

Convergence of Double Fourier Series of Functions from Symmetric Spaces

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Abstract—We establish conditions for the convergence of double Fourier series in the trigonometric system of functions belonging to a symmetric space.

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In the present paper, we prove assertions on the convergence of double Fourier series in the trigonometric system of functions belonging to a symmetric space.

Suppose that $\bar{x} = (x_1, \dots, x_s) \in \mathbb{R}^s$, $I^s = [0, 2\pi]^s$, and $X(\psi)$ is a symmetric function space 2π -periodic in each variable x_j , $j = 1, \dots, s$, with the fundamental function ψ (see [1, p. 123]). The fundamental function $\varphi(t)$ of a symmetric space can be assumed concave, nondecreasing, and continuous on $[0, 1]$, with $\varphi(0) = 0$ (see [1, p. 137]). Such functions are called Φ -functions.

For the function $\psi(t)$, let

$$\alpha_\psi = \lim_{t \rightarrow 0} \frac{\psi(2t)}{\psi(t)}, \quad \beta_\psi = \overline{\lim}_{t \rightarrow 0} \frac{\psi(2t)}{\psi(t)}.$$

It is well known that if $X(\psi)$ is a symmetric space, then $1 \leq \alpha_\psi \leq \beta_\psi \leq 2$.

As examples of symmetric spaces, consider the following:

- $L_q(I^s)$ is a Lebesgue space with norm

$$\|f\|_q = \left(\int_{I^s} |f(\bar{x})|^q d\bar{x} \right)^{1/q}, \quad 1 \leq q < \infty;$$

- $L_{\psi, \infty}(I^s)$ is a Marcinkiewicz space with norm

$$\|f\|_{\psi, \infty} = \sup_{\tau > 0} \frac{\psi(\tau)}{\tau} \int_0^\tau f^*(t) dt,$$

where $f^*(t)$ is a nonincreasing rearrangement of the function $|f(\bar{x})|$, $\bar{x} \in I^s$, and $\psi(t)$ is a Φ -function (see [1, p. 137]).

Let $a_{\bar{n}}(f) = a_{n_1, \dots, n_s}(f)$ be the Fourier coefficients of a function $f \in L_1$ in the multiple system $\{\prod_{j=1}^s e^{in_j x_j}\}$. By $C(\alpha, q, \dots)$ we denote positive constants depending on the parameters indicated above, which are, in general, different in different formulas.

In the proof of the main results, we shall use the following assertions.

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Lemma A ([2, Lemma 3]). *Let the Φ -functions $\varphi(x), \psi(x), x \in [0, 1]$, be given. If $1 < \beta_\psi < \alpha_\varphi < 2$, then, for the function*

$$\theta(x) = \begin{cases} \frac{\varphi(x)}{\psi(x)}, & x \in (0, 1], \\ 0, & x = 0, \end{cases}$$

there exists a Φ -function $\theta_1(x)$ such that $\theta(x) \asymp \theta_1(x), x \in (0, 1]$, and $\alpha_{\theta_1} > 1$.

Lemma B ([2, Lemma 4]). *If $1 < \alpha_\psi \leq \beta_\psi < 2$ for the Φ -functions $\psi(x), x \in [0, 1]$, then, for any $q \in (0, +\infty)$, the following relations hold:*

$$\int_0^x \frac{\psi^q(t)}{t} dt = O(\psi^q(x)), \quad x \rightarrow +0,$$

$$\int_x^1 [t\psi^q(t)]^{-1} dt = O(\psi^{-q}(x)), \quad x \rightarrow +0.$$

Let us present the main results.

Theorem 1. *Suppose that $f \in L_{\psi, \infty}(I^s), \sqrt{2} < \alpha_\psi, \beta_\psi < 2$. Then the following inequality holds:*

$$\psi\left(\frac{1}{|Q|}\right) \left| \sum_{\bar{k} \in Q} a_{\bar{k}}(f) \right| \leq C \cdot \|f\|_{\psi, \infty},$$

where Q is an arbitrary finite subset of \mathbb{Z}^s and $|Q|$ is the number of its elements.

Proof. Suppose that $f \in L_{\psi, \infty}(I^s)$. Set

$$f_1(\bar{x}) = \begin{cases} f(\bar{x}) & \text{if } |f(\bar{x})| \leq f^*(\tau), \quad \tau > 0, \\ 0 & \text{otherwise,} \end{cases} \quad f_0(\bar{x}) = f(\bar{x}) - f_1(\bar{x}), \quad \bar{x} \in I^s.$$

It is well known that (see [3])

$$f_1^*(t) \leq \begin{cases} f^*(\tau), & 0 \leq t < \tau, \\ f^*(t), & \tau \leq t, \end{cases} \quad f_0^*(t) \leq \begin{cases} f^*(t), & 0 \leq t < \tau, \\ 0, & \tau \leq t. \end{cases} \quad (1)$$

Using the property of equimeasurable functions, relations (1), and the definition of the Marcinkiewicz space, we can write

$$\|f_0\|_1 = \|f_0^*\|_1 \leq \frac{\tau}{\psi(\tau)} \cdot \|f\|_{\psi, \infty}. \quad (2)$$

Using the definition of the function f_1 , inequality (1), and Lemmas A and B, and invoking the condition $\sqrt{2} < \alpha_\psi \leq \beta_\psi < 2$, we obtain

$$\|f_1\|_2 = \|f_1^*\|_2 \leq C \cdot \frac{\sqrt{\tau}}{\psi(\tau)} \|f\|_{\psi, \infty}. \quad (3)$$

It follows from inequality (2) that

$$\left| \sum_{\bar{k} \in Q} a_{\bar{k}}(f_0) \right| \leq \sum_{\bar{k} \in Q} \|f_0\|_1 \leq c \cdot |Q| \cdot \frac{\tau}{\psi(\tau)} \|f\|_{\psi, \infty}. \quad (4)$$

Applying Hölder's inequality for the sum, Parseval's inequality, and estimate (3), we find

$$\left| \sum_{\bar{k} \in Q} a_{\bar{k}}(f_1) \right| \leq \sqrt{|Q|} \left(\sum_{\bar{k} \in Q} |a_{\bar{k}}(f_1)|^2 \right)^{1/2} = \sqrt{|Q|} \cdot \|f_1\|_2 \leq C \cdot \sqrt{|Q|} \cdot \frac{\sqrt{\tau}}{\psi(\tau)} \|f\|_{\psi, \infty}. \quad (5)$$

From inequalities (4) and (5), we obtain

$$\left| \sum_{\bar{k} \in Q} a_{\bar{k}}(f) \right| \leq c \frac{\sqrt{|Q|}}{\psi(\tau)} \{ \sqrt{|Q|} \cdot \tau + \sqrt{\tau} \} \cdot \|f\|_{\psi, \infty}.$$

Setting $\tau = |Q|^{-1}$ in this inequality, we obtain the assertion of the theorem. □

Let us now study questions relating to the convergence of the series

$$\sum_{(n_1, n_2) \in \mathbb{N}^2} a_{n_1 n_2} \cdot e^{in_1 x} \cdot e^{in_2 y}, \tag{6}$$

whose coefficients satisfy the conditions

$$a_{m_1 m_2} \geq a_{n_1 n_2} \quad \text{for } m_i \leq n_i, \quad i = 1, 2, \tag{7}$$

$$a_{n_1 n_2} \rightarrow 0 \quad \text{for } n_1 + n_2 \rightarrow +\infty. \tag{8}$$

Questions relating to the convergence of multiple trigonometric series with monotone coefficients were studied by Dyachenko [4] (see also the survey [5]).

The following assertions concerning the Fourier series of functions from symmetric spaces are valid.

Theorem 2. *Suppose that $X(\psi)$ is a symmetric space and $\sqrt{2} < \alpha_\psi, \beta_\psi < 2^{3/4}$. If the Fourier coefficients of a function $f \in X(\psi)$ satisfy condition (7), then its Fourier series is Pringsheim convergent almost everywhere on $(0, 2\pi)^2$.*

Proof. By Lemma 3 from [4], it suffices to prove the convergence of the series

$$\sum_{(k_1, k_2) \in \mathbb{N}^2} \Delta a_{k_1 k_2}(f) \cdot e^{ik_1 x_1} \cdot e^{ik_2 x_2}, \tag{9}$$

where

$$\Delta a_{nm}(f) = a_{nm}(f) - a_{n+1, m}(f) - a_{n, m+1}(f) + a_{n+1, m+1}(f).$$

Using condition (7), Theorem 1, and the inclusion $X(\psi) \subset L_{\psi, \infty}$ (see [1, p. 162]) we obtain

$$a_{n_1 n_2}(f) \leq C \cdot \|f\|_X \cdot [n_1 n_2 \cdot \psi(n_1^{-1} \cdot n_2^{-1})]^{-1}. \tag{10}$$

Since $\beta_\psi < 2^{3/4}$ by the assumption of the theorem, there exists a number p_0 such that the inequality $\beta_\psi < 2^{1/p_0} < 2^{3/4}$ holds. Therefore, by Lemma B, we have

$$\left(\frac{1}{n_1 n_2} \right)^{1/p_0} \cdot \left[\psi \left(\frac{1}{n_1 n_2} \right) \right]^{-1} \leq C \tag{11}$$

for any $(n_1, n_2) \in \mathbb{N}^2$.

Now we choose a number δ so that $2(1 - 1/p_0)(2 - \delta) > 1$. Such a number δ exists, because $4/3 < p_0 < 2$.

Further, using inequalities (10), (11), and Lemma 5 from [4], we can write

$$\sum_{n_1=2}^{\infty} \sum_{n_2=2}^{\infty} |\Delta a_{n_1 n_2}(f)|^2 \ln n_1 \cdot \ln n_2 \leq C \cdot \sum_{n=1}^{\infty} a_{n, n}^{2-\delta}(f) \leq C \cdot \sum_{n=1}^{\infty} \left[\frac{1}{n^{2(1-1/p_0)}} \right]^{2-\delta} < +\infty.$$

Therefore, by the Kaczmarz theorem (see [5, p. 105]), the series (9) is Pringsheim convergent a.e. on $(0, 2\pi)^2$. Therefore, by Lemma 3, from [4] we obtain the assertion of the theorem. □

Remark 1. If $\sqrt{2} < \alpha_\psi, \beta_\psi < 2$, then there exists a series (6) with coefficients satisfying conditions (7), (8) which is the Fourier series of some function $f \in X(\psi)$ divergent in squares on an everywhere dense subset of the square $(0, 2\pi)^2$.

To justify the above assertion, it suffices to note that there exists a number p_0 satisfying the inequality $\sqrt{2} < 2^{1/p_0} < \alpha_\psi < 2$, i.e., $L_{p_0} \subset X(\psi)$, and then use Theorem 7 from [4].

Remark 2. If $2^{3/4} < \alpha_\psi$, $\beta_\psi < 2$, then there exists a series of the form (6) whose coefficients satisfy conditions (7) (8) and

$$\sup_{(n_1, n_2) \in \mathbb{N}^2} \left[n_1 n_2 \psi \left(\frac{1}{n_1 n_2} \right) \right] a_{n_1 n_2} < +\infty,$$

such that each sequence $\{S_{m_k, n_k}(x, y)\}$, where $m_k, n_k \rightarrow +\infty$ as $k \rightarrow +\infty$, is divergent almost everywhere on $(0, 2\pi)^2$.

Explanation. Since $2^{3/4} < \alpha_\psi$, there exists a number p_0 such that $2^{3/4} < 2^{1/p_0} < \alpha_\psi < 2$, i.e., we have $L_{4/3} \subset L_{p_0} \subset X(\psi)$. Now, considering the series constructed in the proof of Theorem 8 from [4] and using the property of its coefficients and Lemma B, we can easily verify that

$$\sup_{(n_1, n_2) \in \mathbb{N}^2} \left[n_1 n_2 \psi \left(\frac{1}{n_1 n_2} \right) \right] a_{n_1 n_2} < C \cdot \sum_{k=4}^{\infty} 2^{-k(2/p_0 - 3/2)} < +\infty.$$

It was proved in [4] that any sequence $\{S_{m_k, n_k}(x, y)\}$ is divergent almost everywhere on $(0, 2\pi)^2$.

Remark 3. In the case $\psi(t) = t^{1/q}$, $1 < q < 2$, Theorem 1 was proved in [6, p. 117] for regular systems.

The conditions for the convergence of double Fourier series of functions from the Orlicz space follow from the results announced in [7].

The condition for the convergence of double trigonometric Fourier series from anisotropic Lebesgue spaces was established in [8].

Theorems 1 and 2 were announced in [9]. Theorem 1 in [9] contains a misprint: instead of $1/|Q|$, one should read $\psi(1/|Q|)$.

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