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Integral transforms and boundary value problems

This article is devoted to determine the solution of the none stationary heat conduction equation for unlimited space and to investigate the two-dimensional Helmholtz equation. The solutions of the considered boundary value problems are obtained with the use of the mixed Fourier transform and of the double Fourier transform. From these line items in work it is illustrated how the integral transforms method can be used to obtain the solution of boundary value problems for partial differential equations of different kinds. In addition, the Green's function is built for the two-dimensional Poisson equation in this article.

Key words: a heat conduction equation, the two-dimensional Helmholtz equation, the two-dimensional Poisson equation, the mixed Fourier transform, the double Fourier transform, a Green's function.

Many boundary value problems in applied mathematics, mathematical physics, and engineering science can be effectively solved by the use of the Fourier transform, the Fourier cosine transform, or the Fourier sine transform. These transforms are very useful for solving partial differential or integral equations for the following reasons. First, these equations are replaced by ordinary differential equations, which enable us to find the solution of the transform function. The solution of the given equation is then obtained in the original variables by inverting the transform solution. Second, the Fourier transform of the elementary source term is used for determination of the fundamental solution that illustrates the basic ideas behind the construction and implementation of Green's functions. Third, the transform solution combined with the convolution theorem provides an elegant representation of the solution for the boundary value and initial value problems [1].

The boundary value problem for $u(x,y,z,t)$ satisfies the following heat conduction equation and boundary conditions

$$u_t = a^2\Delta_3 u, \quad -\infty < x,y < +\infty, \quad 0 < z,t < +\infty; \tag{1}$$

$$u\big|_{z=0} = f(x,y,t), \quad u\big|_{t=0} = g(x,y,z). \tag{2}$$

We use the mixed Fourier transform [2] defined by (3)

$$\tilde{u}(\lambda,\mu,\nu,t) = \frac{1}{\sqrt{2\pi^3}}\int\int_{-\infty}^{+\infty} e^{i[\lambda x+\mu y]}dxdy\int_0^{+\infty} u(x,y,z,t)\sin\nu z dz, \tag{3}$$

to the problem (1), (2) which reduces to

$$\begin{cases} \tilde{u}_t + a^2(\lambda^2+\mu^2+\nu^2)\tilde{u} = \sqrt{\frac{2}{\pi}}a^2\nu F(\lambda,\mu,t); \\ \tilde{u}(\lambda,\mu,\nu,0) = G(\lambda,\mu,\nu). \end{cases} \tag{4}$$

Thus, this transformed problem (4) is solved to obtain

$$\tilde{u}(\lambda,\mu,\nu,t) = G(\lambda,\mu,\nu)e^{-a^2(\lambda^2+\mu^2+\nu^2)t} + \sqrt{\frac{2}{\pi}}a^2\nu\int_0^t F(\lambda,\mu,\tau)e^{-a^2(\lambda^2+\mu^2+\nu^2)(t-\tau)}d\tau = \tilde{u}_1 + \tilde{u}_2. \tag{5}$$

Applying the inverse Fourier transform (6) to (5)

$$u(x,y,z,t)=\frac{1}{\sqrt{2\pi^3}}\int\int_{-\infty}^{+\infty} e^{-i[\lambda x+\mu y]}d\lambda d\mu\int_0^{+\infty}\tilde{u}(\lambda,\mu,\nu,t)\sin\nu z d\nu=u_1+u_2, \qquad (6)$$

we get the solution of this boundary problem.

In order to define such a solution, we first calculate the functions u_1, u_2 given by

$$u_1=\frac{1}{\sqrt{2\pi^3}}\int\int_{-\infty}^{+\infty} e^{-i[\lambda x+\mu y]}d\lambda d\mu\int_0^{+\infty}\tilde{u}_1(\lambda,\mu,\nu,t)\sin\nu z d\nu=$$

$$=\frac{1}{\sqrt{2\pi^3}}\int\int_{-\infty}^{+\infty} e^{-i[\lambda x+\mu y]}d\lambda d\mu\int_0^{+\infty}G(\lambda,\mu,\nu)e^{-a^2(\lambda^2+\mu^2+\nu^2)t}\sin\nu z d\nu=$$

$$=\frac{1}{2\pi^3}\int_{-\infty}^{+\infty}d\xi\int_{-\infty}^{+\infty}e^{-a^2\lambda^2t}e^{-i\lambda(x-\xi)}d\lambda\int_{-\infty}^{+\infty}d\eta\int_{-\infty}^{+\infty}e^{-a^2\mu^2t}\cdot e^{-i\mu(y-\eta)}d\mu\cdot\int_0^{+\infty}g(\xi,\eta,\zeta)d\zeta\cdot\int_0^{+\infty}e^{-a^2\nu^2t}\cdot\sin\nu z\sin\nu\zeta d\nu;$$

$$u_2=\frac{1}{\sqrt{2\pi^3}}\cdot\sqrt{\frac{2}{\pi}}a^2\int_0^t d\tau\int\int_{-\infty}^{+\infty}F(\lambda,\mu,\tau)e^{-a^2(\lambda^2+\mu^2)(t-\tau)}e^{-i(\lambda x+\mu y)}d\lambda d\mu\int_0^{\infty}e^{-a^2\nu^2(t-\tau)}\cdot\nu\sin\nu z dz,$$

where

$$\int_{-\infty}^{+\infty}d\xi\int_{-\infty}^{+\infty}e^{-a^2\lambda^2t}e^{-i\lambda(x-\xi)}d\lambda=\frac{\sqrt{\pi}}{a\sqrt{t}}\int_{-\infty}^{+\infty}e^{-\frac{(x-\xi)^2}{4a^2t}}d\xi;$$

$$\int_{-\infty}^{+\infty}d\eta\int_{-\infty}^{+\infty}e^{-a^2\mu^2t}\cdot e^{-i\mu(y-\eta)}d\mu=\frac{\sqrt{\pi}}{a\sqrt{t}}\cdot\int_{-\infty}^{+\infty}e^{-\frac{(y-\eta)^2}{4a^2t}}d\eta;$$

$$\int_0^{+\infty}g(\xi,\eta,\zeta)d\zeta\cdot\int_0^{+\infty}e^{-a^2\nu^2t}\cdot\sin\nu z\sin\nu\zeta d\nu=\frac{\sqrt{\pi}}{4a\sqrt{t}}\int_0^{+\infty}g(\xi,\eta,\zeta)\left[e^{-\frac{(z-\zeta)^2}{4a^2t}}-e^{-\frac{(z+\zeta)^2}{4a^2t}}\right]d\zeta;$$

$$\int_0^{\infty}e^{-a^2\nu^2(t-\tau)}\cdot\nu\sin\nu z dz=\frac{z\sqrt{\pi}}{4a^3(t-\tau)^{3/2}}e^{-\frac{z^2}{4a^2(t-\tau)}}.$$

Thus, it follows from the inverse Fourier transform that

$$u_1=\frac{1}{(2a\sqrt{\pi t})^3}\int_{-\infty}^{+\infty}d\xi\int_{-\infty}^{+\infty}d\eta\int_0^{+\infty}g(\xi,\eta,\zeta)\cdot\left[e^{-\frac{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}{4a^2t}}-e^{-\frac{(x+\xi)^2+(y+\eta)^2+(z+\zeta)^2}{4a^2t}}\right]d\zeta;$$

$$u_2=\frac{z\sqrt{\pi}a^2}{4a^3\pi^2}\int_0^t\frac{e^{-\frac{z^2}{4a^2(t-\tau)}}d\tau}{(t-\tau)^{3/2}}\cdot\int\int_{-\infty}^{+\infty}F(\lambda,\mu,\tau)e^{-a^2(\lambda^2+\mu^2)(t-\tau)}e^{-i(\lambda x+\mu y)}d\lambda d\mu=$$

$$=\frac{z}{(2a\sqrt{\pi})^3}\int_0^t\frac{d\tau}{(t-\tau)^{5/2}}\int\int_{-\infty}^{+\infty}f(\xi,\eta,\tau)e^{-\frac{(x-\xi)^2+(y-\eta)^2+z^2}{4a^2(t-\tau)}}d\xi d\eta.$$

The inverse Fourier transform gives the solution of this boundary problem

$$u(x,y,z,t)=u_1+u_2==\frac{1}{(2a\sqrt{\pi t})^3}\int_{-\infty}^{+\infty}d\xi\int_{-\infty}^{+\infty}d\eta\int_0^{+\infty}g(\xi,\eta,\zeta)\left[e^{-\frac{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}{4a^2t}}-e^{-\frac{(x+\xi)^2+(y+\eta)^2+(z+\zeta)^2}{4a^2t}}\right]d\zeta+$$

$$+\frac{z}{(2a\sqrt{\pi})^3}\int_0^t\frac{d\tau}{(t-\tau)^{5/2}}\int\int_{-\infty}^{+\infty}f(\xi,\eta,\tau)e^{-\frac{(x-\xi)^2+(y-\eta)^2+z^2}{4a^2(t-\tau)}}d\xi d\eta.$$

Now we find the fundamental solution of the two-dimensional Helmholtz equation

$$-\nabla^2u+a^2u=\delta(x-k)\delta(y-l),\quad -\infty<x,y<\infty. \qquad (7)$$

It is convenient to make the change of variables

$$x-\kappa=x^*,\ y-l=y^*.$$

Consequently, (7) reduces to the form, dropping the asterisks,

$$u_{xx}+u_{yy}-a^2u=-\delta(x)\delta(y). \qquad (8)$$

Application of the double Fourier transform $\tilde{u}(\xi,\eta)=F\{u(x,y)\}$ [3] to (8) gives

$$\tilde{u}(\zeta)=\frac{1}{2\pi(\bar{\zeta}^2+a^2)},$$

where $\bar{\zeta}=(\xi,\eta)$ and $\bar{\zeta}^2=\xi^2+\eta^2.$

The inverse Fourier transform yields the solution

$$u(x,y)=\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\bar{\zeta}^2+a^2\right)^{-1}e^{i\bar{\zeta}\cdot\bar{x}}d\xi d\eta. \tag{9}$$

In terms of polar coordinates $(x,y)=r(\cos\theta,\sin\theta),\ (\xi,\eta)=\rho(\cos\varphi,\sin\varphi),$ the integral solution (9) becomes

$$u(x,y)=\frac{1}{4\pi^2}\int_0^{\infty}\frac{\rho\cdot d\rho}{(\rho^2+a^2)}\int_0^{\infty}e^{ir\rho\cos(\varphi-\theta)}d\varphi,$$

which is, replacing the second integral by $2\pi J_0(r\rho),$

$$u(x,y)=\frac{1}{2\pi}\int_0^{\infty}\frac{\rho\cdot J_0(r\rho)d\rho}{(\rho^2+a^2)}. \tag{10}$$

In terms of the original coordinates, the fundamental solution of (7) is given by

$$u(\bar{r},\bar{k})=\frac{1}{2\pi}\int_0^{\infty}\frac{\rho\cdot J_0\left[\rho\left\{(x-k)^2+(y-l)^2\right\}^{\frac{1}{2}}\right]d\rho}{(\rho^2+a^2)}. \tag{11}$$

Accordingly, the solution of the inhomogeneous equation

$$(\nabla^2-a^2)v=-f(x,y) \tag{12}$$

is

$$v(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}u(\bar{r},\bar{k})f(\bar{k})d\bar{k},$$

where $u(\bar{r},\bar{k})$ is given by (11).

Since the integral solution (10) does not exist for $a=0,$ Green's function for the two-dimensional Poisson equation (12) cannot be derived from (11). Instead, we differentiate (10) with respect to r to obtain

$$\frac{\partial u}{\partial r}=\frac{1}{2\pi}\int_0^{\infty}\frac{\rho^2J_0'(r\rho)d\rho}{(\rho^2+a^2)},$$

which is, for $a=0,$

$$\frac{\partial u}{\partial r}=\frac{1}{2\pi}\int_0^{\infty}J_0'(r\rho)d\rho=-\frac{1}{2\pi\cdot r}.$$

Integrating this result gives

$$u(r,\theta)=-\frac{1}{2\pi}\log r.$$

In terms of the original coordinates, the Green's function becomes

$$u(\bar{r},\bar{k})=-\frac{1}{4\pi}\log\left[(x-k)^2+(y-l)^2\right].$$

This is Green's function for the two-dimensional Poisson equation $\nabla^2v=-f(x,y).$

Thus, the solution of the Poisson equation is.

$$v(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}u(\bar{r},\bar{k})f(\bar{k})d\bar{k}.$$

As shown above the importance of integral transforms is that they provide powerful methods for solving boundary value problems [3].

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Г.А.Есенбаева

Интегралдық түрлендірулер және шеттік есептер

Мақала шектелмеген кеңістіктер үшін жылуөткізгіштік стационар емес теңдеулерді шешуге және Гельмгольцтің екі өлшемді теңдеулерін зерттеуге арналған. Қарастырылып отырған шеттік есептердің шешімдері аралас және екі еселі Фурье түрлендірулерінің көмегімен алынған. Сонымен қатар әр түрлі типтегі дербес туындылар теңдеулері үшін шеттік есептердің шешімдерін алуда интегралдық түрлендірулер әдісін қалай қолдануға болатыны көрсетілген. Пуассон екі өлшемді теңдеулері үшін Грин функциясы құрылған.

Г.А.Есенбаева

Интегральные преобразования и граничные задачи

Статья посвящена определению решения нестационарного уравнения теплопроводности для неограниченного пространства и исследованию двумерного уравнения Гельмгольца. Решения рассматриваемых граничных задач получены с помощью смешанного и двукратного преобразований Фурье. Исходя из этого, в работе проиллюстрировано, как метод интегральных преобразований может быть использован для получения решения граничных задач для уравнений в частных производных различных типов. Также в работе построена функция Грина для двумерного уравнения Пуассона.

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Application of Cantor pairing function in the two simplest tasks

In this paper application of Cantor pairing function is considered in solving the following two problems. 1. In determining the main Pythagorean triples. 2. When calculating the sum of the digits of *n*-digit numbers in the decimal number system. When calculating the sum of digits of all unequivocal to n-digit numbers in the decimal number system. For these two problems program was written in the programming language Borland Delphi 7.

Key words: Diophantine equation, Pythagorean equation, Pythagorean triple, Cantor pairing function, tuple, linear order, Cantor's number.

In this paper considered Diophantine problems (Diophantine equations). Diophantine equations are called algebraic equations or systems of algebraic equations with rational (natural) coefficients, which are found in integer or rational numbers. Such equations have the number of unknowns (variables) which exceeds the number of equations. The theory of Diophantine equations is the most important section of number theory.

It is known that in general the problem of the set of solutions of a system of Diophantine equations algorithmically unsolvable (Yu.B. Matiyasevich, 1970).