v_{lN}(t,x),\quad x\in[0,\omega], \tag{15}$$

$$\lim_{s\to t_s-0} v_s(t,x) = v_{s+1}(t_s,x),\quad x\in[0,\omega],\ s=\overline{1,lN-1}, \tag{16}$$

where condition (16) is continuity conditions of solution in the internal lines of partitioning the domain Ω.

A solution of problems (12) to (16) is a system functions $v([t],x) = (v_1(t,x), v_2(t,x), \dots, v_{lN}(t,x))' \in C\left(\Omega, t_r, R^{nlN}\right)$, with functions $v_r(t,x)$, $r=\overline{1,l}$, are continuous on $[-t_{l-(r-1)},-t_{l-r})\times[0,\omega]$, satisfy condition (14), and functions $v_r(t,x)$, $r=\overline{1,lN}$, are continuously differentiable on Ω_r, satisfy of system of differential equations with finite delay (12), (13) and conditions (15), (16). Right-hand side derivative of function $v_r(t,x)$ satisfies to differential equations with delay (12), (13) at the lines $t = t_{r-1}$.

By $\lambda_r(x)$, denote the value of $v_r(t,x)$ under $t = t_{r-1}$, $r=\overline{1,lN}$. We replace $v_r(t,x)$ by $\widetilde{v}_r(t,x)+\lambda_r(x)$ in each domain Ω_r.

Then problems (12) to (16) is equivalent to the problem with unknown functions $\lambda_r(x)$:

$$\frac{\partial\widetilde{v}_r(t,x)}{\partial t} = A(t,x)(\widetilde{v}_r(t,x)+\lambda_r(x)) + A_0(t,x)diag[\varphi_r(t-\tau)]\lambda_1(x) + g(t,x), \tag{17}$$

$$(t,x)\in\Omega_r,\quad r=\overline{1,l},$$

$$\frac{\partial\widetilde{v}_r(t,x)}{\partial t} = A(t,x)(\widetilde{v}_r(t,x)+\lambda_r(x)) + A_0(t,x)(\widetilde{v}_{r-l}(t-\tau,x)+\lambda_{r-l}(x)) + g(t,x), \tag{18}$$

$$(t,x)\in\Omega_r,\quad r=\overline{l+1,lN}$$

$$\widetilde{v}_r(t_{r-1},x)=0, \quad r=\overline{1,lN}, \quad x\in[0,\omega], \tag{19}$$

$$\lambda_1(x)=\lambda_{lN}(x)+\lim_{t\to T-0}\widetilde{v}_{lN}(t,x), \quad x\in[0,\omega], \tag{20}$$

$$\lambda_s(x)+\lim_{t\to t_s-0}\widetilde{v}_s(t,x)=\lambda_{s+1}(x), \quad s=\overline{1,lN-1}, \quad x\in[0,\omega]. \tag{21}$$

The pair $(\lambda^*(x),\tilde{v}^*([t],x))$, where $\lambda^*(x)=(\lambda_1^*(x),\ldots,\lambda_{lN}^*(x))'\in C([0,\omega],R^{nlN})$, and $\tilde{v}^*([t],x)=(\tilde{v}_1^*(t,x),\ldots,\tilde{v}_{lN}^*(t,x))'\in C(\Omega,t_r,R^{nlN})$ is a solution to (17) to (20) if the functions $\tilde{v}_r^*(t,x)$ are continuous on $[-t_{l-(r-1)},-t_{l-r})\times[0,\omega]$, $r=\overline{1,l}$, continuously differentiable on Ω_r, $r=\overline{1,lN}$, satisfies the system of differential equations (17), (18) under $\lambda_r(x)=\lambda_r^*(x)$ for all $(t,x)\in\Omega_r$, initial conditions (19), conditions (20), (21) for all $x\in[0,\omega]$, and $r=\overline{1,lN}$.

Problems (12) to (16) and (17) to (21) are equivalent in the following sense. If the pair $(\lambda^*(x),\tilde{v}^*([t],x))$, where $\lambda^*(x)=(\lambda_1^*(x),\ldots,\lambda_{lN}^*(x))'$, $\tilde{v}^*([t],x)=(\widetilde{v}_1^*(t,x),\ldots,\widetilde{v}_{lN}^*(t,x))'$ is a solution to (17) to (21), then the system functions $v^*([t],x)=(\lambda_1^*(x)+\widetilde{v}_1^*(t,x),\lambda_2^*(x)+\widetilde{v}_2^*(t,x),\ldots,\lambda_{lN}^*(x)+\widetilde{v}_{lN}^*(t,x))'$ is a solution to problems (12) to (16). Conversely, if the system functions $v^{**}([t],x)=(v_1^{**}(t,x),v_2^{**}(t,x),\ldots,v_{lN}^{**}(t,x))'$ is a solution to problems (12) to (16), then the pair $(\lambda^{**}(x),\tilde{v}^{**}([t],x))$, where $\lambda^{**}(x)=(v_1^{**}(t_0,x),v_2^{**}(t_1,x),\ldots,v_{lN}^{**}(t_{lN-1}x))'$,
$\tilde{v}^{**}([t],x)=(v_1^{**}(t,x)-v_1^{**}(t_0,x),v_2^{**}(t,x)-v_2^{**}(t_1,x),\ldots,v_{lN}^{**}(t,x)-v_{lN}^{**}(t_{lN-1},x))'$ is a solution to problems (17) to (21).

In problems (17) to (21), we have the initial conditions $\widetilde{v}_r(t_{r-1},x)=0$. This allows us to determine the unknown functions from the family of system Volterra integral equations second kind:

at fixed $\lambda_r(x)$ the function $\tilde{v}_r(t,x)$, $t\in\Omega_r$, $r=\overline{1,l}$, determine from equation

$$\tilde{v}_r(t,x)=\int_{t_{r-1}}^{t}A(s,x)[\tilde{v}_r(s,x)+\lambda_r(x)]ds+\int_{t_{r-1}}^{t}A_0(s,x)\Phi_r(s-\tau)\lambda_1(x)ds+\int_{t_{r-1}}^{t}g(s,x)ds, \tag{22}$$

where $\Phi_r(t-\tau)=diag[\varphi_r(t-\tau)]$ is diagonal matrix on dimension $(n\times n)$;

at fixed $\lambda_r(x)$, $\lambda_{r-l}(x)$, $\tilde{v}_{r-l}(t-\tau,x)$ the function $\tilde{v}_r(t,x)$, $t\in\Omega_r$, $r=\overline{l+1,lN}$, determine from equation

$$\tilde{v}_r(t,x)=\int_{t_{r-1}}^{t}A(s,x)[\tilde{v}_r(s,x)+\lambda_r(x)]ds+\int_{t_{r-1}}^{t}A_0(s,x)[\tilde{v}_{r-l}(s-\tau,x)+\lambda_{r-l}(x)]ds++\int_{t_{r-1}}^{t}g(s,x)ds, \tag{23}$$

where the pair $(\lambda_r(x),\tilde{v}_r(t,x))$, $r=\overline{1,l}$ satisfies (22), and the pair $(\lambda_{r-l}(x),\tilde{v}_{r-l}(t,x))$, $r=l+1,\ldots,l(N-1)$ satisfies to equation

$$\begin{aligned}\tilde{v}_{r-l}(t,x)=&\int_{t_{r-l-1}}^{t}A(s,x)[\tilde{v}_{r-l}(s,x)+\lambda_{r-l}(x)]ds+\\&+\int_{t_{r-l-1}}^{t}A_0(s,x)[\tilde{v}_{r-2l}(s-\tau,x)+\lambda_{r-2l}(x)]ds+\int_{t_{r-l-1}}^{t}g(s,x)ds,\quad t\in[t_{r-l-1},t_{r-l}),\quad x\in[0,\omega].\end{aligned}$$

Substituting $\tilde{v}_r(\tau,x)$ in the right-hand side (22) and repeating the process ν times, and $\nu\in\mathbb{N}$, we obtain the representation for function $\tilde{v}_r(t,x)$

$$\tilde{v}_r(t,x)=D_{\nu r}(t,t,x)\cdot\lambda_r(x)+E_{\nu r}(t,t,x)\cdot\lambda_1(x)+F_{\nu r}(t,x,g(t,x))+G_{\nu r}(t,x,\tilde{v}_r(t,x)),\quad t\in\Omega_r,\ r=\overline{1,l}. \tag{24}$$

Similarly, applying the right-hand side of (23), and substituting the expressions for the previously found functions $\tilde{v}_{il+j}(t,x)$, $t\in[t_{il+j-1},t_{il+j})$, $x\in[0,\omega]$, $i=0,1,\ldots,N-2$, $j=1,2,\ldots,l$, we obtain the representation for functions $\tilde{v}_{il+j}(t,x)$ of the following form

$$\begin{aligned}\tilde{v}_{il+j}(t,x)=&D_{\nu,il+j}(t,t,x)\cdot\lambda_{il+j}(x)+P^i_{\nu,il+j}\left[t,E_{\nu,il+j}(t,t-i\tau,x)\right]\cdot\lambda_1(x)+\\&+\sum_{k=1}^{i}P^{k-1}_{\nu,il+j}\left[t,H_{\nu,il+j}(t,t-(k-1)\tau,x)+P_{\nu,il+j}[t,D_{\nu,il+j}(t,t-k\tau,x)]\right]\cdot\lambda_{(i-k)l+j}(x)+\\&+\sum_{k=0}^{i}P^{k}_{\nu,il+j}\left[t,F_{\nu,il+j}(t,x,g(t,x))+G_{\nu,il+j}(t,x,\tilde{v}_{(i-k)l+j}(t-k\tau,x)\right],\end{aligned} \tag{25}$$

$t \in [t_{il+j-1}, t_{il+j}), x \in [0, \omega], i = 1, 2, \ldots, N-1, j = 1, 2, \ldots, l$, where

$$D_{\nu,il+j}(t, t-m\tau, x) = \sum_{k=0}^{\nu-1} \int_{t_{il+j-1}}^{t} A(s_1 - m\tau, x) \ldots \int_{t_{il+j-1}}^{s_k} A(s_{k+1} - m\tau, x) ds_{k+1} \ldots ds_1,$$

$$H_{\nu,il+j}(t, t-m\tau, x) = \int_{t_{il+j-1}}^{t} A_0(s_1 - m\tau, x) ds_1 + \sum_{k=1}^{\nu-1} \int_{t_{il+j-1}}^{t} A(s_1 - m\tau, x) \ldots$$

$$\ldots \int_{t_{il+j-1}}^{s_{k-1}} A(s_k - m\tau, x) \int_{t_{il+j-1}}^{s_k} A_0(s_{k+1} - m\tau, x) ds_{k+1} ds_k \ldots ds_1,$$

$$F_{\nu,il+j}(t, x, g(t-m\tau, x)) = \int_{t_{il+j-1}}^{t} g(s_1 - m\tau, x) ds_1 + \sum_{k=1}^{\nu-1} \int_{t_{il+j-1}}^{t} A(s_1 - m\tau, x) \ldots$$

$$\ldots \int_{t_{il+j-1}}^{s_{k-1}} A(s_k - m\tau, x) \int_{t_{il+j-1}}^{s_k} g(s_{k+1} - m\tau, x) ds_{k+1} ds_k \ldots ds_1,$$

$$G_{\nu,il+j}(t, x, \tilde{v}_{il+j}(t-m\tau, x)) = \int_{t_{il+j-1}}^{t} A(s_1 - m\tau, x) \ldots$$

$$\ldots \int_{t_{il+j-1}}^{s_{\nu-2}} A(s_{\nu-1} - m\tau, x) \int_{t_{il+j-1}}^{s_{\nu-1}} A(s_\nu - m\tau, x) \tilde{v}_{il+j}(s_\nu, x) ds_\nu ds_{\nu-1} \ldots ds_1,$$

$$P_{\nu,il+j}(t, x, \tilde{v}_{(i-1)l+j}(t-m\tau, x)) = \int_{t_{il+j-1}}^{t} A_0(s_1 - (m-1)\tau, x) \tilde{v}_{(i-1)l+j}(s_1 - m\tau, x) ds_1 +$$

$$+ \sum_{k=1}^{\nu-1} \int_{t_{il+j-1}}^{t} A(s_1 - (m-1)\tau, x) \ldots \int_{t_{il+j-1}}^{s_{k-1}} A(s_k - (m-1)\tau, x) \times$$

$$\times \int_{t_{il+j-1}}^{s_k} A_0(s_{k+1} - (m-1)\tau, x) \tilde{v}_{(i-1)l+j}(s_{k+1} - m\tau, x) ds_{k+1} ds_k \ldots ds_1,$$

$$E_{\nu,il+j}(t, t-m\tau, x) = \int_{t_{il+j-1}}^{t} A_0(s_1 - m\tau, x) \Phi_j(s_1 - (m+1)\tau) ds_1 +$$

$$+ \sum_{k=1}^{\nu-1} \int_{t_{il+j-1}}^{t} A(s_1 - m\tau, x) \ldots \int_{t_{il+j-1}}^{s_{k-1}} A(s_k - m\tau, x) \times$$

$$\times \int_{t_{il+j-1}}^{s_k} A_0(s_{k+1} - m\tau, x) \Phi_j(s_{k+1} - (m+1)\tau) ds_{k+1} ds_k \ldots ds_1,$$

$m = \overline{0, i}$, $i = \overline{1, N-1}$, $j = \overline{1, l}$, $P^0[t, x, y] = y$, $P^k(t, x, y) = P[t, x, P^{k-1}[t, x, y]]$.

Existence of the limits $\lim_{t \to t_r - 0} D_{\nu r}(t, t, x)$, $\lim_{t \to t_r - 0} F_{\nu r}(t, x, g)$, $\lim_{t \to t_r - 0} E_{\nu r}(t, x)$, $\lim_{t \to t_r - 0} H_{\nu r}(t, x)$, it follows from the continuity of function $\varphi(t)$ on $[-\tau, 0]$ and $A(t, x)$, $A_0(t, x)$, $g(t, x)$ on Ω (thereby also on Ω_r). Since the function $\tilde{v}_r(t, x)$ is continuous on Ω_r and exists $\lim_{t \to t_r - 0} u_r(t, x)$, then after determining $\tilde{v}_r(t, x)$ at $t = t_r$ by its left-hand side limit, we obtain that its is also continuous on $[t_{r-1}, t_r] \times [0, \omega]$. From this and the continuity of the $A(t, x)$, $A_0(t, x)$, and $\varphi(t)$ it follows that the existence of limits $\lim_{t \to t_r - 0} G_{\nu r}(t, x, \tilde{v})$, $\lim_{t \to t_r - 0} P_{\nu r}(t, x, \tilde{v}(t - \tau, x))$.

Passing to the limit in (24), (25) as $t \to t_r - 0$, we find

$$\lim_{t \to t_r - 0} \tilde{v}_r(t, x) = D_{\nu r}(t_r, t, x) \lambda_r(x) + E_{\nu r}(t_r, t, x) \lambda_1(x) + F_{\nu r}(t_r, x, g(t, x)) + G_{\nu r}(t_r, x, \tilde{v}_r(t, x)), \quad r = \overline{1, l},$$
$$\lim_{t \to t_{il+j} - 0} \tilde{v}_{il+j}(t, x) = D_{\nu, il+j}(t_{il+j}, t, x) \lambda_{il+j}(x) + \quad (26)$$

$$+\sum_{k=1}^{i}P^{k-1}_{\nu,il+j}\left[t_{il+j},H_{\nu,il+j}(t,t-(k-1)\tau,x)+P_{\nu,il+j}[t,D_{\nu,il+j}(t,t-k\tau,x)]\right]\lambda_{(i-k)l+j}(x)+$$
$$+P^{i}_{\nu,il+j}\left[t_{il+j},E_{\nu,il+j}(t,t-i\tau,x)\right]\lambda_1(x)+\sum_{k=0}^{i}P^{k}_{\nu,il+j}\leq[t_{il+j},F_{\nu,il+j}(t,x,g(t,x))+$$
$$+G_{\nu,il+j}(t,x,\tilde{v}_{(i-k)l+j}(t-k\tau,x)\Big],\quad i=\overline{1,N-1},\quad j=\overline{1,l}. \tag{27}$$

Then, substituting their into (20) and (21) instead of $\lim_{t\to t_r-0}\tilde{v}_r(t,x)$, $r=\overline{1,lN}$, and $\tilde{v}_{r+1}(t_r,x)$, respectively, we obtain the system of equations with respect to functional parameters $\lambda_1(x),\lambda_2(x),\ \dots,\lambda_{lN}(x)$ the following form

$$\left(I-P^{N-1}_{\nu,lN}[T,E_{\nu,lN}(t,t-(N-1)\tau,x)]\right)\lambda_1(x)-\sum_{k=1}^{N-1}P^{k-1}_{\nu,lN}\left[T,H_{\nu,lN}(t,t-(k-1)\tau,x)\right.$$
$$\left.+P_{\nu,lN}[t,D_{\nu,lN}(t,t-k\tau,x)]\right]\lambda_{(N-1-k)l+l}(x)-\left(I+D_{\nu,lN}(T,t,x)\right)\lambda_{lN}(x)$$
$$=\sum_{k=0}^{N-1}P^{k}_{\nu,lN}\left[T,F_{\nu,lN}(t,x,g(t-k\tau,x))+G_{\nu,lN}(t,x,\tilde{v}_{(i-k)l+l}(t-k\tau,x)\right], \tag{28}$$

$$E_{\nu,j}(t_j,t,x)\lambda_1(x)+\left(I+D_{\nu,j}(t_j,t,x)\right)\lambda_j(x)-\lambda_{j+1}(x)$$
$$=-F_{\nu,j}(t_j,x,g(t,x))-G_{\nu,j}(t_j,t,\tilde{v}_j(t,x),\quad j=\overline{1,l}, \tag{29}$$

$$P^{i}_{\nu,il+j}\left[t_{il+j},E_{\nu,il+j}(t,t-i\tau,x)\right]\lambda_1(x)+\left(I+D_{\nu,il+j}(t_{il+j},t,x)\right)\lambda_{il+j}(x)-\lambda_{il+j+1}(x)$$
$$+\sum_{k=1}^{i}P^{k-1}_{\nu,il+j}\left[t_{il+j},H_{\nu,il+j}(t,t-(k-1)\tau,x)+P_{\nu,il+j}[t,D_{\nu,il+j}(t,t-k\tau,x)]\right]\lambda_{(i-k)l+j}(x)$$
$$=-\sum_{k=0}^{i}P^{k}_{\nu,il+j}\left[t_{il+j},F_{\nu,il+j}(t,x,g(t-k\tau,x))+G_{\nu,il+j}(t,x,\tilde{v}_{(i-k)l+j}(t-k\tau,x))\right], \tag{30}$$

where in (30) the index $j=\overline{1,l}$ for $i=\overline{1,N-2}$, and the index $j=\overline{1,l-1}$ for $i=N-1$.

System of Equations (28) to (30) write in the form

$$Q_\nu(l,x)\lambda(x)=-\tilde{F}_\nu(l,x)-\tilde{G}_\nu(\tilde{v},l,x), \tag{31}$$

where the $(nlN\times nlN)$ matrix $Q_\nu(l,x)$ composed of coefficients of the unknown parameters $\lambda_r(x)$, $r=\overline{1,lN}$ of system (28) to (30), $\lambda(x)=(\lambda_1(x),\lambda_2(x),\ \dots,\lambda_{lN}(x))'\in C([0,\omega],R^{nlN})$,

$$\tilde{F}_\nu(l,x)=\left(-\tilde{F}_{\nu,lN}(T,x),\tilde{F}_{\nu1}(t_1,x),\tilde{F}_{\nu2}(t_2,x),\ \dots,\tilde{F}_{\nu,lN-1}(t_{lN-1},x)\right)'\in C([0,\omega],R^{nlN}),$$
$$\tilde{G}_\nu(\tilde{v},l)=\left(-\tilde{G}_{\nu,lN}(\tilde{v},T,x),\tilde{G}_{\nu1}(\tilde{v},t_1,x),\tilde{G}_{\nu2}(\tilde{v},t_2,x),\ \dots,\tilde{G}_{\nu,lN-1}(\tilde{v},t_{lN-1},x)\right)'\in R^{nlN},$$

and $\tilde{G}_{\nu,il+j}(\tilde{v},t_{il+j},x)=\sum_{k=0}^{i}P^{k}_{\nu,il+j}[t_{il+j},G_{\nu,il+j}(t,x,\tilde{v}_{(i-k)l+j}(t-k\tau,x))]$,

$$\tilde{F}_{\nu,il+j}(t_{il+j},x)=\sum_{k=0}^{i}P^{k}_{\nu,il+j}[t_{il+j},F_{\nu,il+j}(t,x,g(t-k\tau,x))],\quad i=\overline{0,N-1},\ j=\overline{1,l}.$$

If we know $\tilde{v}([t],x)\in C(\Omega,t_r,R^{nlN})$ with components $\tilde{v}_r(t,x)$, then from (31) we find $\lambda(x)\in C([0,\omega],R^{nlN})$ with components $\lambda_r(x)$, and $r=\overline{1,lN}$. Conversely, if we know $\lambda(x)\in C([0,\omega],R^{nlN})$, then from (22), (23) we can find $\tilde{v}([t],x)\in C(\Omega,t_r,R^{nlN})$. Since the $\tilde{v}([t],x)$ and $\lambda(x)$ are unknown to find a solution to problems (17) to (21), we use the iterative method. A pair $(\lambda^*(x),\widetilde{v}^*([t],x))\in C([0,\omega],R^{nlN})\times C(\Omega,t_r,R^{nlN})$, with components $(\lambda^*_r(x),\widetilde{v}^*_r(t,x))$, and $r=\overline{1,lN}$ we determine as limit sequence $(\lambda^{(k)}(x),\widetilde{v}^{(k)}([t],x))\in C([0,\omega],R^{nl})\times C(\Omega,t_r,R^{nlN})$ with components $(\lambda^{(k)}_r(x),\widetilde{v}^{(k)}_r(t,x))$, and $r=\overline{1,lN}$, and $k=0,1,2,\ldots$ by the following algorithm:

Step 0. (a) We assume that for some $\nu, l \in \mathbb{N}$ the matrix $Q_\nu(l,x) : R^{nlN} \to R^{nlN}$ invertible for all $x \in [0,\omega]$. The zero approximation with respect to functional parameter $\lambda^{(0)}(x) = (\lambda_1^{(0)}(x), \ldots, \lambda_{lN}^{(0)}(x))' \in C([0,\omega], R^{nlN})$ we define from the system of linear equations

$Q_\nu(l,x)\lambda(x) = -\tilde{F}_\nu(l,x)$, i.e. $\lambda^{(0)}(x) = -[Q_\nu(l,x)]^{-1}\tilde{F}_\nu(l,x)$. b) Solving the family of Cauchy problems (17), (19) for $\lambda_r(x) = \lambda_r^{(0)}(x)$ on Ω_r, we find $\widetilde{v}_r^{(0)}(t,x)$, $r = 1, 2, \ldots, l$. Further, solving the family of Cauchy problems (18), (19) for $\lambda_r(x) = \lambda_r^{(0)}(x)$, $\lambda_{r-l}(x) = \lambda_{r-l}^{(0)}(x)$, and $\widetilde{v}_{r-l}(t-\tau,x) = \widetilde{v}_{r-l}^{(0)}(t-\tau,x)$ on Ω_r, we find $\widetilde{v}_r^{(0)}(t,x)$, $r = \overline{l+1, lN}$.

Step 1. (a) Replacing function $\widetilde{v}([t],x)$ by $\widetilde{v}^{(0)}([t],x)$ in system (31), we determine $\lambda^{(1)}(x) = (\lambda_1^{(1)}(x), \ldots, \lambda_{lN}^{(1)}(x))' \in C([0,\omega], R^{nlN})$. (b) From the family of Cauchy problems (17), (19) for $\lambda_r(x) = \lambda_r^{(1)}(x)$ on Ω_r we find $\widetilde{v}_r^{(1)}(t,x)$, $r = 1, 2, \ldots, l$. Further, solving the family of Cauchy problems (18), (19) for $\lambda_r(x) = \lambda_r^{(1)}(x)$, $\lambda_{r-l}(x) = \lambda_{r-l}^{(1)}(x)$, and $\widetilde{v}_{r-l}(t-\tau,x) = \widetilde{v}_{r-l}^{(1)}(t-\tau,x)$ on Ω_r, we find $\widetilde{v}_r^{(1)}(t,x)$, $r = \overline{l+1, lN}$.

And so on.

Step k. (a) Replacing function $\widetilde{v}([t],x)$ by $\widetilde{v}^{(k-1)}([t],x)$ in system (31), we determine $\lambda^{(k)}(x) = (\lambda_1^{(k)}(x), \ldots, \lambda_{lN}^{(k)}(x))' \in C([0,\omega], R^{nlN})$. (b) From the family of Cauchy problems (17), (19) for $\lambda_r(x) = \lambda_r^{(k)}(x)$ on Ω_r we find $\widetilde{v}_r^{(k)}(t,x)$, $r = 1, 2, \ldots, l$. Further, solving the family of Cauchy problems (18), (19) for $\lambda_r(x) = \lambda_r^{(k)}(x)$, $\lambda_{r-l}(x) = \lambda_{r-l}^{(k)}(x)$, and $\widetilde{v}_{r-l}(t-\tau,x) = \widetilde{v}_{r-l}^{(k)}(t-\tau,x)$ on Ω_r, we find $\widetilde{v}_r^{(k)}(t,x)$, $r = \overline{l+1, lN}$, $k = 1, 2, \ldots$.

The method of parametrization divides the process of finding unknown functions into two parts: (a) From the system of functional Equation (31), we find the previously introduced parameters $\lambda_r(x)$. (b) From a family of Cauchy problems for ordinary differential equations (17), (18), (19), we find the unknown functions $\widetilde{v}_r(t,x)$.

Now we state the main theorem of the realization and convergence of the proposed algorithm. This assertion also provides sufficient conditions for the unique solvability of problem (9)-(11).

Theorem 1. *Suppose that for some $l \in \mathbb{N}$ and $\nu \in \mathbb{N}$, the matrix $Q_\nu(l,x) : R^{nlN} \to R^{nlN}$ is invertible for all $x \in [0,\omega]$ and the following inequalities hold:*

(a) $\|[Q_\nu(l,x)]^{-1}\| \le \gamma_\nu(l,x)$, and $\gamma_\nu(l,x)$ is a positive continuous function in $x \in [0,\omega]$

(b) $$q_\nu(l,x) = \gamma_\nu(l,x)\frac{1}{\nu!}\left[\frac{\alpha(x)\tau}{l}\right]^\nu \max_{i=\overline{0,N-1}} \sum_{\rho=0}^{i} \frac{1}{\rho!} \cdot \left[\frac{\alpha_0(x)\tau}{l} \sum_{k_1=0}^{\nu-1} \frac{1}{k_1!}\left[\frac{\alpha(x)\tau}{l}\right]^{k_1}\right]^\rho P(l,x) \le \chi < 1,$$

where

$$P(l,x) = \max\left[\max_{1\le j\le l} \sup_{t\in[t_{j-1},t_j)} \left\{ e^{\frac{\alpha(x)\tau}{l}} - 1 + \frac{\alpha_0(x)\tau}{l} e^{\frac{\alpha(x)\tau}{l}} \|\Phi_j(t-\tau)\| \right\},\right.$$
$$\left.\max_{\substack{1\le i\le N-1\\1\le j\le l}} \sup_{t\in[t_{il+j-1},t_{il+j})} \left\{ e^{\frac{\alpha(x)\tau}{l}} \sum_{k_1=1}^{i} \left[\frac{\alpha_0(x)\tau}{l} e^{\frac{\alpha(x)\tau}{l}}\right]^{k_1} + e^{\frac{\alpha(x)\tau}{l}} - 1 + \left[\frac{\alpha_0(x)\tau}{l} e^{\frac{\alpha(x)\tau}{l}}\right]^{i+1} \|\Phi_j(t-(i+1)\tau)\| \right\}\right],$$

and χ – const.

Then the sequence of pairs $(\lambda^{(k)}(x), u^{(k)}([t],x))$ converges to $(\lambda^*(x), u^*([t],x))$ as $k \to \infty$, and the pair $(\lambda^*(x), u^*([t],x))$ is a unique solution to problem (17)-(21), and the estimates hold:

$$\|\lambda^*(x) - \lambda^{(k)}(x)\| \le \frac{[q_\nu(l,x)]^k}{1-q_\nu(l,x)}\|\lambda^{(1)}(x) - \lambda^{(0)}(x)\| \le$$

$$\le \frac{[q_\nu(l,x)]^k}{1-q_\nu(l,x)}\gamma_\nu(l,x)\frac{1}{\nu!}\left[\frac{\alpha(x)\tau}{l}\right]^\nu \max_{i=\overline{0,N-1}} \sum_{\rho=0}^{i}\frac{1}{\rho!}\left[\frac{\alpha_0(x)\tau}{l}\sum_{k=0}^{\nu-1}\frac{1}{k!}\left[\frac{\alpha(x)\tau}{l}\right]^k\right]^\rho \Psi(l,x),$$

$$\|u^*([\cdot],x) - u^{(k)}([\cdot],x)\|_2 \le$$

$$\le P(l,x)\frac{[q_\nu(l,x)]^k}{1-q_\nu(l,x)}\gamma_\nu(l,x)\frac{1}{\nu!}\left[\frac{\alpha(x)\tau}{l}\right]^\nu \max_{i=\overline{0,N-1}} \sum_{\rho=0}^{i}\frac{1}{\rho!}\left[\frac{\alpha_0(x)\tau}{l}\sum_{k=0}^{\nu-1}\frac{1}{k!}\left[\frac{\alpha(x)\tau}{l}\right]^k\right]^\rho \Psi(l,x),$$

where

$$\Psi(l,x) = \max_{i=\overline{0,N-1}}\max_{j=\overline{1,l}} \left\{ \left[\left[\frac{\alpha_0(x)\tau}{l} e^{\frac{\alpha(x)\tau}{l}} \right]^{i+1} \sup_{t\in[t_{il+j-1},t_{il+j})} \|\Phi_j(t-(i+1)\tau)\| - 1 \right.\right.$$

$$+ e^{\frac{\alpha(x)\tau}{l}} \sum_{k_1=0}^{i} \left[\frac{\alpha_0(x)\tau}{l} e^{\frac{\alpha(x)\tau}{l}} \right]^{k_1} \Bigg] \gamma_\nu(l,x) \frac{\tau}{l} \sum_{k_1=0}^{\nu-1} \frac{1}{k_1!} \left[\frac{\alpha(x)\tau}{l} \right]^{k_1} \sup_{t\in[t_{il+j-1},t_{il+j})} \|g(t-i\tau,x)\|$$

$$\left. \times \sum_{\rho=0}^{i} \frac{1}{\rho!} \left\{ \frac{\alpha_0(x)\tau}{l} \sum_{k_1=0}^{\nu-1} \frac{1}{k_1!} \left[\frac{\alpha(x)\tau}{l} \right]^{k_1} \right\}^{\rho} + \frac{\tau}{l} e^{\frac{\alpha(x)\tau}{l}} \sum_{k_1=0}^{i} \left[\frac{\alpha_0(x)\tau}{l} e^{\frac{\alpha(x)\tau}{l}} \right]^{k_1} \sup_{t\in[t_{il+j-1},t_{il+j})} \|g(t-i\tau,x)\| \right\}.$$

Proof. Suppose that conditions (a) and (b) of theorem are valid. Then, we have the following estimates

$$\|F_{\nu,il+j}(t_{il+j},x,g(t-i\tau,x))\| \le \frac{\tau}{l} \sum_{k=0}^{\nu-1} \frac{1}{k!} \left[\frac{\alpha(x)\tau}{l} \right]^k \sup_{t\in[t_{il+j-1},t_{il+j})} \|g(t-i\tau,x)\|,$$

$$\|G_{\nu,il+j}(t_{il+j},x,\tilde{v}_{il+j}(t-i\tau,x))\| \le \frac{1}{\nu!} \left[\frac{\alpha(x)\tau}{l} \right]^\nu \sup_{t\in[t_{il+j-1},t_{il+j})} \|\tilde{v}_{il+j}(t-i\tau,x)\|,$$

$$\|P_{\nu,il+j}(t_{il+j},x,\tilde{v}_{(i-1)l+j}(t-i\tau,x))\| \le \frac{\alpha_0(x)\tau}{l} \sum_{k=0}^{\nu-1} \frac{1}{k!} \left[\frac{\alpha(x)\tau}{l} \right]^k \sup_{t\in[t_{il+j-1},t_{il+j})} \|\tilde{v}_{(i-1)l+j}(t-i\tau,x)\|,$$

$$\|E_{\nu,il+j}(t_{il+j},t-i\tau,x)\| \le \frac{\alpha_0(x)\tau}{l} \sum_{k=0}^{\nu-1} \frac{1}{k!} \left[\frac{\alpha(x)\tau}{l} \right]^k \max_{j=\overline{1,l}} \sup_{t\in[t_{il+j-1},t_{il+j})} \|\Phi_j(t-(i+1)\tau)\|,$$

$$\|D_{\nu,il+j}(t_{il+j},t-i\tau,x)\| \le \frac{1}{\nu!} \left[\frac{\alpha(x)\tau}{l} \right]^\nu,$$

$$\|H_{\nu,il+j}(t_{il+j},t-i\tau,x)\| \le \frac{\alpha_0(x)\tau}{l} \sum_{k=0}^{\nu-1} \frac{1}{k!} \left[\frac{\alpha(x)\tau}{l} \right]^k.$$

The continuity of matrices $A(t,x)$ and $A_0(t,x)$ implies the continuity of matrix $Q_\nu(l,x)$ in $x \in [0,\omega]$. From our conditions of theorem and the following inequality

$$||[Q_\nu(l,x)]^{-1} - [Q_\nu(l,\bar{x})]^{-1}|| \le ||[Q_\nu(l,x)]^{-1}|| \cdot ||Q_\nu(l,x) - Q_\nu(l,\bar{x})|| \cdot ||[Q_\nu(l,\bar{x})]^{-1}||,$$

where $x,\bar{x} \in [0,\omega]$, we get that the matrix $[Q_\nu(l,x)]^{-1}$ is continuous for all $x \in [0,\omega]$. Then there exists a unique $\lambda^{(0)}(x)$ with components $\lambda_r^{(0)}(x) \in C([0,\omega],R^n)$, and $r = \overline{1,lN}$, and we have $\|\lambda^{(0)}(x)\| = \max_{r=\overline{1,lN}} \|\lambda_r^{(0)}(x)\| \le$

$$\le \gamma_\nu(l,x)\|\tilde{F}_\nu(l,x)\| \le \gamma_\nu(l,x) \max_{\substack{i=\overline{0,N-1}\\ j=\overline{1,l}}} \|\tilde{F}_{\nu,il+j}(t_{il+j},x)\| \le \gamma_\nu(l,x) \frac{\tau}{l} \sum_{k=0}^{\nu-1} \frac{1}{k!} \left[\frac{\alpha(x)\tau}{l} \right]^k$$

$$\times \max_{\substack{i=\overline{0,N-1}\\ j=\overline{1,l}}} \left\{ \sup_{t\in[t_{il+j-1},t_{il+j})} \|g(t-i\tau,x)\| \sum_{\rho=0}^{i} \frac{1}{\rho!} \left[\frac{\alpha_0(x)\tau}{l} \sum_{k=0}^{\nu-1} \frac{1}{k!} \left[\frac{\alpha(x)\tau}{l} \right]^k \right]^\rho \right\}. \tag{32}$$

Functions $\tilde{v}_r^{(0)}(t,x)$, $r = \overline{1,l}$ are a solution to Cauchy problems (17), (18) for $\lambda_r(x) = \lambda_r^{(0)}(x)$, and we find their by method of successive approximations:

$$\tilde{v}_r^{(0,0)}(t,x) = 0, \tilde{v}_r^{(0,1)}(t,x) = \int_{t_{r-1}}^{t} A(s,x)ds\lambda_r^{(0)}(x) + \int_{t_{r-1}}^{t} A_0(s,x)\Phi_r(s-\tau)ds\lambda_1^{(0)}(x)$$

$$+\int_{t_{r-1}}^{t} g(s,x)ds, t\in[t_{r-1},t_r),\quad x\in[0,\omega],\quad r=\overline{1,l}.$$

From continuity of the matrices $A(t,x)$, $A_0(t,x)$, the function $g(t,x)$ on Ω, and the function $\varphi(t)$ on $[-\tau,0]$, it follows that the existence of limit

$$\lim_{t\to t_r-0}\tilde{v}_r^{(0,1)}(t,x)=\int_{t_{r-1}}^{t_r}A(s,x)ds\lambda_r^{(0)}(x)+\int_{t_{r-1}}^{t_r}A_0(s,x)\Phi_r(s-\tau)ds\lambda_1^{(0)}(x)+\int_{t_{r-1}}^{t_r}g(s,x)ds.$$

For difference $\tilde{v}_r^{(0,1)}(t,x)-\tilde{v}_r^{(0,0)}(t,x)$ we get the following inequality

$$\|\tilde{v}_r^{(0,1)}(t,x)-\tilde{v}_r^{(0,0)}(t,x)\|$$

$$\le\int_{t_{r-1}}^{t}\|A(s,x)\|ds\|\lambda_r^{(0)}(x)\|+\int_{t_{r-1}}^{t}\|A_0(s,x)\|\|\Phi_r(s-\tau)\|ds\|\lambda_1^{(0)}(x)\|+\int_{t_{r-1}}^{t}\|g(s,x)\|ds.$$

Determining the function $\tilde{v}_r^{(0,1)}(t,x)$ at the line $t=t_r$ by equality $\tilde{v}_r^{(0,1)}(t_r,x)==\lim_{t\to t_r-0}\tilde{v}_r^{(0,1)}(t,x)$, we obtain the function $\tilde{v}_r^{(0,1)}$ is continuous on the rectangle $[t_{r-1},t_r]\times[0,\omega]$. Continuing the iterative process, at the mth step, we obtain

$$\tilde{v}_r^{(0,m+1)}(t,x)=\int_{t_{r-1}}^{t}A(s,x)\tilde{v}_r^{(0,m)}(s,x)ds+\int_{t_{r-1}}^{t}A(s,x)ds\lambda_r^{(0)}(x)$$
$$+\int_{t_{r-1}}^{t}A_0(s,x)\Phi_r(s-\tau)ds\lambda_1^{(0)}(x)+\int_{t_{r-1}}^{t}g(s,x)ds,\quad t\in[t_{r-1},t_r),\ r=\overline{1,l}. \tag{33}$$

Supposing existence of limit $\lim_{t\to t_r-0}\tilde{v}_r^{(0,m)}(t,x)$, based on (33), we establish the existence of limit

$$\lim_{t\to t_r-0}\tilde{v}_r^{(0,m+1)}(t,x)=\int_{t_{r-1}}^{t_r}A(s,x)\tilde{v}_r^{(0,m)}(s,x)ds+\int_{t_{r-1}}^{t_r}A(s,x)ds\lambda_r^{(0)}(x)$$
$$+\int_{t_{r-1}}^{t_r}A_0(s,x)\Phi_r(s-\tau)ds\lambda_1^{(0)}(x)+\int_{t_{r-1}}^{t_r}g(s,x)ds,\quad m=1,2,\ \dots.$$

And the estimates hold

$$\|\tilde{v}_r^{(0,m+1)}(t,x)-\tilde{v}_r^{(0,m)}(t,x)\|\le\int_{t_{r-1}}^{t}\|A(s,x)\|\dots\int_{t_{r-1}}^{s_{m-2}}\|A(s_{m-1},x)\|$$
$$\times\|\tilde{v}_r^{(0,1)}(s_{m-1},x)-\tilde{v}_r^{(0,0)}(s_{m-1},x)\|ds_{m-1}\dots ds,\qquad t\in[t_{r-1},t_r),\quad x\in[0,\omega],\ r=\overline{1,l}.$$

Hence

$$\max_{r=\overline{1,l}}\sup_{t\in[t_{r-1},t_r)}\|\tilde{v}_r^{(0,m+1)}(t,x)-\tilde{v}_r^{(0,m)}(t,x)\|\le\frac{1}{(m+1)!}\left[\frac{\alpha(x)\tau}{l}\right]^{m+1}\|\lambda^{(0)}(x)\|$$

$$+\frac{1}{m!}\left[\frac{\alpha(x)\tau}{l}\right]^{m}\left[\frac{\alpha_0(x)\tau}{l}\max_{r=\overline{1,l}}\sup_{t\in[t_{r-1},t_r)}\|\Phi_r(t-\tau)\|\cdot\|\lambda^{(0)}(x)\|+\frac{\tau}{l}\max_{r=\overline{1,l}}\sup_{t\in[t_{r-1},t_r)}\|g(t,x)\|\right]. \tag{34}$$

From inequality (34), it is clear that the sequence of functions $\tilde{v}_r^{(0,m)}(t,x)$ converges to function $\tilde{v}_r^{(0)}(t,x)$ as $m\to\infty$, and $(t,x)\in\Omega_r)$, $r=\overline{1,l}$. Using the inequality of Gronwall - Bellman, we have the following estimate

$$\|\tilde{v}_r^{(0)}(t,x)\|\le\left[e^{\alpha(x)(t-t_{r-1})}-1\right]\|\lambda_r^{(0)}(x)\|+$$

$$+e^{\alpha(x)(t-t_{r-1})}\cdot \sup_{t\in[t_{r-1},t_r)}\left(\int_{t_{r-1}}^{t}\|A_0(s,x)\|\cdot\|\Phi_r(s-\tau)\|ds\|\lambda_1^{(0)}(x)\|+\int_{t_{r-1}}^{t}\|g(s,x)\|ds\right),\ r=\overline{1,l},$$

whence we get

$$\sup_{t\in[t_{r-1},t_r)}\|\tilde{v}_r^{(0)}(t,x)\|\le\left[e^{\frac{\alpha(x)\tau}{l}}-1\right]\|\lambda_r^{(0)}(x)\|$$

$$+e^{\frac{\alpha(x)\tau}{l}}\left[\frac{\alpha_0(x)\tau}{l}\sup_{t\in[t_{r-1},t_r)}\|\Phi_r(t-\tau)\|\cdot\|\lambda_1^{(0)}(x)\|+\frac{\tau}{l}\sup_{t\in[t_{r-1},t_r)}\|g(t,x))\|\right],\ r=\overline{1,l}. \tag{35}$$

Functions $\tilde{v}_r^{(0)}(t,x)$, $r=\overline{l+1,2l}$ are a solution to Cauchy problems (18), (19) for $\lambda_r(x)=\lambda_r^{(0)}(x)$, $\tilde{v}_{r-l}(t-\tau,x)=\tilde{v}_{r-l}^{(0)}(t-\tau,x)$, and we find their by method of successive approximations: $\tilde{v}_r^{(0,0)}(t,x)=0$,

$$\tilde{v}_r^{(0,m+1)}(t,x)=\int_{t_{r-1}}^{t}A(s,x)\tilde{v}_r^{(0,m)}(s,x)ds+\int_{t_{r-1}}^{t}A(s,x)ds\lambda_r^{(0)}(x)$$

$$+\int_{t_{r-1}}^{t}A_0(s,x)[\tilde{v}_{r-l}^{(0)}(s-\tau,x)+\lambda_{r-l}^{(0)}(x)]ds+\int_{t_{r-1}}^{t}g(s,x)ds,\ t\in[t_{r-1},t_r),\ \ r=\overline{l+1,2l}.$$

From continuity of the matrices $A(t,x)$, $A_0(t,x)$, the functions $g(t,x)$, $\tilde{v}_{r-l}^{(0)}(t-\tau,x)$ on Ω it follows that the existence of limit

$$\lim_{t\to t_r-0}\tilde{v}_r^{(0,1)}(t,x)=\int_{t_{r-1}}^{t_r}A(s,x)ds\lambda_r^{(0)}(x)+\int_{t_{r-1}}^{t_r}A_0(s,x)[\tilde{v}_{r-l}^{(0)}(s-\tau,x)+\lambda_{r-l}^{(0)}(x)]ds+\int_{t_{r-1}}^{t_r}g(s,x)ds.$$

For difference $\tilde{v}_r^{(0,1)}(t,x)-\tilde{v}_r^{(0,0)}(t,x)$ we get the following inequality

$$\|\tilde{v}_r^{(0,1)}(t,x)-\tilde{v}_r^{(0,0)}(t,x)\|$$

$$\le\int_{t_{r-1}}^{t}\|A(s,x)\|ds\|\lambda_r^{(0)}(x)\|+\int_{t_{r-1}}^{t}\|A_0(s,x)\|\|\tilde{v}_{r-l}^{(0)}(s-\tau,x)+\lambda_{r-l}^{(0)}(x)\|ds+\int_{t_{r-1}}^{t}\|g(s,x)\|ds.$$

Determining the function $\tilde{v}_r^{(0,1)}(t,x)$ at the line $t=t_r$ by equality $\tilde{v}_r^{(0,1)}(t_r,x)==\lim_{t\to t_r-0}\tilde{v}_r^{(0,1)}(t,x)$, we obtain the function $\tilde{v}_r^{(0,1)}(t,x)$ is continuous on the rectangle $[t_{r-1},t_r]\times[0,\omega]$.

At the mth step supposing of the existence $\lim_{t\to t_r-0}\tilde{v}_r^{(0,m)}(t,x)$, we establish the existence of limit

$$\lim_{t\to t_r-0}\tilde{v}_r^{(0,m+1)}(t,x)=\int_{t_{r-1}}^{t_r}A(s,x)\tilde{v}_r^{(0,m)}(s,x)ds$$

$$+\int_{t_{r-1}}^{t_r}A(s,x)ds\lambda_r^{(0)}(x)+\int_{t_{r-1}}^{t_r}A_0(s,x)[\tilde{v}_{r-l}^{(0)}(s-\tau,x)+\lambda_{r-l}^{(0)}(x)]ds+\int_{t_{r-1}}^{t_r}f(s,x)ds.$$

And, we get the estimate

$$\|\tilde{v}_r^{(0,m+1)}(t,x)-\tilde{v}_r^{(0,m)}(t,x)\|$$

$$\le\int_{t_{r-1}}^{t}\|A(s,x)\|\dots\int_{t_{r-1}}^{s_{m-2}}\|A(s_{m-1},x)\|\cdot\|\tilde{v}_r^{(0,1)}(s_{m-1},x)-\tilde{v}_r^{(0,0)}(s_{m-1},x)\|ds_{m-1}\dots ds,$$

$t\in[t_{r-1},t_r), x\in[0,\omega]$, $r=\overline{l+1,2l}$. Hence

$$\max_{r=\overline{l+1,2l}}\sup_{t\in[t_{r-1},t_r)}\|\tilde{v}_r^{(0,m+1)}(t,x)-\tilde{v}_r^{(0,m)}(t,x)\|\le\frac{1}{(m+1)!}\left[\frac{\alpha(x)\tau}{l}\right]^{m+1}\|\lambda^{(0)}(x)\|+\frac{1}{m!}\left[\frac{\alpha(x)\tau}{l}\right]^{m}\cdot$$

$$\cdot\left[\frac{\alpha_0(x)\tau}{l}\ \{\max_{r=\overline{l+1,2l}}\sup_{t\in[t_{r-1},t_r)}\|\tilde{v}_{r-l}^{(0)}(t-\tau,x)\|+\|\lambda^{(0)}(x)\|+\frac{\tau}{l}\max_{r=\overline{1,l}}\sup_{t\in[t_{r-1},t_r)}\|f(t,x)\|\right]. \tag{36}$$

From inequality (36), it follows that the sequence of functions $\tilde{v}_r^{(0,m)}(t,x)$, $r=\overline{l+1,2l}$ uniformly converges to function $\tilde{v}_r^{(0)}(t,x)$ as $m\to\infty$, and $t\in[t_{r-1},t_r), x\in[0,\omega]$, $r=\overline{l+1,2l}$. Using the inequality of Gronwall - Bellman, we have the estimate

$$\|\tilde{v}_r^{(0)}(t,x)\|\le\left[e^{\alpha(x)(t-t_{r-1})}-1\right]\|\lambda_r^{(0)}(x)\|$$

$$+e^{\alpha(x)(t-t_{r-1})}\sup_{t\in[t_{r-1},t_r)}\left(\int_{t_{r-1}}^{t}\|A_0(s,x)\|\cdot\|\tilde{v}_{r-l}^{(0)}(s-\tau,x)+\lambda_{r-l}^{(0)}(x)\|ds+\int_{t_{r-1}}^{t}\|g(s,x)\|ds\right),$$

$r=\overline{l+1,2l}$, whence we get

$$\begin{aligned}\sup_{t\in[t_{r-1},t_r)}\|\tilde{v}_r^{(0)}(t,x)\|&\le\left[e^{\frac{\alpha(x)\tau}{l}}-1\right]\|\lambda_r^{(0)}(x)\|\\&+e^{\frac{\alpha(x)\tau}{l}}\left[\frac{\alpha_0(x)\tau}{l}\sup_{t\in[t_{r-1},t_r)}\|\tilde{v}_{r-l}^{(0)}(t-\tau,x)\|+\frac{\alpha_0(x)\tau}{l}\|\lambda_{r-l}^{(0)}(x)\|+\frac{\tau}{l}\sup_{t\in[t_{r-1},t_r)}\|f(t,x)\|\right]\\&\le\left[e^{\frac{\alpha(x)\tau}{l}}-1\right]\|\lambda_r^{(0)}(x)\|+e^{\frac{2\alpha(x)\tau}{l}}\left[\frac{\alpha_0(x)\tau}{l}\right]^2\sup_{t\in[t_{r-1},t_r)}\|\Phi_{r-l}(t-2\tau)\|\cdot\|\lambda_1^{(0)}(x)\|\\&+e^{\frac{2\alpha(x)\tau}{l}}\frac{\alpha_0(x)\tau}{l}\|\lambda_{r-l}^{(0)}(x)\|+e^{\frac{2\alpha(x)\tau}{l}}\left[\frac{\tau}{l}\right]^2\sup_{t\in[t_{r-1},t_r)}\|f(t-\tau,x)\|+e^{\frac{\alpha(x)\tau}{l}}\frac{\tau}{l}\sup_{t\in[t_{r-1},t_r)}\|g(t,x)\|,\end{aligned}$$

$r=\overline{l+1,2l}$. Similarly, for finding of functions $\tilde{v}_{il+j}^{(0)}(t,x)$, $i=\overline{2,N-1}$, $j=\overline{1,l}$, are the solutions to Cauchy problems (18), (19) for $\lambda_{il+j}(x)=\lambda_{il+j}^{(0)}(x)$, $\lambda_{(i-1)l+j}(x)=\lambda_{(i-1)l+j}^{(0)}(x)$, $\tilde{v}_{(i-1)l+j}(t-\tau,x)=\tilde{v}_{(i-1)l+j}^{(0)}(t-\tau,x)$, we construct the iterative process. We establish the existence of finite left-hand side limits

$$\lim_{t\to t_{il+j}-0}\tilde{v}_{il+j}^{(0,m+1)}(t,x)=\int_{t_{il+j-1}}^{t_{il+j}}A(s,x)\tilde{v}_{il+j}^{(0,m)}(s,x)ds+\int_{t_{il+j-1}}^{t_{il+j}}A(s,x)ds\lambda_{il+j}^{(0)}(x)$$

$$+\int_{t_{il+j-1}}^{t_{il+j}}A_0(s,x)[\tilde{v}_{(i-1)l+j}^{(0)}(s-\tau,x)+\lambda_{(i-1)l+j}^{(0)}(x)]ds+\int_{t_{il+j-1}}^{t_{il+j}}g(s,x)ds$$

for all $m=0,1,2,\ \dots$. Convergence of sequence functions $\tilde{v}_{il+j}^{(0,m)}(t,x)$ to the function $\tilde{v}_{il+j}^{(0)}(t,x)$ as $m\to\infty$ follows from the estimate

$$\begin{aligned}\max_{r=\overline{l+1,2l}}\sup_{t\in[t_{il+j-1},t_{il+j})}\|\tilde{v}_{il+j}^{(0,m+1)}(t,x)-\tilde{v}_{il+j}^{(0,m)}(t,x)\|&\le\frac{1}{(m+1)!}\left[\frac{\alpha(x)\tau}{l}\right]^{m+1}\|\lambda^{(0)}(x)\|\\&+\frac{1}{m!}\left[\frac{\alpha(x)\tau}{l}\right]^m\left[\frac{\alpha_0(x)\tau}{l}\max_{\substack{i=\overline{2,N-1}\\ j=\overline{1,l}}}\sup_{t\in[t_{il+j-1},t_{il+j})}\|\tilde{v}_{(i-1)l+j}^{(0)}(t-\tau,x)\|\right.\\&\left.+\frac{\alpha_0(x)\tau}{l}\|\lambda^{(0)}(x)\|+\frac{\tau}{l}\max_{\substack{i=\overline{2,N-1}\\ j=\overline{1,l}}}\sup_{t\in[t_{il+j-1},t_{il+j})}\|g(t,x)\|\right].\end{aligned}$$

From inequalities (34), (36) and hence it follows that the sequence of systems functions $\tilde{v}^{(0,m)}([t],x)$ converges to system of functions $\tilde{v}^{(0)}([t],x)$ by the norm of space $C(\Omega,t_r,R^{nlN})$. Taking into account that the completeness of this space, we obtain $\tilde{v}^{(0)}([t],x)\in C(\Omega,t_r,R^{nlN})$. Using the inequality of Gronwall - Bellman, we have the following estimate

$$\begin{aligned}\max_{\substack{i=\overline{0,N-1}\\ j=\overline{1,l}}}\sup_{t\in[t_{il+j-1},t_{il+j})}\|\tilde{v}_{il+j}^{(0)}(t,x)\|&\le\max_{\substack{i=\overline{0,N-1}\\ j=\overline{1,l}}}\left\{\left[left[\frac{\alpha_0(x)\tau}{l}e^{\frac{\alpha(x)\tau}{l}}\right]^{i+1}\sup_{t\in[t_{il+j-1},t_{il+j})}\|\Phi_j(t-(i+1)\tau)\|\right.\\&\left.-1+e^{\frac{\alpha(x)\tau}{l}}\sum_{k=0}^{i}\left[\frac{\alpha_0(x)\tau}{l}e^{\frac{\alpha(x)\tau}{l}}\right]^k\right]\|\lambda^{(0)}(x)\|++\frac{\tau}{l}e^{\frac{\alpha(x)\tau}{l}}\sum_{k=0}^{i}\left[\frac{\alpha_0(x)\tau}{l}e^{\frac{\alpha(x)\tau}{l}}\right]^k\sup_{t\in[t_{il+j-1},t_{il+j})}\|g(t-i\tau,x)\|\right\},\end{aligned}\tag{37}$$

whence taking into account that (32), we get

$$\|\tilde{v}^{(0)}([\cdot],x)\|_2 \le M(l,x). \tag{38}$$

On the first step of the algorithm, we determine $\lambda^{(1)}(x)$ from equation

$$Q_\nu(l,x)\lambda(x) = -\tilde{F}_\nu(l,x) - \tilde{G}_\nu(\tilde{v}^{(0)},l,x), \quad \lambda \in R^{nlN}, \text{ ie,}$$
$$\lambda^{(1)}(x) = [Q_\nu(l,x)]^{-1}[-\tilde{F}_\nu(l,x) - \tilde{G}_\nu(\tilde{v}^{(0)},l,x)],$$

and calculate the norm of difference $\lambda^{(1)}(x) - \lambda^{(0)}(x)$:

$$\begin{aligned}
&\|\lambda^{(1)}(x) - \lambda^{(0)}(x)\| \le \gamma_\nu(l,x)\|\tilde{G}_\nu(\tilde{v}^{(0)},l,x)\| \le \gamma_\nu(l,x) \max_{i=\overline{0,N-1},j=\overline{1,l}} \|\tilde{G}_{\nu,il+j}(\tilde{v}^{(0)},t_{il+j},x)\| \\
&\le \gamma_\nu(l,x)\frac{1}{\nu!}\left[\frac{\alpha(x)\tau}{l}\right]^\nu \max_{i=\overline{0,N-1}} \sum_{\rho=0}^{i} \frac{1}{\rho!}\left[\frac{\alpha_0(x)\tau}{l}\sum_{k=0}^{\nu-1}\frac{1}{k!}\left[\frac{\alpha(x)\tau}{l}\right]^k\right]^\rho \|\tilde{v}^{(0)}([\cdot],x)\|_2 \\
&\le \gamma_\nu(l,x)\frac{1}{\nu!}\left[\frac{\alpha(x)\tau}{l}\right]^\nu \max_{i=\overline{0,N-1}} \sum_{\rho=0}^{i} \frac{1}{\rho!}\left[\frac{\alpha_0(x)\tau}{l}\sum_{k=0}^{\nu-1}\frac{1}{k!}\left[\frac{\alpha(x)\tau}{l}\right]^k\right]^\rho \Psi(l,x).
\end{aligned} \tag{39}$$

For finding $\tilde{v}_r^{(1)}(t,x)$, $r = \overline{1,l}$, is solution of Cauchy problem (17), (19) at $\lambda_r(x) = \lambda_r^{(1)}(x)$ we use the following iterative process $\tilde{v}_r^{(1,0)}(t,x) = \tilde{v}_r^{(0)}(t,x)$,

$$\begin{aligned}
\tilde{v}_r^{(1,m+1)}(t,x) &= \int_{t_{r-1}}^{t} A(s,x)\tilde{v}_r^{(1,m)}(s,x)ds + \int_{t_{r-1}}^{t} A(s,x)ds\lambda_r^{(1)}(x) \\
&+ \int_{t_{r-1}}^{t} A_0(s,x)\Phi_r(s-\tau)ds\lambda_1^{(1)}(x) + \int_{t_{r-1}}^{t} g(s,x)ds, \quad t \in [t_{r-1},t_r),\ r = \overline{1,l}.
\end{aligned} \tag{40}$$

Then

$$\begin{aligned}
\tilde{v}_r^{(1,1)}(t,x) &= \int_{t_{r-1}}^{t} A(s,x)\tilde{v}_r^{(1,0)}(s,x)ds + \int_{t_{r-1}}^{t} A(s,x)ds\lambda_r^{(1)}(x) \\
&+ \int_{t_{r-1}}^{t} A_0(s,x)\Phi_r(s-\tau)ds\lambda_1^{(1)}(x) + \int_{t_{r-1}}^{t} g(s,x)ds.
\end{aligned}$$

From the existence of left-hand side finite limits $\lim_{t\to t_r-0} \tilde{v}_r^{(0)}(t,x)$ for all $r = \overline{1,l}$, and determining functions $\tilde{v}_r^{(0)}(t,x)$ at the line $t = t_r$ by their left-hand side limit, we obtain the functions $\tilde{v}_r^{(0)}(t,x)$ are continuous on $[t_{r-1},t_r] \times [0,\omega]$. From continuity of $A(t,x)$, $A_0(t,x)$, $g(t,x)$ on $[0,T] \times [0,\omega]$, and $\varphi(t)$ on $[-\tau,0]$, it follows the existence

$$\begin{aligned}
&\lim_{t\to t_r-0} \tilde{v}_r^{(1,1)}(t,x) = \int_{t_{r-1}}^{t_r} A(s,x)\tilde{v}_r^{(0)}(s,x)ds+ \\
&+ \int_{t_{r-1}}^{t_r} A(s,x)ds\lambda_r^{(1)} + \int_{t_{r-1}}^{t_r} A_0(s,x)\Phi_r(s-\tau)ds\lambda_1^{(1)}(x) + \int_{t_{r-1}}^{t_r} g(s,x)ds.
\end{aligned}$$

Determining of function $\tilde{v}_r^{(1,1)}(t,x)$ at the line $t = t_r$ by equality $\tilde{v}_r^{(1,1)}(t_r,x) = \lim_{t\to t_r-0} \tilde{v}_r^{(1,1)}(t,x)$, we get the function $\tilde{v}_r^{(1,1)}(t,x)$ is continuous on $[t_{r-1},t_r] \times [0,\omega]$.

Using iterative process (40), we get the existence of finite left-hand side limits $\lim_{t\to t_r-0} \tilde{v}_r^{(1,m)}(t,x)$, $r = \overline{1,l}$, for all $m = 2,3,\ldots$.

Now, we show the convergence of sequence of functions $\tilde{v}_r^{(1,m)}(t,x)$, $r=\overline{1,l}$, to the function $\tilde{v}_r^{(1)}(t,x)$ as $m\to\infty$. For this, we evaluate of differences $\tilde{v}_r^{(1,m+1)}(t,x)-\tilde{v}_r^{(1,m)}(t,x)$, $\tilde{v}_r^{(1,m+1)}(t,x)-\tilde{v}_r^{(1,0)}(t,x)$, $m=0,1,2,\ldots$:

$$\begin{aligned}
&\|\tilde{v}_r^{(1,1)}(t,x)-\tilde{v}_r^{(1,0)}(t,x)\|\le\int_{t_{r-1}}^{t}\|A(s,x)\|ds\|\lambda_r^{(1)}(x)-\lambda_r^{(0)}(x)\|\\
&+\int_{t_{r-1}}^{t}\|A_0(s,x)\|\cdot\|\Phi_r(s-\tau)\|ds\|\lambda_1^{(1)}(x)-\lambda_1^{(0)}(x)\|,\ t\in[t_{r-1},t_r),\ r=\overline{1,l},\\
&\|\tilde{v}_r^{(1,m+1)}(t,x)-\tilde{v}_r^{(1,m)}(t,x)\|\le\int_{t_{r-1}}^{t}\|A(s,x)\|\cdot\|\tilde{v}_r^{(1,m)}(s,x)-\tilde{v}_r^{(1,m-1)}(s,x)\|ds\\
&\le\int_{t_{r-1}}^{t}\|A(s,x)\ldots\|\int_{t_{r-1}}^{s_{m-2}}\|A(s_{m-1},x)\|\cdot\|\tilde{v}_r^{(1,1)}(s_{m-1},x)-\tilde{v}_r^{(1,0)}(s_{m-1},x)\|ds_{m-1}\ldots ds,\\
&\max_{r=\overline{1,l}}\sup_{t\in[t_{r-1},t_r)}\|\tilde{v}_r^{(1,m+1)}(t,x)-\tilde{v}_r^{(1,m)}(t,x)\|\le\left[\frac{1}{(m+1)!}\left(\frac{\alpha(x)\tau}{l}\right)^{m+1}\right.\\
&\left.+\frac{1}{m!}\left(\frac{\alpha(x)\tau}{l}\right)^{m}\frac{\alpha_0(x)\tau}{l}\max_{r=\overline{1,l}}\sup_{t\in[t_{r-1},t_r)}\|\Phi_r(t-\tau)\|\right]\|\lambda^{(1)}(x)-\lambda^{(0)}(x)\|,
\end{aligned}\tag{41}$$

$$\begin{aligned}
&\|\tilde{v}_r^{(1,2)}(t,x)-\tilde{v}_r^{(1,0)}(t,x)\|\le\int_{t_{r-1}}^{t}\|A(s,x)\|\cdot\|\tilde{v}_r^{(1,1)}(s,x)-\tilde{v}_r^{(1,0)}(s,x)\|ds\\
&+\int_{t_{r-1}}^{t}\|A(s,x)\|ds\|\lambda_r^{(1)}(x)-\lambda_r^{(0)}(x)\|ds+\int_{t_{r-1}}^{t}\|A_0(s,x)\|\cdot\|\Phi_r(s-\tau)\|ds\|\lambda_1^{(1)}(x)-\lambda_1^{(0)}(x)\|\\
&\le\left[\int_{t_{r-1}}^{t}\|A(s,x)\|ds+\int_{t_{r-1}}^{t}\|A(s,x)\|\int_{t_{r-1}}^{s}\|A(s_1,x)\|ds_1ds\right]\|\lambda_r^{(1)}(x)-\lambda_r^{(0)}(x)\|\\
&+\left[\int_{t_{r-1}}^{t}\|A_0(s,x)\|\cdot\|\Phi_r(s-\tau)\|ds+\int_{t_{r-1}}^{t}\|A(s,x)\|\int_{t_{r-1}}^{s}\|A_0(s_1,x)\|\cdot\|\Phi_r(s_1-\tau)\|ds_1ds\right]\\
&\times\|\lambda_1^{(1)}(x)-\lambda_1^{(0)}(x)\|,\\
&\|\tilde{v}_r^{(1,m+1)}(t,x)-\tilde{v}_r^{(1,0)}(t,x)\|\le\sum_{k=1}^{m+1}\frac{1}{k!}\left(\int_{t_{r-1}}^{t}\|A(s,x)\|ds\right)^k\|\lambda_r^{(1)}(x)-\lambda_r^{(0)}(x)\|\\
&+\int_{t_{r-1}}^{t}\|A_0(s,x)\|\cdot\|\Phi_r(s-\tau)\|ds\sum_{k=0}^{m-1}\frac{1}{k!}\left(\int_{t_{r-1}}^{t}\|A(s,x)\|ds\right)^k\|\lambda_1^{(1)}(x)-\lambda_1^{(0)}(x)\|,\\
&\max_{r=\overline{1,l}}\sup_{t\in[t_{r-1},t_r)}\|\tilde{v}_r^{(1,m+1)}(t,x)-\tilde{v}_r^{(1,0)}(t,x)\|\le\left[\sum_{k=1}^{m+1}\frac{1}{k!}\left(\frac{\alpha(x)\tau}{l}\right)^k\right.\\
&\left.+\frac{\alpha_0(x)\tau}{l}\max_{r=\overline{1,l}}\sup_{t\in[t_{r-1},t_r)}\|\Phi_r(t-\tau)\|\sum_{k=0}^{m-1}\frac{1}{k!}\left(\frac{\alpha(x)\tau}{l}\right)^k\right]\|\lambda^{(1)}(x)-\lambda^{(0)}(x)\|.
\end{aligned}\tag{42}$$

From inequality (41), it follows that the convergence of sequence of functions $\tilde{v}_r^{(1,m)}(t,x)$, $r=\overline{1,l}$, to the function $\tilde{v}_r^{(1)}(t,x)$, $r=\overline{1,l}$. Passing to limit in (42) as $m\to\infty$, we obtain the following estimate

$$\max_{r=\overline{1,l}}\sup_{t\in[t_{r-1},t_r)}\|\tilde{v}_r^{(1)}(t,x)-\tilde{v}_r^{(0)}(t,x)\|\le P_1(l)\|\lambda^{(1)}(x)-\lambda^{(0)}(x)\|.\tag{43}$$

Functions $\tilde{v}_r^{(1)}(t,x),\ r=\overline{l+1,2l}$ are the solution to Cauchy problems (18), (19) at $\lambda_r(x)=\lambda_r^{(1)}(x)$, $\lambda_{r-l}(x)=\lambda_{r-l}^{(1)}(x)$, $\tilde{v}_{r-l}(t-\tau,x)=\tilde{v}_{r-l}^{(1)}(t-\tau,x)$, we find by the method of successive approximations: $\tilde{v}_r^{(1,0)}(t,x)=\tilde{v}_r^{(0)}(t,x)$,

$$\tilde{v}_r^{(1,m+1)}(t,x)=\int_{t_{r-1}}^{t}A(s,x)\tilde{v}_r^{(1,m)}(s,x)ds+\int_{t_{r-1}}^{t}A(s,x)ds\lambda_r^{(1)}(x)+$$
$$+\int_{t_{r-1}}^{t}A_0(s,x)[\tilde{v}_{r-l}^{(1)}(s-\tau,x)+\lambda_{r-l}^{(1)}(x)]ds+\int_{t_{r-1}}^{t}f(s,x)ds,\quad (t,x)\in\Omega_r,\ r=\overline{l+1,2l}.$$

Since the left-hand side limits $\lim_{t\to t_r-0}\tilde{v}_r^{(0)}(t,x)$ exist for all $r=\overline{l+1,2l}$, then determining the functions $\tilde{v}_r^{(0)}(t,x)$ at the line $t=t_r$ by their left-hand side limit, we obtain continuous on $[t_{r-1},t_r]\times[0,\omega]$ functions $\tilde{v}_r^{(0)}(t,x)$. From continuity $A(t,x),\ A_0(t,x),\ g(t,x),\ \tilde{v}_{r-l}^{(1)}(t-\tau,x)$ on Ω it follows that the existence

$$\lim_{t\to t_r-0}\tilde{v}_r^{(1,1)}(t,x)=\int_{t_{r-1}}^{t_r}A(s,x)\tilde{v}_r^{(0)}(s,x)ds+\int_{t_{r-1}}^{t_r}A(s,x)ds\lambda_r^{(1)}(x)+$$
$$+\int_{t_{r-1}}^{t_r}A_0(s,x)[\tilde{v}_{r-l}^{(1)}(s-\tau,x)+\lambda_{r-l}^{(1)}(x)]ds+\int_{t_{r-1}}^{t_r}g(s,x)ds.$$

Determining function $\tilde{v}_r^{(1,1)}(t,x)$ at the line $t=t_r$ by equality $\tilde{v}_r^{(1,1)}(t_r,x)=\lim_{t\to t_r-0}\tilde{v}_r^{(1,1)}(t,x)$, we get continuous on $[t_{r-1},t_r]\times[0,\omega]$ function. Using the iterative process we get the existence of finite left-hand side limits $\lim_{t\to t_r-0}\tilde{v}_r^{(1,m)}(t,x)$, $r=\overline{l+1,2l}$, for all $m=3,4,\ldots$. Convergence of sequence of functions $\tilde{v}_r^{(1,m)}(t,x)$, $r=\overline{l+1,2l}$, to function $\tilde{v}_r^{(1)}(t,x)$, $r=\overline{l+1,2l}$, it follows that from the estimate

$$\max_{r=\overline{l+1,2l}}\sup_{t\in[t_{r-1},t_r)}\|\tilde{v}_r^{(1,m+1)}(t,x)-\tilde{v}_r^{(1,m)}(t,x)\|\le\left[\frac{1}{(m+1)!}\left(\frac{\alpha(x)\tau}{l}\right)^{m+1}\right.$$
$$\left.+(1+P_1(l,x))\frac{\alpha_0(x)\tau}{l}\frac{1}{m!}\left(\frac{\alpha(x)\tau}{l}\right)^{m}\right]\|\lambda^{(1)}(x)-\lambda^{(0)}(x)\|.\tag{44}$$

In the inequality

$$\max_{r=\overline{1,l}}\sup_{t\in[t_{r-1},t_r)}\|\tilde{v}_r^{(1,m+1)}(t,x)-\tilde{v}_r^{(1,0)}(t,x)\|\le\left[\sum_{k=1}^{m+1}\frac{1}{k!}\left(\frac{\alpha(x)\tau}{l}\right)^{k}+\frac{\alpha_0(x)\tau}{l}\sum_{k=0}^{m-1}\frac{1}{k!}\left(\frac{\alpha(x)\tau}{l}\right)^{k}\right.$$
$$\left.\times\left(1+\max_{r=\overline{l+1,2l}}\sup_{t\in[t_{r-1},t_r)}\|\tilde{v}_{r-l}^{(1)}(t-\tau,x)-\tilde{v}_{r-l}^{(0)}(t-\tau,x)\|\right)\right]\|\lambda^{(1)}(x)-\lambda^{(0)}(x)\|,\tag{45}$$

passing to limit as $m\to\infty$, we obtain the estimate

$$\max_{r=\overline{l+1,2l}}\sup_{t\in[t_{r-1},t_r)}\|\tilde{v}_r^{(1)}(t,x)-\tilde{v}_r^{(0)}(t,x)\|\le P_2(l,x)\|\lambda^{(1)}(x)-\lambda^{(0)}(x)\|.\tag{46}$$

Analogously, for finding the functions $\tilde{v}_{il+j}^{(1)}(t,x),\ i=\overline{2,N-1},\ j=\overline{1,l}$, are solutions of Cauchy problems (18), (19) at $\lambda_{il+j}(x)=\lambda_{il+j}^{(1)}(x)$, $\lambda_{(i-1)l+j}(x)=\lambda_{(i-1)l+j}^{(1)}(x)$, $\tilde{v}_{(i-1)l+j}(t-\tau,x)=\tilde{v}_{(i-1)l+j}^{(1)}(t-\tau,x)$, we construct the iterative process. We establish of the existence of finite left-hand side limits

$$\lim_{t\to t_{il+j}-0}\tilde{v}_{il+j}^{(1,m+1)}(t,x)=\int_{t_{il+j-1}}^{t_{il+j}}A(s,x)\tilde{v}_{il+j}^{(1,m)}(s,x)ds+\int_{t_{il+j-1}}^{t_{il+j}}A(s,x)ds\lambda_{il+j}^{(1)}(x)$$
$$+\int_{t_{il+j-1}}^{t_{il+j}}A_0(s,x)[\tilde{v}_{(i-1)l+j}^{(1)}(s-\tau,x)+\lambda_{(i-1)l+j}^{(1)}(x)]ds+\int_{t_{il+j-1}}^{t_{il+j}}g(s,x)ds,\quad\text{for all}\quad m=0,1,2,\ldots.$$

Convergence of the sequence of functions $\tilde{v}_{il+j}^{(1,m)}(t,x)$ as $m \to \infty$ to the function $\tilde{v}_{il+j}^{(1)}(t,x)$, it follows that from estimate founded as estimates (44) to (46), ie,

$$\max_{\substack{i=\overline{2,N-1}\\ j=\overline{1,l}}} \sup_{t\in[t_{il+j-1},t_{il+j})} \|\tilde{v}_{il+j}^{(1,m+1)}(t,x) - \tilde{v}_{il+j}^{(1,m)}(t,x)\| \leq \left[\frac{1}{(m+1)!}\left(\frac{\alpha(x)\tau}{l}\right)^{m+1} + (1+P_i(l,x))\frac{\alpha_0(x)\tau}{l}\frac{1}{m!}\left(\frac{\alpha(x)\tau}{l}\right)^{m}\right]\|\lambda^{(1)}(x)-\lambda^{(0)}(x)\|, \tag{47}$$

$$\max_{\substack{i=\overline{2,N-1}\\ j=\overline{1,l}}} \sup_{t\in[t_{il+j-1},t_{il+j})} \|\tilde{v}_{il+j}^{(1,m+1)}(t,x) - \tilde{v}_{il+j}^{(1,0)}(t,x)\| \leq \left[\sum_{k=1}^{m+1}\frac{1}{k!}\left(\frac{\alpha(x)\tau}{l}\right)^{k} + \frac{\alpha_0(x)\tau}{l}\sum_{k=0}^{m-1}\frac{1}{k!}\left(\frac{\alpha(x)\tau}{l}\right)^{k}\cdot \left(1+\max_{r=\overline{l+1,2l}}\sup_{t\in[t_{r-1},t_r)}\|\tilde{v}_{(i-1)l+j}^{(1)}(t-\tau,x)-\tilde{v}_{(i-1)l+j}^{(0)}(t-\tau,x)\|\right)\right]\|\lambda^{(1)}(x)-\lambda^{(0)}(x)\|, \tag{48}$$

$$\max_{\substack{i=\overline{2,N-1}\\ j=\overline{1,l}}} \sup_{t\in[t_{il+j-1},t_{il+j})} \|\tilde{v}_{il+j}^{(1)}(t,x) - \tilde{v}_{il+j}^{(0)}(t,x)\| \leq P_{i+1}(l,x)\|\lambda^{(1)}(x)-\lambda^{(0)}(x)\|. \tag{49}$$

From inequalities (41), (44), (47), it follows that the sequence of systems of functions $\tilde{v}^{(1,m)}([t],x)$ converges to the system of functions $\tilde{v}^{(1)}([t],x)$ by norm of space $C(\Omega,t_r,R^{nlN})$. Taking into account that the completeness of this space, we get $\tilde{v}^{(1)}([t],x) \in C(\Omega,t_r,R^{nlN})$.

Continuing the proposed algorithm, we obtain the sequence of pairs $(\lambda^{(k)}(x),\tilde{v}^{(k)}([t],x))$, where $\lambda^{(k)}(x) \in C([0,\omega],R^{nlN})$, $\tilde{v}^{(k)}([t],x)\in C(\Omega,t_r,R^{nlN})$, $k=1,2,\ \dots$, and applying the inequality of Gronwall - Bellman, we evaluate the difference of solutions to Cauchy problems by the difference of parameters

$$\|\tilde{v}_r^{(k)}(t,x)-\tilde{v}_r^{(k-1)}(t,x)\| \leq \left(e^{\alpha(x)(t-t_{r-1})}-1\right)\|\lambda_r^{(k)}(x)-\lambda_r^{(k-1)}(x)\|$$
$$+ e^{\alpha(x)(t-t_{r-1})}\sup_{t\in[t_{r-1},t_r)}\int_{t_{r-1}}^{t}\|A_0(s,x)\|\cdot\|\Phi_r(s-\tau)\|ds\|\lambda_1^{(k)}(x)-\lambda_1^{(k-1)}(x)\|,\ t\in[t_{r-1},t_r),\ r=\overline{1,l},$$
$$\|\tilde{v}_{il+j}^{(k)}(t,x)-\tilde{v}_{il+j}^{(k-1)}(t,x)\| \leq \left(e^{\alpha(x)(t-t_{il+j-1})}-1\right)\|\lambda_{il+j}^{(k)}-\lambda_{il+j}^{(k-1)}\|$$
$$+ e^{\alpha(x)(t-t_{il+j-1})}\cdot\sup_{t\in[t_{il+j-1},t_{il+j})}\int_{t_{r-1}}^{t}\|A_0(s,x)\|\cdot\Big(\|\tilde{v}_{(i-1)l+j}^{(k)}(s-\tau,x)-\tilde{v}_{(i-1)l+j}^{(k-1)}(s-\tau,x)\|$$
$$+\|\lambda_{(i-1)l+j}^{(k)}(x)-\lambda_{(i-1)l+j}^{(k-1)}(x)\|\Big)ds,\ t\in[t_{il+j-1},t_{il+j}),\ i=\overline{1,N-1},\ j=\overline{1,l},\ k=1,2,...$$

Hence

$$\|\tilde{v}^{(k)}([\cdot],x)-\tilde{v}^{(k-1)}([\cdot],x)\| \leq P(l,x)\|\lambda^{(k)}(x)-\lambda^{(k-1)}(x)\|. \tag{50}$$

Since $\lambda^{(k+1)}(x)$, and $\lambda^{(k)}(x)$ are the solutions of equation (31) for right-hand sides $G_\nu(\tilde{v}^{(k)},2h,x)$, and $G_\nu(\tilde{v}^{(k-1)},2h,x)$, respectively, then for norm their difference, we have the following estimate

$$\|\lambda^{(k+1)}(x)-\lambda^{(k)}(x)\| \leq \gamma_\nu(l,x)\cdot\|\tilde{G}_\nu(\tilde{v}^{(k)},l,x)-\tilde{G}_\nu(\tilde{v}^{(k-1)},l,x)\|$$
$$\leq \gamma_\nu(l,x)\frac{1}{\nu!}\left(\frac{\alpha(x)\tau}{l}\right)^{\nu}\max_{i=\overline{0,N-1}}\sum_{\rho=0}^{i}\frac{1}{\rho!}\left(\frac{\alpha_0(x)\tau}{l}\sum_{k=0}^{\nu-1}\frac{1}{k!}\left(\frac{\alpha(x)\tau}{l}\right)^{k}\right)^{\rho}\|\tilde{v}^{(k)}([\cdot],x)-\tilde{v}^{(k-1)}([\cdot],x)\|_2.$$

Substituting of right-hand side of inequality (1.46) instead of $\|\tilde{v}^{(k)}([\cdot],x)-\tilde{v}^{(k-1)}([\cdot],x)\|_2$, we get the inequality

$$\|\lambda^{(k+1)}(x)-\lambda^{(k)}(x)\| \leq q_\nu(l,x)\|\lambda^{(k)}(x)-\lambda^{(k-1)}(x)\|,\quad k=1,2,\ \dots\quad . \tag{51}$$

From condition $q_\nu(l,x)<1$ and inequality (51) we obtain that the convergence of sequence $\lambda^{(k)}(x)$ to $\lambda^*(x)$.

Then, from (50), it follows that the convergence of sequence of functions $\tilde{v}^{(k)}([t],x)$ by the norm of space $C(\Omega, t_r, R^{nN})$ to the function $\tilde{v}^*([t],x)$. Taking into account that the completeness of this space we obtain that the system of functions $\tilde{v}^*([t],x) \in C(\Omega, t_r, R^{nN})$. Now, we show that an uniqueness. Let exists $\left(\tilde{\lambda}(x), \tilde{v}([t],x)\right)$ is another solution to problem (17)-(21). As estimates (50),(51), we establish

$$\|\tilde{v}^*([\cdot],x) - \tilde{v}([\cdot],x)\|_2 \le P(l,x)\|\lambda^*(x) - \tilde{\lambda}(x)\|, \quad \|\lambda^*(x) - \tilde{\lambda}(x)\| \le q_\nu(l,x)\|\lambda^*(x) - \tilde{\lambda}(x)\|.$$

By virtue of condition $q_\nu(l,x) < 1$, we have the equalities $\lambda^*(x) = \tilde{\lambda}(x)$, $\tilde{v}^*([t],x) = \tilde{v}([t],x)$, i.e. $\lambda_r^*(x) = \tilde{\lambda}_r(x)$, $\tilde{v}_r^*(t,x) = \tilde{v}_r(t,x)$ for all $(t,x) \in \Omega_r$, $r = \overline{1, lN}$. Theorem 1 is proved.

We determine the functions $v^{(k)}(t,x)$ by the equalities

$$v^{(k)}(t,x) = \lambda_r^{(k)}(x) + \tilde{v}_r^{(k)}(t,x), \quad t \in \Omega_r, \quad r = \overline{1, lN},$$
$$v^{(k)}(T,x) = \lambda_{lN}^{(k)}(x) + \lim_{t \to T-0} \tilde{v}_{lN}^{(k)}(t,x), \quad x \in [0,\omega], k = 0, 1, 2, \ldots .$$

From the equivalence of problems (9) to (11) and (17) to (23), it follows □

Theorem 2. *Suppose that for some $l \in \mathbb{N}$ and $\nu \in \mathbb{N}$, the matrix $Q_\nu(l,x) : R^{nlN} \to R^{nlN}$ is invertible for all $x \in [0,\omega]$ and the conditions (a) and (b) of Theorem 1 are fulfilled. Then problems (9) to (11) has a unique solution $v^*(t,x)$ and the estimates hold:*

$$\|v^*(t,x) - v^{(k)}(t,x)\| \le \gamma_\nu(l,x)\frac{[q_\nu(l,x)]^{(k)}}{1 - q_\nu(l,x)}\frac{1}{\nu!}\left[\frac{\alpha(x)\tau}{l}\right]^\nu$$
$$\times \max_{i=\overline{0,N-1}} \sum_{\rho=0}^{i} \frac{1}{\rho!}\left[\frac{\alpha_0(x)\tau}{l}\sum_{k=0}^{\nu-1}\frac{1}{k!}\left[\frac{\alpha(x)\tau}{l}\right]^k\right]^\rho (1 + P(l,x))\Psi(l,x), \ (t,x) \in \Omega, \tag{52}$$

where the function $v^{(k)}(t,x)$ is piecewise continuously differentiable on Ω, for its the function $\lambda_r^{(k)}(x) + \tilde{v}_r^{(k)}(t,x)$, $r = \overline{1, lN}, k = 0, 1, 2, \ldots$ is restriction on Ω_r, $r = \overline{1, lN}$.

The main condition for unique solvability of the problem is invertibility the matrix $Q_\nu(l,x)$ for all $x \in [0,\omega]$ under some numbers $l \in \mathbb{N}$ and $\nu \in \mathbb{N}$.

Rewrite the matrix $Q_\nu(l,x)$ in the following form $\begin{pmatrix} Q_{11}(\nu,l,x) & Q_{12}(\nu,l,x) \\ Q_{21}(\nu,l,x) & Q_{22}(\nu,l,x) \end{pmatrix}$, where $Q_{11}(\nu,l,x)$ is matrix on dimension $n \times n$, $Q_{12}(\nu,l,x)$ is matrix on dimension $n \times (lN-1)n$, $Q_{21}(\nu,l,x)$ is matrix on dimension $(lN-1)n \times n$, and $Q_{22}(\nu,l,x)$ is matrix on dimension $(lN-1)n \times (lN-1)n$.

Since $(nlN \times nlN)$- matrix $Q_\nu(l,x)$ for $N \ge 2$ has a special block-band structure, we have

Lemma 1. *The $(nlN \times nlN)$- matrix $Q_\nu(l,x)$ is invertible at $x \in [0,\omega]$ if and only if so is the $(n \times n)$- matrix $M_\nu(l,x) = Q_{11}(\nu,l,x) - Q_{12}(\nu,l,x)[Q_{22}(\nu,l,x)]^{-1}Q_{21}(\nu,l,x)$.*

Lemma 2. *If the matrix $M_\nu(l,x)$ is invertible for all $x \in [0,\omega]$, then*

$$[Q_\nu(l,x)]^{-1} = \begin{pmatrix} \Lambda_{11}(\nu,l,x) & \Lambda_{12}(\nu,l,x) \\ \Lambda_{21}(\nu,l,x) & \Lambda_{22}(\nu,l,x) \end{pmatrix}, \tag{53}$$

where

$$\Lambda_{11}(\nu,l,x) = M_\nu^{-1}(l,x),$$
$$\Lambda_{12}(\nu,l,x) = -M_\nu^{-1}(l,x)Q_{12}(\nu,l,x)[Q_{22}(\nu,l,x)]^{-1},$$
$$\Lambda_{21}(\nu,l,x) = -[Q_{22}(\nu,l,x)]^{-1}Q_{21}(\nu,l,x)M_\nu^{-1}(l,x),$$
$$\Lambda_{22}(\nu,l,x) = [Q_{22}(\nu,l,x)]^{-1}\{I + Q_{21}(\nu,l,x)M_\nu^{-1}(l,x)Q_{12}(\nu,l,x)[Q_{22}(\nu,l,x)]^{-1}\}.$$

3 | THE WELL-POSEDNESS OF PERIODIC PROBLEMS FAMILY (9) TO (11) AND ITS RELATIONSHIP WITH SOLVABILITY OF PROBLEMS (1) TO (4)

In this section, the sufficient and necessary conditions of unique solvability of the periodic problems family for system of ordinary differential equations with finite delay are established in the terms of initial data. The coefficient criteria of unique solvability of the periodic problem for the system of hyperbolic equations with finite time delay (1) to (4) are obtained.

We establish the assertion about equivalence of well-posedness of periodic problem (1) to (4) and well-posedness of the family of periodic problems for the system of ordinary differential equations with finite delay.

$C^1([0,T],R^n)$ be a space of continuously differentiable on $[0,T]$ vector functions $\psi(t)$ with the norm $||\psi||_{1,0} = \max\left(\max_{t\in[0,T]}||\psi(t)||, \max_{t\in[0,T]}||\dot{\psi}(t)||\right)$;

$C^{1,1}(\Omega,R^n)$ be a space of functions $u(t,x)\in C(\Omega,R^n)$ with continuous on Ω partial derivatives $\frac{\partial u(t,x)}{\partial x}, \frac{\partial u(t,x)}{\partial t}, \frac{\partial^2 u(t,x)}{\partial t\partial x}$ with the norm $||u||_{1,1} = \max\left(||u||_0, \left|\left|\frac{\partial u}{\partial x}\right|\right|_0, \left|\left|\frac{\partial u}{\partial t}\right|\right|_0, \left|\left|\frac{\partial^2 u}{\partial t\partial x}\right|\right|_0\right)$.

Lemma 3. *If problems (9) to (11) have a solution for arbitrary $g(t,x)\in C(\Omega,R^n)$, then this solution is unique.*

Proof. We show that the family of homogeneous periodic problems

$$\frac{\partial v}{\partial t} = A(t,x)v + A_0(t,x)v(t-\tau,x), \quad t\in[0,T], \quad x\in[0,\omega], \quad v\in R^n, \tag{54}$$

$$v(z,x) = diag[v(0,x)]\cdot\varphi(z), \quad z\in[-\tau,0], \quad x\in[0,\omega], \tag{55}$$

$$v(0,x) = v(T,x), \quad x\in[0,\omega] \tag{56}$$

has only the zero solution.

Let $\hat{v}(t,x)$ be a solution of problems (54) to (56) and there exists a point $(t_0,x_0)\in\Omega$ such that $||\hat{v}(t_0,x_0)||\neq 0$. Consider the periodic problem for the system of ordinary differential equations with delay

$$\frac{dv}{dt} = A(t,x_0)v + A_0(t,x_0)v(t-\tau,x_0) + \tilde{f}(t), \quad t\in[0,T], \quad v\in R^n, \tag{57}$$

$$v(z,x_0) = diag[v(0,x_0)]\cdot\varphi(z), \quad z\in[-\tau,0], \tag{58}$$

$$v(0,x_0) = v(T,x_0). \tag{59}$$

By assumption, problems (57) to (59) have a solution for arbitrary $\tilde{f}(t)\in C([0,T],R^n)$. Hence, it follows that the homogeneous boundary problem

$$\frac{dv}{dt} = A(t,x_0)v + A_0(t,x_0)v(t-\tau,x_0), \quad t\in[0,T], \quad v\in R^n, \tag{60}$$

$$v(z,x_0) = diag[v(0,x_0)]\cdot\varphi(z), \quad z\in[-\tau,0], \tag{61}$$

$$v(0,x_0) = v(T,x_0) \tag{62}$$

has only the zero solution. This contradicts with our assumption that the function $\hat{v}(t,x_0)$ is a nonzero solution of problems (60) to (62). Therefore, if $\hat{v}(t,x)$ is a solution to the homogeneous periodic problems (54) to (56), then $\hat{v}(t,x) = 0$ for all $(t,x)\in\Omega$. Lemma 3 is proved. □

Definition 1. Problems (9) to (11) are called well-posed if for arbitrary $g(t,x)\in C(\Omega,R^n)$ it has a unique solution $v(t,x)$ and for it the estimate holds

$$\max_{t\in[0,T]}||v(t,x)|| \leq K\max_{t\in[0,T]}||g(t,x)||, \tag{63}$$

where the constant K independent of $g(t,x)$ and $x\in[0,\omega]$.

Lemma 4. *If $v(t,x)$ is a solution to problems (9) to (11), and the estimate holds*

$$||v||_0 \leq K||g||_0, \tag{64}$$

where K is constant independent of the function $F(t,x)$, then for every $x \in [0,\omega]$ the inequality (63) is valid.

Proof. Take an arbitrary $\hat{x} \in [0,\omega]$, and consider the problem

$$\frac{dv}{dt} = A(t,\hat{x})v + A_0(t,\hat{x})v(t-\tau,\hat{x}) + g(t,\hat{x}), \quad t \in [0,T], \tag{65}$$

$$v(z,\hat{x}) = diag[v(0,\hat{x})] \cdot \varphi(z), \quad z \in [-\tau, 0], \tag{66}$$

$$v(0,\hat{x}) = v(T,\hat{x}). \tag{67}$$

By assumption, $\hat{v}(t) = v(t,\hat{x})$ is the solution to problems (65 to (67). Let $\tilde{v}(t,x)$ be a solution to the family of periodic problems

$$\frac{\partial v}{\partial t} = A(t,x)v + A_0(t,x)v(t-\tau,x) + \tilde{g}(t,x), \quad t \in [0,T], \quad x \in [0,\omega],$$

$$v(z,x) = \text{diag}[v(0,x)] \cdot \varphi(z), z \in [-\tau,0], x \in [0,\omega],$$

$$v(0,x) = v(T,x), x \in [0,\omega],$$

where $\tilde{g}(t,x) = g(t,\hat{x})$. From the uniqueness of solution for problems (65) to (67), we get that $\tilde{v}(t,\hat{x}) = v(t,\hat{x})$. Then, using (64), we have

$\max_{t\in[0,T]} ||v(t,\hat{x})|| = \max_{t\in[0,T]} ||\tilde{v}(t,\hat{x})|| \leq K||\tilde{g}||_0 = K \max_{t\in[0,T]} ||g(t,\hat{x})||$. By the arbitrariness of $\hat{x} \in [0,\omega]$, we obtain the statement of lemma. Lemma 4 is proved.

Denote by $\Omega^\eta = [0,T] \times [0,\eta]$ and $||u||_{0,\eta} = \max_{(t,x)\in\Omega^\eta} ||u(t,x)||$. □

Definition 2. Boundary value problems (1) to (4) are called well-posed if for arbitrary $f(t,x) \in C(\Omega, R^n)$ and $\psi(t) \in C^1([-\tau,T], R^n)$ it has a unique classical solution $u(t,x)$ and this solution satisfies the following estimate

$$\max\left(||u||_{0,\eta}, \left|\left|\frac{\partial u}{\partial x}\right|\right|_{0,\eta}, \left|\left|\frac{\partial u}{\partial t}\right|\right|_{0,\eta}\right) \leq \tilde{K} \max\left(||f||_{0,\eta}, ||\psi||_{1,0}\right),$$

where constant $\tilde{K}$ independent of $f(t,x)$ and $\psi(t)$ and $\eta \in [0,\omega]$.

Theorem 3. *The periodic problems (1) to (4) are well-posed if and only if so are problem (9) to (11).*

Proof of Theorem 3 is similar to the proof of theorem 1 in Asanova and Dzhumabaev[31] taking into account the properties of the delayed argument.

From Theorem 3, it follows that the well-posedness of periodic problems (1) to (4) is equivalent to the well-posedness of problems (9) to (11). From Theorems 1 and 2, it follows

Theorem 4. *Suppose that for some $l \in \mathbb{N}$ and $\nu \in \mathbb{N}$ the matrix $Q_\nu(l,x) : R^{nlN} \to R^{nlN}$ is invertible for all $x \in [0,\omega]$ and the conditions (a) and (b) of Theorem 1 are fulfilled. Then problems (9) to (11) have a unique solution $v^*(t,x)$ and the estimates hold:*

$$\|v^*(t,x)\| \leq \Psi_\nu(l,x) \max_{t\in[0,T]} \|g(t,x)\|, \tag{68}$$

where

$$\Psi_\nu(l,x) = \gamma_\nu(l,x) \sum_{k_1=0}^{\nu-1} \frac{1}{k_1!}\left(\frac{\alpha(x)\tau}{l}\right)^{k_1} \sum_{\rho=0}^{N-1} \frac{1}{\rho!}\left(\frac{\alpha_0(x)\tau}{l} \sum_{k_1=0}^{\nu-1} \frac{1}{k_1!}\left(\frac{\alpha(x)\tau}{l}\right)^{k_1}\right)^{\rho} \{1 + P(l,x)$$
$$\times \left(1 + \frac{\gamma_\nu(l,x)}{1-q_\nu(l,x)} \frac{1}{\nu!}\left(\frac{\alpha\tau}{l}\right)^{\nu} \sum_{\rho=0}^{N-1} \frac{1}{\rho!}\left(\frac{\alpha_0(x)\tau}{l} \sum_{k_1=0}^{\nu-1} \frac{1}{k_1!}\left(\frac{\alpha(x)\tau}{l}\right)^{k_1}\right)^{\rho} (1 + P(l,x))\right) \cdot \frac{\tau}{l} + e^{\frac{\alpha(x)\tau}{l}}$$
$$\times \sum_{\rho=0}^{N-1}\left(\frac{\alpha_0(x)\tau}{l} e^{\frac{\alpha(x)\tau}{l}}\right)^{\rho} \left(1 + \frac{\gamma_\nu(l,x)}{1-q_\nu(l,x)} \frac{1}{\nu!}\left(\frac{\alpha(x)\tau}{l}\right)^{\nu} \sum_{\rho=0}^{N-1} \frac{1}{\rho!}\left(\frac{\alpha_0(x)\tau}{l} \sum_{k_1=0}^{\nu-1} \frac{1}{k_1!}\left(\frac{\alpha(x)\tau}{l}\right)^{k_1}\right)^{\rho} (1 + P(l,x))\right) \frac{\tau}{l}.$$

The function $\Psi_\nu(l,x)$ in the inequality (68) is bounded and continuous in x on $[0,\omega]$ for given $l \in \mathbb{N}$ and $\nu \in \mathbb{N}$, and independent of the function $g(t,x)$. Then, from Theorem 4, it follows that well-posedness of problems (9) to (11).

The following statements is proved as theorems 2 to 4 in Solodushkin et al.[24]

Theorem 5. *If problems (9) to (11) are well-posed, then for all l, $l = 1, 2, \dots$, there exists $\nu = \nu(l)$, and the matrix $Q_\nu(l,x) : R^{nlN} \to R^{nlN}$ is invertible for all $x \in [0,\omega]$ and inequalities (a) and (b) in Theorem 1 are valid.*

Theorem 6. *If problems (9) to (11) are well-posed, then for all $\nu \in \mathbb{N}$, there exists $l = l(\nu) > 0$, and the matrix $Q_\nu(l,x) : R^{nlN} \to R^{nlN}$ is invertible for all $x \in [0,\omega]$ and inequalities (a), (b) of Theorem 1 are valid.*

Theorem 7. *Problems (9) to (11) are well-posed if and only if for all $\nu \in \mathbb{N}$ there exists $l_0 = l_0(\nu)$ such that the matrix $Q_\nu(l,x) : R^{nlN} \to R^{nlN}$ is invertible for all $x \in [0,\omega]$, and for every $l \geq l_0(\nu)$ the inverse matrix $[Q_\nu(l,x)]^{-1}$ satisfies the following estimate*

$$||[Q_\nu(l,x)]^{-1}|| \leq \gamma \frac{l}{\tau}, \tag{69}$$

where γ is a constant independent of l.

Moreover, if it is known a constant well-posedness of problems (9) to (11), the number K, then for all $\varepsilon > 0$ there exists $\bar{l}(\varepsilon,\nu)$ and estimate (69) is satisfied with constant $\gamma = (1+\varepsilon)K$ under $l \geq \bar{l}(\varepsilon,\nu)$. Conversely, if the inequality (69) is hold, then $K = \gamma$.

For constructing the algorithms of finding approximate solutions to the periodic problem for system of hyperbolic equations with finite delay 1 to 4 results of paper Assanova et al[34] *are used.*

Thus, the necessary and sufficient conditions of well-posedness of problems (9) to (11) are established.

Therefore, using Theorems 1 to 7, we obtain the following statements.

Theorem 8. *Periodic problems (1) to (4) are well-posed if and only if for all $\nu \in \mathbb{N}$, there exists $l = l(\nu) > 0$, and the matrix $Q_\nu(l,x) : R^{nlN} \to R^{nlN}$ is invertible for all $x \in [0,\omega]$, and the inequalities (a) and (b) of Theorem 1 are valid.*

Theorem 9. *Periodic problems (1) to (4) are well-posed if and only if for all for all $\nu \in \mathbb{N}$, there exists $l = l(\nu) > 0$, and the matrix $Q_\nu(l,x) : R^{nlN} \to R^{nlN}$ is invertible for all $x \in [0,\omega]$, and inequalities (a) and (b) of Theorem 1 are valid.*

Theorem 10. *Periodic problems (1) to (4) are well-posed if and only if for all $\nu \in \mathbb{N}$, there exists $l_0 = l_0(\nu)$ such that the matrix $Q_\nu(l,x) : R^{nlN} \to R^{nlN}$ is invertible for all $x \in [0,\omega]$, and for every $l \geq l_0(\nu)$ the inverse matrix $[Q_\nu(l,x)]^{-1}$ satisfies the estimate (69).*

Example. Consider periodic problem for hyperbolic equation with delay on $[0,1]\times[0,1]$

$$\frac{\partial^2 u(t,x)}{\partial t \partial x} = -\frac{\partial u(t-0.5,x)}{\partial x} + 1, \quad (t,x) \in [0,1]\times[0,1], \tag{70}$$

$$\frac{\partial u(s,x)}{\partial x} = \frac{\partial u(0,x)}{\partial x} \cdot 1, \quad s \in [-0.5,0], \tag{71}$$

$$\frac{\partial u(0,x)}{\partial x} = \frac{\partial u(1,x)}{\partial x}, \tag{72}$$

$$u(t,0) = 0, \qquad t \in [-0.5,1]. \tag{73}$$

Corresponding family of periodic problems has the form

$$\frac{\partial v(t,x)}{\partial t} = -v(t-0.5,x) + 1, \quad t \in [0,1], \quad x \in [0,1], \tag{74}$$

$$v(s,x) = v(0,x)\cdot 1, \quad s \in [-0.5,0], \quad x \in [0,1], \tag{75}$$

$$v(0,x) = v(1,x), \quad x \in [0,1]. \tag{76}$$

Problems (74) to (76) have a unique solution $v(t,x) = 1$.

We use the parametrization method for $N = 2$, $l = 1$. Let $\lambda_1(x) = v_1(0,x), \lambda_2(x) = v_2(0.5,x)$, and we introduce the functions $\tilde{v}_1(t,x) = v_1(t,x) - \lambda_1(x)$, $t \in [0,0.5) \times [0,1]$, $\tilde{v}_2(t,x) = v_2(t,x) - \lambda_2(x)$, $t \in [0.5,1) \times [0,1]$. We get

$$\frac{\partial \tilde{v}_1(t,x)}{\partial t} = -\lambda_1(x) + 1,\ \tilde{v}_1(0,x) = 0, \quad t \in [0,0.5) \times [0,1],$$
$$\frac{\partial \tilde{v}_2(t,x)}{dt} = -\tilde{v}_1(t-0.5,x) - \lambda_1(x) + 1,\ \tilde{v}_2(0.5,x) = 0, \quad t \in [0.5,1) \times [0,1],$$
$$\lambda_1(x) = \lambda_2(x) + \lim_{t\to 1-0} \tilde{v}_2(t,x),$$
$$\lambda_1(x) + \lim_{t\to 0.5-0} \tilde{v}_1(t,x) = \lambda_2(x).$$

Cauchy problems are equivalent to integral equations

$$\tilde{v}_1(t,x) = -\int_0^t \lambda_1(x)ds + \int_0^t ds,\ t \in [0,0.5),$$
$$\tilde{v}_2(t,x) = -\int_{0.5}^t \tilde{v}_1(s-0.5,x)ds - \int_{0.5}^t \lambda_1(x)ds + \int_{0.5}^t ds,\ t \in [0.5,1).$$

For $l = 1,\ \nu = 1$ the (2×2) matrix $Q_1(1,x)$ has the form: $Q_1(1,x) = \begin{pmatrix} 1.375 & -1 \\ 0.5 & -1 \end{pmatrix}$ and is invertible $[Q_1(1,x)]^{-1} = \begin{pmatrix} \frac{8}{7} & -\frac{8}{7} \\ \frac{4}{7} & -\frac{11}{7} \end{pmatrix}$.

We check the conditions (a) and (b) of Theorem 1: $\|[Q_1(1,x)]^{-1}\| \le \frac{16}{7}$, $\quad q_1(1,x) = 0 < 1$.

Therefore, problems (74) to (76) have a unique solution. Using algorithm, we find it. From system equations

$$\begin{cases} 1.375\lambda_1(x) - \lambda_2(x) = 0.375, \\ 0.5\lambda_1(x) - \lambda_2(x) = -0.5, \end{cases}$$

we get $\lambda_1^{(0)}(x) = 1$, $\lambda_2^{(0)}(x) = 1$. Solving Cauchy problems for $\lambda_1(x) = \lambda_1^{(0)}(x) = 1$, $\lambda_2(x) = \lambda_2^{(0)}(x) = 1$, we have $\tilde{v}_1^{(0)}(t,x) = 0$, $\tilde{v}_2^{(0)}(t,x) = 0$.

Then $v(t,x) = 1$. From Theorem 2, it follows that problems (70) to (73) have a unique solution $u(t,x) = x$.

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ORCID

Anar T. Assanova https://orcid.org/0000-0001-8697-8920

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