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Multipliers in weighted Sobolev spaces on the axis

This work establishes necessary and sufficient conditions for the boundedness of one variable differential operator acting from a weighted Sobolev space $W^l_{p,v}$ to a weighted Lebesgue space on the positive real half line. The coefficients of differential operators are often assumed to be pointwise multipliers of function spaces. The author introduces pointwise multipliers in weighted Sobolev spaces; obtains the description of the space of multipliers $M(W_1 \to W_2)$ for a pair of weighted Sobolev spaces (W_1, W_2) with weights of general type.

Keywords: Sobolev space, pointwise multiplier, weighted space, differential operator, admissible function, slow variation condition, Otelbaev function.

The results obtained in this paper can be regarded as a natural extension of certain results (in dimension one) of the monograph "Theory of multipliers in spaces of differentiable functions" by the authors V.G. Maz'ya and T.O. Shaposhnikova [1]. Such a book is currently the only work in which the theory of pointwise multipliers in unweighted spaces of differentiable functions is treated systematically. A part of the chapters of this work are devoted to multipliers in classical Sobolev spaces $W^k_p,\ k \geq 1-$integer, $1 \leq p < \infty$.

For the latest developments of pointwise multipliers we refer to the monographs [1], [2], which are entirely devoted to this topic. Let us point out some specific directions through the works [3–6].

Let X, Y be Banach spaces whose elements are functions $y\colon \Omega \to \mathbb{R}\ (\mathbb{C})$. We say that a function $z\colon \Omega \to \mathbb{R}\ (\mathbb{C})$ such that a multiplication operator

$$Ty = zy,\ y \in X,$$

is bounded from X to Y, is a multiplier for the pair (X, Y). We denote by $M(X \to Y)$ the space of all multipliers for the pair (X, Y). We introduce the norm

$$\|z;\, M(X \to Y)\| = \|T;\, X \to Y\|\,,$$

in $M(X \to Y)$ [1]. Different kinds of problems arise in the theory of multipliers. The first problem is the problem of describing the space $M(X \to Y)$ for the pair (X, Y). Further, there are problems with studying differential operators as operators acting in the space of multipliers such as the problem of norm evaluation.

We denote by $L_{q,\omega}(I),\ I = [0, \infty)$, the weighted Lebesgue space of all measurable functions in I with the norm

$$\|f\|_{q,\omega} = \|f;\, L_{q,\omega}(I)\| = \left(\int_I |f(x)|^q\, \omega(x)\, dx\right)^{\frac{1}{q}} < \infty \quad (1 \leq q < \infty)\,,$$

$L_q(I) = L_{q,\omega}(I), \quad \omega \equiv 1$. Here $\omega(\cdot)$ is a weight in I, i.e., it is an almost everywhere positive locally integrable function.

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Below $A^l(I)$ is a class of all functions y in I having absolutely continuous derivatives up to order $l-1$ in I.

Let $\omega_0,\ \omega_1$ be weighted functions in I. Let $l \geq 1$ be an integer. We denote by $W^l_{q,\omega_0,\omega_1}(I)$ the weighted Sobolev space of all functions $y \in A^l(I)$ equipped with the following weighted norm

$$\left\|y;\, W^l_{q,\omega_0,\omega_1}(I)\right\| = \left\|y^{(l)};\, L_{q,\omega_0}(I)\right\| + \|y;\, L_{q,\omega_1}(I)\|\,.$$

The purpose of this paper is to obtain the description of the space $M(W^l_{p,v}(I) \to W^m_{q,\omega_0,\omega_1}(I))$.

We define as a length function in I any positive and right-continuous function $h(\cdot)$ ($h(\cdot)$ is a l.f.). We denote by $\Delta(x)$ the segment $[x, x+h(x)]$ for the l.f. $h(\cdot)$.

Definition 1. A weighted function v in I is called admissible with respect to the length function $h(\cdot)$, if there exist $0<\delta<1$, $0<\tau\leq 1$, such that the following inequality is true

$$h(x)^{l-\frac{1}{p}} \inf_{\{e\}} \left(\int\limits_{\Delta(x)\setminus e} v(t)\,dt \right)^{\frac{1}{p}} \geq \tau \tag{1}$$

for all $\Delta(x)$, $x \in I$. In (1) the infimum is taken over all measurable subset e of $\Delta(x)$ with Lebesgue measure $|e| \leq \delta\,|\Delta(x)|$. We denote by $\Pi_{l,p}(\delta,\tau)$ the set of admissible weights v with respect to the l.f. $h(\cdot)$.

Let us give some examples.

Example 1. Since

$$h(x)^{l-\frac{1}{p}} \inf_{\{e\}} \left(\int\limits_{\Delta(x)\setminus e} v(t)\,dt \right)^{\frac{1}{p}} \geq (1-\delta)^{\frac{1}{p}} = \tau,$$

the function $v \equiv 1$ is admissible with respect to the l.f. $h(\cdot)=1$.

Definition 2. We say that a function $\omega(\cdot) > 0$ satisfies the slow variation condition with respect to the l.f. $h(\cdot)$, if there exist constants $0<b_1<1<b_2$ such that

$$b_1\omega(x) \leq \omega(t) \leq b_2\omega(x) \quad \text{for all} \quad t \in \Delta(x). \tag{2}$$

Example 2. Let $v(\cdot)>0$ satisfy the slow variation condition (2) with respect to the l.f. $h(x) = v(x)^{-\frac{1}{lp}}$. Then v is admissible with respect to the l.f. $h(x) = v(x)^{-\frac{1}{lp}}$ with $\tau^p = b_1(1-\delta)$. The proof is trivial.

Every power function $v\,(x) = (1+x)^{\mu}$ $(x \geq 0)$, $0<\mu<+\infty$ satisfies the slow variation condition with respect to the l.f. $h\,(x) = (1+x)^{-\frac{\mu}{lp}}$ in I. Indeed,

$$\left(\frac{1+t}{1+x}\right)^{\mu} \leq 2^{\mu} = b_2,\ \left(\frac{1+t}{1+x}\right)^{\mu} \geq 1 > 2^{-\mu} = b_1$$

for all $t \in \Delta(x)$.

Definition 3. We say that a weight v satisfies the condition $A_{(\delta,\beta)}$ $(0<\delta,\beta<1)$ with respect to the length function $h(\cdot)$ in I, if for any interval $\Delta=[a,b] \subset \Delta(x) = [x, x+h(x)]$ $(x \geq 0)$ and any measurable subset e of Δ with the Lebesgue measure $|e| \leq \delta|\Delta|$ the following inequality holds

$$\int\limits_e v(t)\,dt \leq \beta \int\limits_\Delta v(t)\,dt.$$

We denote by $A_{(\delta,\beta)}$ the set of all weights v which satisfy the condition $A_{(\delta,\beta)}$ with respect to the l.f. $h(\cdot)$. For example, if $b_2 b_1^{-1}\delta < 1$ in (2), then $v \in A_{(\delta,\beta)}$ with $\beta = b_2 b_1^{-1}\delta$.

Let v^* be an Otelbaev function [7]. Namely

$$v^*(x) = \sup\left\{ h > 0\colon h^{lp-1}\int\limits_x^{x+h} v(t)\,dt \le 1 \right\}.$$

We first show that $0 < v^*(x) < \infty$ for all $x \ge 0$. To do this, we note that

$$M(x,h;v) \overset{\text{def}}{=} h^{lp-1}\int\limits_x^{x+h} v(t)\,dt \xrightarrow[h\to 0+]{} 0$$

and that $M(x,h;v) \to \infty$ if $h \to \infty$. Hence, there exist $\delta_x > 0$ and $T_x > 0$, such that

$$M(x,h;v) \le 1, \quad \text{if} \quad 0 < h \le \delta_x, M(x,h;v) > 1, \quad \text{if} \quad h \ge T_x.$$

Therefore, we obtain

$$(0,\delta_x) \subset H_{x,v} = \{h > 0\colon M(x,h;v) \le 1\} \subset (0,\, T_x), \delta_x \le \sup H_{x,v} = v^*(x) \le T_x.$$

The function $v^*(\cdot)$ is right-continuous in I. By using absolute continuity property of the integral, we can imply that

$$v^*(x)^{lp-1}\int\limits_x^{x+v^*(x)} v(t)\,dt = 1.$$

Example 3. Any weight $v \in A_{(\delta,\beta)}$ (with respect to the l.f. $h(x) = v^*(x)$) in I is admissible with respect to the l.f. $h(x) = v^*(x)$. Thus, for all $e \subset \Delta^*(x) = [x, x+v^*(x)]$ with the measure $|e| \le \delta|\Delta^*(x)|$, we have

$$v^*(x)^{lp-1}\inf_{\{e\}}\int\limits_{\Delta^*(x)\setminus e} v(t)\,dt = v^*(x)^{lp-1}\inf_{\{e\}}\left(\int\limits_{\Delta^*(x)} v(t)\,dt - \int\limits_e v(t)\,dt\right) \ge$$

$$\ge (1-\beta)v^*(x)^{lp-1}\int\limits_{\Delta^*(x)} v(t)\,dt = 1-\beta = \tau.$$

Let $C^l[a,b]$ $(-\infty < a < b < \infty)$ be a space of all functions y, having continuous derivatives up to order l in $[a,b]$.

Lemma 1. [8] Let v belong to $\Pi_{l,p}(\delta,\tau)$ with respect to the l.f. $h(\cdot)$. Then there exists a constant $C^* = C^*(\delta,\tau) > 1$ such that

$$h(x)^{-lp}\int\limits_x^{x+h(x)} |y|^p\,dt \le C^*\int\limits_x^{x+h(x)}\left(\left|y^{(l)}\right|^p + |y|^p\,v(t)\right)dt \quad (x \ge 0)$$

for all $y \in C^l(\Delta)$, where $\Delta = [x, x+h(x)]$.

Lemma 2. Let $1 \le p,q < \infty$. Let $0 \le j < l$ be integers. Let $v \in \Pi_{l,p}(\delta,\tau)$ with respect to the l.f. $h(\cdot)$. Let $\omega \in L^+_{loc}(I)$, $d\omega(t) = \omega(t)\,dt$. Then

$$\max_{[x,x+h(x)]}|y^{(j)}(t)| \le (c^*+1)A(l,j,p)\times$$

$$\times h(x)^{l-j-1/p}\left(\int_x^{x+h(x)}|y^{(l)}(t)|^p\,dt+\int_x^{x+h(x)}|y(t)|^p v(t)\,dt\right)^{1/p}, \tag{3}$$

$$\left(\int_x^{x+h(x)}|y^{(j)}(t)|^q\,d\omega(t)\right)^{1/q}\le (c^*+1)A(l,j,p,q)h(x)^{l-j-1/p}\times$$

$$\times\left(\int_x^{x+h(x)}\omega(t)\,dt\right)^{1/q}\left(\int_x^{x+h(x)}|y^{(l)}(t)|^p\,dt+\int_x^{x+h(x)}|y(t)|^p v(t)\,dt\right)^{1/p}.$$

Here we consider a differential operator of the form

$$Ly=\sum_{k=0}^{m}\rho_k(x)y^{(k)}\quad (x\ge 0), \tag{4}$$

where $\rho_k(\cdot)\in L_{loc}(I),\ I=[0,\infty),\ m\ge 1$ is an integer. In the sequel, we assume that L is defined on a subspace $D(L)$ of $W^l_{p,v}$. Here we will investigate the boundedness of the operator $L\colon W^l_{p,v}\to L_{q,\omega}$, $l>m\ge 1$.

Theorem 1. Let $l>m\ge 1$ be integers. Let $1<p\le q<\infty$. Let v belong to $\Pi_{l,p}(\delta,\tau)$ with respect to the l.f. $h(\cdot)$. Let $(d\,\omega(t)=\omega(t)\,dt)$

$$R_k=\sup_{x\ge 0}h(x)^{l-k-\frac{1}{p}}\left\{\int_x^{x+h(x)}|\rho_k(t)|^q\,d\omega(t)\right\}^{\frac{1}{q}}<\infty$$

for $k=0,1,\ldots,m$. Then the operator L in (4) is bounded from $W^l_{p,v}(I)$ to $L_{q,\omega}(I)$. Here the norm satisfies the inequality

$$\left\|L;\,W^l_{p,v}(I)\to L_{q,\omega}(I)\right\|\le c\sum_{k=0}^{m}R_k.$$

Proof. Let $y\in D(L)\subset W^l_{p,v}$. For the k-th summand in (4), we have

$$\left\|\rho_k y^{(k)}\right\|^q_{q,\omega}=\int_0^\infty\left|\rho_k y^{(k)}\right|^q\,d\omega(t)=\sum_{j=0}^{\infty}\int_{\Delta_j}\left|\rho_k y^{(k)}\right|^q\,d\omega(t),$$

where the system of segments $\{\Delta_j\}$, $j\ge 0$, is constructed as follows

$$\Delta_{j+1}=[x_j,x_{j+1}]\,,\ x_{j+1}=x_j+h(x_j)\ \ (x_0=0)\,.$$

By virtue of (3), we obtain

$$\int_0^\infty\left|\rho_k(t)y^{(k)}\right|^q\,d\omega(t)=\sum_{j=0}^{\infty}\int_{\Delta_j}\left|\rho_k(t)y^{(k)}\right|^q\,d\omega(t)\le$$

$$\le\sum_{j=0}^{\infty}\left(\max_{\Delta_j}\left|y^{(k)}\right|\right)^q\int_{\Delta_j}|\rho_k(t)|^q\,d\omega(t)\le$$

$$\le\sum_{j=0}^{\infty}\left((1+c^*)^{\frac{1}{p}}A(l,k,p)\,|\Delta_j|^{l-k-\frac{1}{p}}\left[\int_{\Delta_j}\left(\left|y^{(l)}\right|^p+v(t)\,|y|^p\right)\,dt\right]^{\frac{1}{p}}\right)^q\times$$

$$\times \int_{\Delta_j} |\rho_k(t)|^q \, d\omega(t) \le \widetilde{c}^q_{l,k,p} \sum_{j=0}^{\infty} \left[|\Delta_j|^{l-k-\frac{1}{p}} \left(\int_{\Delta_j} |\rho_k(t)|^q \, d\omega(t) \right)^{\frac{1}{q}} \right]^q \times$$

$$\times \left[\int_{\Delta_j} \left(\left| y^{(l)} \right|^p + v(t) \, |y|^p \right) \, dt \right]^{\frac{q}{p}} \le \widetilde{c}^q_{l,k,p} R^q_k \left\| y; \, W^l_{p,v}(I) \right\|^q ,$$

where $\widetilde{c}_{l,k,p} = A(l,k,p) \, (1+c^*)^{\frac{1}{p}}$.

As a result, we have

$$\| Ly; \, L_{q,\omega}(I) \| \le \sum_{k=0}^{m} \left\| \rho_k y^{(k)}; \, L_{q,\omega} \right\| \le c \sum_{k=0}^{m} R_k \left\| y; \, W^l_{p,v}(I) \right\| .$$

Thus the proof of Theorem 1 is complete.

Let us assume that the operator L in (4) is bounded as an operator from $W^l_{p,v}$ to $L_{q,\omega}$, i.e., $D(L) \subset W^l_{p,v}$ and there exists a constant $b > 0$ such that

$$\left(\int_I |Ly|^q \, d\omega(t) \right)^{\frac{1}{q}} \le b \, \|y; \, W^l_{p,v}\| \quad (y \in D(L)). \tag{5}$$

We take the function $\eta \in C^\infty_0(I)$, $0 \le \eta \le 1$, with $\operatorname{supp}(\eta) \subset [0,1]$, such that $\eta = 1$ in $\left[\frac{1}{4}, \frac{3}{4}\right]$. Let $\Delta = [x, x+h(x)]$, $h(x) = v^*(x)$, $\widetilde{\Delta} = [x + \frac{h}{4}, x + \frac{3h}{4}]$. We set $y_0(t) = \eta\left(\frac{t-x}{h}\right)$. Then $y_0(t) = 1$, $Ly_0(t) = \rho_0(t)$ for all $t \in \widetilde{\Delta}$. Therefore,

$$\left(\int_{\widetilde{\Delta}} |\rho_0|^q \, d\omega(t) \right)^{\frac{1}{q}} = \left(\int_{\widetilde{\Delta}} |Ly_0|^q \, d\omega(t) \right)^{\frac{1}{q}} \le b \, \|y_0; \, W^l_{p,v}(\Delta)\|. \tag{6}$$

Moreover,

$$\begin{aligned} \|y_0; \, W^l_{p,v}(\Delta)\| &= \left(\int_\Delta \left| y_0^{(l)}(t) \right|^p \, dt \right)^{\frac{1}{p}} + \left(\int_\Delta |y_0(t)|^p v(t) \, dt \right)^{\frac{1}{p}} = \\ &= \left(\int_\Delta h^{-lp} \left| \eta^{(l)} \left(\frac{t-x}{h} \right) \right|^p \, dt \right)^{\frac{1}{p}} + \left(\int_\Delta v(t) \, dt \right)^{\frac{1}{p}} \le \\ &\le h^{-l+\frac{1}{p}} \, (c^*_l + 1) , \end{aligned} \tag{7}$$

where $c^*_l = \|\eta^{(l)}; \, C[0,1]\| = \max\limits_{t \in [0,1]} |\eta^{(l)}(t)|$.

Recall that the following equality holds in Δ

$$h^{l-\frac{1}{p}} \left(\int_\Delta v(t) \, dt \right)^{\frac{1}{p}} = 1.$$

By (6), (7), we obtain

$$\left(\int_{\widetilde{\Delta}} |\rho_0|^q \, d\omega(t) \right)^{\frac{1}{q}} \le \widetilde{c}_0 \, b \, h^{-l+\frac{1}{p}}, \tag{8}$$

where $\widetilde{c}_0 = c^*_l + 1$. We take the function $y_1(t) = (t-x) y_0(t)$. We have $|y_1(t)| = \left|(t-x)\eta\left(\frac{t-x}{h}\right)\right| = |t-x| \le h$, $|y_1'(t)| = |(t-x)y_0'(t) + y_0(t)| = 1$, $\left| y_1^{(k)}(t) \right| = 0$ for any $t \in \widetilde{\Delta}$, when $k \ge 2$. Therefore, from (5) it follows that

$$\left(\int_{\widetilde{\Delta}} |\rho_1|^q \, d\omega(t) \right)^{\frac{1}{q}} = \left\| Ly_1 - \rho_0 y_1; \, L_{q,\omega}(\widetilde{\Delta}) \right\| \le$$

$$\leq \|Ly_1;\, L_{q,\omega}(\widetilde{\Delta})\| + \|\rho_0 y_1;\, L_{q,\omega}(\widetilde{\Delta})\| \leq b\,\|y_1;\, W^l_{p,v}(\Delta)\| + \widetilde{c}_0\, b\, h^{1-l+\frac{1}{p}}. \tag{9}$$

We have

$$\left|y_1^{(l)}(t)\right| = \left|\sum_{j=0}^{l}\binom{l}{j}(t-x)^{(j)}h^{-(l-j)}\eta^{(l-j)}\left(\frac{t-x}{h}\right)\right| \leq$$

$$\leq \left|\binom{l}{0}(t-x)h^{-l}\eta^{(l)}\left(\frac{t-x}{h}\right)\right| + \left|\binom{l}{1}h^{-(l-1)}\eta^{(l-1)}\left(\frac{t-x}{h}\right)\right| \leq h^{-l+1}\left[\binom{l}{0}c_l^* + \binom{l}{1}c_{l-1}^*\right].$$

Hence,

$$\left\|y_1;\, W^l_{p,v}(\Delta)\right\| = \left\|y_1^{(l)};\, L_p(\Delta)\right\| + \left(\int_\Delta\left|(t-x)\eta\left(\frac{t-x}{h}\right)\right|^p v(t)\,dt\right)^{\frac{1}{p}} \leq$$

$$\leq h^{1-l+\frac{1}{p}}\left\{\binom{l}{0}c_l^* + \binom{l}{1}c_{l-1}^* + 1\right\} = h^{1-l+\frac{1}{p}}\left\{1 + \sum_{j=0}^{1}\binom{l}{j}c_{l-j}^*\right\}. \tag{10}$$

By (8)–(10), we have

$$\left(\int_{\widetilde{\Delta}}|\rho_1|^q\,d\omega(t)\right)^{\frac{1}{q}} \leq b\,h^{1-l+\frac{1}{p}}\left\{1 + \sum_{j=0}^{1}\binom{l}{j}c_{l-j}^*\right\} + \widetilde{c}_0\,b\,h^{1-l+\frac{1}{p}} =$$

$$= b\,h^{1-l+\frac{1}{p}}\left\{\sum_{j=0}^{1}\binom{l}{j}c_{l-j}^* + \widetilde{c}_0 + 1\right\} = \widetilde{c}_1\,b\,h^{1-l+\frac{1}{p}}.$$

Let us assume that for any k $(1 \leq k < m)$ following estimates hold

$$\left\|\rho_j;\, L_{q,\omega}(\widetilde{\Delta})\right\| \leq b\,h^{j-l+\frac{1}{p}}\widetilde{c}_j \quad (0 \leq j \leq k-1).$$

Then we take $y_k(t) = (t-x)^k y_0(t)$, and we have

$$y_k(t) = (t-x)^k,$$

$$y_k^{(j)}(t) = k(k-1)\ldots(k-j+1)(t-x)^{k-j} \quad (1 \leq j \leq k),$$

$$y_k^{(j)}(t) = 0 \quad (j > k)$$

for all $t \in \widetilde{\Delta}$. Thus,

$$\left(\int_{\widetilde{\Delta}}|\rho_k(t)|^q\,d\omega(t)\right)^{\frac{1}{q}} = \frac{1}{k!}\left(\int_{\widetilde{\Delta}}\left|\rho_k(t)y_k^{(k)}\right|^q\,d\omega(t)\right)^{\frac{1}{q}} =$$

$$= \frac{1}{k!}\left(\int_{\widetilde{\Delta}}\left|Ly_k(t) - \sum_{j=0}^{k-1}\rho_j(t)y_k^{(j)}\right|^q\,d\omega(t)\right)^{\frac{1}{q}} \leq$$

$$\leq \frac{1}{k!}\left\{\left\|Ly_k;\, L_{q,\omega}(\widetilde{\Delta})\right\| + \sum_{j=0}^{k-1}\left\|\rho_j y_k^{(j)};\, L_{q,\omega}(\widetilde{\Delta})\right\|\right\} \leq$$

$$\leq \frac{1}{k!}\,b\,\left\|y_k;\, W^l_{p,v}(\Delta)\right\| + \frac{1}{k!}\sum_{j=0}^{k-1}\left\|\rho_j y_k^{(j)};\, L_{q,\omega}(\widetilde{\Delta})\right\| \leq$$

$$\leq \frac{1}{k!}\, b\, \left\{ \left\| y_k^{(l)};\, L_p(\Delta) \right\| + \| y_k;\, L_{p,v}(\Delta) \| \right\} + \sum_{j=0}^{k-1} \frac{1}{(k-j)!} h^{k-j} \left\| \rho_j;\, L_{q,\omega}(\widetilde{\Delta}) \right\| \leq$$

$$\leq \frac{1}{k!}\, b \left\{ \left(\int_\Delta \left| \sum_{j=0}^{k} \binom{l}{j} \left((t-x)^k \right)^{(j)} h^{j-l} \eta^{(l-j)} \left(\frac{t-x}{h} \right) \right|^p dt \right)^{\frac{1}{p}} + \right.$$

$$\left. + \left(\int_\Delta |y_k|^p v(t)\, dt \right)^{\frac{1}{p}} \right\} +$$

$$+ \sum_{j=0}^{k-1} \frac{1}{(k-j)!} h^{k-j} \left(\int_{\widetilde{\Delta}} |\rho_j|^q \, d\omega(t) \right)^{\frac{1}{q}} \leq b\, h^{k-l+\frac{1}{p}} \widetilde{c}_k.$$

So, we have

$$\left(\int_{x+h/4}^{x+3h/4} |\rho_k(t)|^q \, d\omega(t) \right)^{1/q} \ll b\, h^{k-l+\frac{1}{p}} \quad (h = v^*(x),\ 0 \leq k \leq m).$$

Theorem 2. Let $l > m \geq 1$ be integers, $1 < p < \infty$, $1 \leq q < \infty$, $lp > 1$. Let the operator L in (4) be bounded from $W^l_{p,v}$ to $L_{q,\omega}$. Then $(d\,\omega(t) = \omega(t)\, dt)$

$$\widetilde{R}_k = \sup_{x \geq 0} v^*(x)^{l-k-\frac{1}{p}} \left\{ \int\limits_{x+\frac{v^*(x)}{4}}^{x+\frac{3v^*(x)}{4}} |\rho_k(t)|^q \, d\omega(t) \right\}^{\frac{1}{q}} \leq \widetilde{c}_k \left\| L;\, W^l_{p,v} \to L_{q,\omega} \right\|. \tag{11}$$

Proof. We have the fulfillment of condition (5) with $b = \left\| L;\, W^l_{p,v} \to L_{q,\omega} \right\|$. In this case, we have shown that the following inequality holds

$$v^*(x)^{l-k-\frac{1}{p}} \left\{ \int\limits_{x+\frac{v^*(x)}{4}}^{x+\frac{3v^*(x)}{4}} |\rho_k(t)|^q \, d\omega(t) \right\}^{\frac{1}{q}} \leq \widetilde{c}_k \left\| L;\, W^l_{p,v} \to L_{q,\omega} \right\|$$

for all $x \geq 0$. Then it follows the validity of inequality (11). The proof of Theorem 2 is complete.

We set $R^* = \sum\limits_{k=0}^{m} R^*_k$, where $R^*_k = R_k$ with $h(x) = v^*(x)$, and $\widetilde{R}^* = \sum\limits_{k=0}^{m} \widetilde{R}_k$.

Theorem 3. Let $l > m \geq 1$ be integers, $1 < p \leq q < \infty$. Let v be in $A_{(\delta,\beta)}$. Let $R^* < \infty$. Then the operator L in (4) is bounded from $W^l_{p,v}$ to $L_{q,\omega}$. Furthermore,

$$c_0 \widetilde{R}^* \leq \left\| L;\, W^l_{p,v} \to L_{q,\omega} \right\| \leq c_1 R^*.$$

The statements of Theorem 3 are direct consequences of Theorem 1 and Theorem 2.

Theorem 4. Let $l > m \geq 1$ be integers, $1 < p \leq q < \infty$. Let $v \in \Pi_{l,p}(\delta,\tau)$ with respect to the l.f. $h(\cdot)$ in I. Let $\mu \in A^m(I)$. If

$$M_{k,\mu,\omega_0} = \sup_{x \geq 0} h(x)^{l-k-\frac{1}{p}} \left\{ \int\limits_{x}^{x+h(x)} \left| \mu^{(m-k)}(t) \right|^q d\omega_0(t) \right\}^{\frac{1}{q}} < \infty \quad (k = 0, 1, \ldots, m),$$

$$M_{0,\mu,\omega_1} = \sup_{x\geq 0} h(x)^{l-\frac{1}{p}} \left\{ \int\limits_{x}^{x+h(x)} |\mu(t)|^q \, d\omega_1(t) \right\}^{\frac{1}{q}} < \infty,$$

then $\mu \in M(W^l_{p,v} \to W^m_{q,\omega_0,\omega_1})$. Moreover,

$$\left\| \mu; \, M(W^l_{p,v} \to W^m_{q,\omega_0,\omega_1}) \right\| \leq C \left[\sum_{k=0}^{m} M_{k,\mu,\omega_0} + M_{0,\mu,\omega_1} \right],$$

where $C = C(n,l,p,q) > 0$.

Proof. We have

$$\|\mu y; W^m_{q,\omega_0,\omega_1}\|^q = \int_0^\infty \left(|(\mu y)^{(m)}|^q \omega_0 + |\mu y|^q \omega_1 \right) dt.$$

Since $(\mu y)^{(m)}(t) = Ly$, $\rho_k = \frac{m!}{k!(m-k)!}\mu^{(m-k)}$, it follows that

$$\int_0^\infty |(\mu y)^{(m)}|^q \omega_0(t)\, dt = \int_0^\infty |Ly|^q \omega_0(t)\, dt = \|Ly; L_{q,\omega_0}\|^q$$

and

$$\int_0^\infty |\mu y|^q \omega_1(t)\, dt \leq c \sum_j h_j^{(l-1/p)q} \left(\int_{\Delta_j} |\mu|^q \omega_1 \right) \|y; W^l_{p,v}\|^q \leq$$

$$\leq \left(\sup_x h(x)^{l-1/p} \left(\int_{\Delta_j} |\mu|^q \omega_1 \right)^{1/q} \right)^q \|y; W^l_{p,v}\|^q.$$

Thus, the proof of Theorem 4 follows the lines of the proof of Theorem 1.

Theorem 5. Let $1 < p \leq q < \infty$. Let $l > m \geq 1$ be integers. If $\mu \in M(W^l_{p,v} \to W^m_{q,\omega_0,\omega_1})$, then

$$\left\| \mu; \, M(W^l_{p,v} \to W^m_{q,\omega_0,\omega_1}) \right\| \geq C \left[\sum_{k=0}^{m} M^*_{k,\mu,\omega_0} + M^*_{0,\mu,\omega_1} \right],$$

where

$$M^*_{k,\mu,\omega_0} = \sup_{x\geq 0} v^*(x)^{l-k-\frac{1}{p}} \left\{ \int\limits_{x+\frac{v^*(x)}{4}}^{x+\frac{3v^*(x)}{4}} \left| \mu^{(m-k)}(t) \right|^q d\omega_0(t) \right\}^{\frac{1}{q}} < \infty,$$

$$M^*_{0,\mu,\omega_1} = \sup_{x\geq 0} v^*(x)^{l-\frac{1}{p}} \left\{ \int\limits_{x+\frac{v^*(x)}{4}}^{x+\frac{3v^*(x)}{4}} |\mu(t)|^q \, d\omega_1(t) \right\}^{\frac{1}{q}} < \infty.$$

The constant C does not depend on $h(\cdot), v$ and μ.

Proof. By $\mu \in M(W^l_{p,v} \to W^m_{q,\omega_0,\omega_1})$ it follows that

$$\|\mu; M(W^l_{p,v} \to W^m_{q,\omega_0,\omega_1})\| \geq \frac{\|(\mu y)^{(m)}; L_{q,\omega_0}\|}{\|y; W^l_{p,v}\|} + \frac{\|\mu y; L_{q,\omega_1}\|}{\|y; W^l_{p,v}\|}.$$

Then

$$\|\mu; M(W^l_{p,v} \to W^m_{q,\omega_0,\omega_1})\| \geq \sup_{0\neq y\in W^l_{p,v}} \frac{\|(\mu y)^{(m)}; L_{q,\omega_0}\|}{\|y; W^l_{p,v}\|} = \|L; W^l_{p,v} \to L_{q,\omega_0}\|,$$

where $Ly = \sum\limits_{k=0}^{m} \rho_k y^{(k)}$, $\rho_k = c_k \mu^{(m-k)}$. By Theorem 3, we obtain

$$\|\mu; M(W^l_{p,v} \to W^m_{q,\omega_0,\omega_1})\| \ge c_0 \sum_{k=0}^{m} \sup_{x\ge 0} M^*_{k,\mu,\omega_0}$$

Next, we take a function $y_0(t)$ defined as in Theorem 2. Then

$$\|\mu; M(W^l_{p,v} \to W^m_{q,\omega_0,\omega_1})\| \ge \frac{\left(\int\limits_{x+v^*(x)/4}^{x+3v^*(x)/4} |\mu|^q \, d\omega_1\right)^{1/q}}{\left(\int\limits_{x}^{x+v^*(x)} |y_0^{(l)}|^p \, dt\right)^{1/p} + \left(\int\limits_{x}^{x+v^*(x)} |y_0|^p v(t) \, dt\right)^{1/p}} \ge$$

$$\ge \frac{\left(\int\limits_{x+v^*(x)/4}^{x+3v^*(x)/4} |\mu|^q \, d\omega_1\right)^{1/q}}{\left(h^{1-lp}\int\limits_{0}^{1} |\eta^{(l)}|^p \, d\xi\right)^{1/p} + \left(\int\limits_{x}^{x+v^*(x)} |y_0|^p v(t) \, dt\right)^{1/p}} \ge c_1 M^*_{0,\mu,\omega_1}.$$

Thus, the proof of Theorem 5 is complete.

Corollary 1. Let $l > m \ge 1,\ 1 < p \le q < \infty$. Let $\mu \in C^m(I)$. Then $\mu \in M(W^l_p \to W^m_{q,\omega_0,\omega_1})$ if and only if

$$U_k = \sup_{x\ge 1} \int\limits_{x}^{x+1} |\mu^{(m-k)}|^q \, d\omega_1 < \infty \quad (k = 0, 1, \ldots, m),$$

$$V = \sup_{x\ge 1} \int\limits_{x}^{x+1} |\mu|^q \, d\omega_0 < \infty.$$

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References

1 Maz'ya V.G.Theory of multipliers in spaces of differentiable functions / V.G. Maz'ya, T.O. Shaposhnikova. — Boston: Pitman, 1985. — 344 p.

2 Maz'ya V.G.Theory of Sobolev multipliers: with applications to differential and integral operators / V.G. Maz'ya, T.O. Shaposhnikova. — Berlin: Grundlehren der mathematischen Wissenschaften (Fundamental Principles of Mathematical Sciences), Springer-Verlag, 2009. — 337. — xiv+614 p.

3 Нейман-Заде М.И. Операторы Шрёдингера с сингулярными потенциалами из пространств мультипликаторов / М.И. Нейман-Заде, А.А. Шкаликов // Математические заметки. — 1999. — 66. — 5. — С. 723–733.

4 Беляев А.А. Мультипликаторы в пространствах бесселевых потенциалов: случай индексов гладкости разного знака / А.А. Беляев, А.А. Шкаликов // Алгебра и анализ. — 2018. — 30. — 2. — С. 76–96.

5 Кусаинова Л.К. Об ограниченности оператора Шрёдингера в весовых пространствах Соболева / Л.К. Кусаинова, А.Х. Мырзагалиева, Я.Т. Султанаев // Математические заметки. — 2016. — 99. — 6. — С. 945–949.

6 Kussainova L. On Multipliers from Weighted Sobolev Spaces to Lebesgue Spaces / L. Kussainova, A. Myrzagaliyeva // Springer Proceedings in Mathematics and Statisticsthis. — 2017. — 216. — P. 52–57.

7 Отелбаев М. Теоремы вложения пространств с весом и их приложения к изучению спектра оператора Шрёдингера / М. Отелбаев // Тр. Мат. ин-та им. В.А. Стеклова. — 1979. — 150. — С. 265–305.

8 Кусаинова Л.К. Теоремы вложения и интерполяции весовых пространств Соболева: дис. д-ра физ.-мат. наук: 01.01.01 – «Вещественный, комплексный и функциональный анализ (алгебра, логика и теория чисел)» / Лейли Кабиденовна Кусаинова. — Алматы, 1999. — 255 с.

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Осьтегі салмақты Соболев кеңістіктеріндегі мультипликаторлар

Жұмыста $W^l_{p,v}$ салмақты Соболев кеңістігінен салмақты Лебег кеңістігіне оң нақты жартылай түзуде әсер ететін бір айнымалы дифференциалдық оператордың шектелуі үшін қажетті және жеткілікті шарттар анықталған. Дифференциалдық операторлардың коэффициенттерін мультипликаторлар ретінде қарастыру заңды екені белгілі. Салмақты Соболев кеңістіктерінде нүктелік мультипликаторлар енгізілген. Жалпы типті салмақтары бар (W_1, W_2) салмақты Соболев кеңістіктерінің жұбы үшін $M(W_1 \to W_2)$ кеңістігінің сипаттамасы алынған.

Кілт сөздер: Соболев кеңістігі, нүктелік көбейткіш, салмақты кеңістік, дифференциалдық оператор, рұқсат етілген функция, баяу вариация шарты, Өтелбаев функциясы.

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Мультипликаторы в весовых пространствах Соболева на оси

В статье установлены необходимые и достаточные условия ограниченности дифференциального оператора одной переменной, действующего из весового пространства Соболева $W^l_{p,v}$ в весовое пространство Лебега на положительной вещественной полупрямой. Хорошо известно, что коэффициенты дифференциальных операторов естественно рассматривать как мультипликаторы. Мы вводим точечные мультипликаторы в весовых пространствах Соболева. Получено описание пространства $M(W_1 \to W_2)$ для пары весовых пространств Соболева (W_1, W_2) с весами общего типа.

Ключевые слова: пространство Соболева, точечный мультипликатор, весовое пространство, дифференциальный оператор, допустимая функция, условие медленного колебания, функция Отельбаева.

References

1 Maz'ya, V.G., & Shaposhnikova, T.O. (1985). *Theory of multipliers in spaces of differentiable functions.* Boston: Pitman.

2 Maz'ya, V.G., & Shaposhnikova, T.O. (2009). *Theory of Sobolev multipliers: with applications to differential and integral operators.* Berlin: Grundlehren der mathematischen Wissenschaften (Fundamental Principles of Mathematical Sciences), Springer-Verlag.

3 Neiman-Zade, M.I., & Shkalikov, A.A. (1999). Operatory Shredingera s singuliarnymi potentsialami iz prostranstv multiplikatorov [Schrödinger Operators with Singular Potentials from Spaces of Multipliers]. *Matematicheskie zametki — Mathematical Notes, 66 (5),* 723-733 [in Russian].

4 Beliaev, A.A., & Shkalikov, A.A. (2018). Multiplikatory v prostranstvakh besselevykh potentsialov: sluchai indeksov gladkosti raznogo znaka [Multipliers in Bessel potential spaces: the case of smoothness indices of different sign]. *Algebra i analiz – Algebra and Analysis, 30*(2), 76–96 [in Russian].

5 Kusainova, L.K., Myrzagalieva, A.Kh., & Sultanaev, Ya.T. (2016). Ob ogranichennosti operatora Shredingera v vesovykh prostranstvakh Soboleva [On the boundedness of the Shrödinger operator in weighted Sobolev spaces]. *Matematicheskie zametki — Mathematical Notes, 99*(6), 945–949 [in Russian].

6 Kussainova, L., & Myrzagaliyeva, A. (2017). On Multipliers from Weighted Sobolev Spaces to Lebesgue Spaces. *Springer Proceedings in Mathematics and Statisticsthis, 216,* 52–57.

7 Otelbaev, M. (1979). Teoremy vlozheniia prostranstv s vesom i ikh primeneniia k izucheniiu spektra operatora Shredingera [Imbedding theorems for spaces with a weight and their application to the study of the spectrum of a Schrödinger operator]. *Trudy Matematicheskogo instituta imeni V.A. Steklova — Proceedings of the V.A. Steklov Institute of Mathematics, 150,* 265–305 [in Russian].

8 Kusainova, L.K. (1999). Teoremy vlozheniia i interpoliatsii vesovykh prostranstv Soboleva [Embedding and interpolation theorems of weighted Sobolev spaces]. *Doctor's thesis.* Almaty [in Russian].