

On Approximation Orders of Functions of Several Variables in the Lorentz Space

G. Akishev¹

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Abstract—We consider the anisotropic Lorentz space of periodic functions. Sufficient conditions are proved for a function to belong to the anisotropic Lorentz space. Estimates for the order of approximation by trigonometric polynomials of the Nikol'skii–Besov class in the anisotropic Lorentz space are established.

Keywords: Lorentz space, Nikol'skii–Besov class, best approximation.

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INTRODUCTION

Let $\bar{x} = (x_1, \dots, x_m) \in I^m = [0, 2\pi]^m$, $\bar{p} = (p_1, \dots, p_m)$, $\bar{\theta} = (\theta_1, \dots, \theta_m)$, and $\theta_j, p_j \in [1, +\infty)$ for $j = 1, \dots, m$. Denote by $L_{\bar{p}, \bar{\theta}}^*(I^m)$ the Lorentz space of all Lebesgue measurable functions $f(\bar{x})$ that are 2π -periodic in each variable and such that the value

$$\|f\|_{\bar{p}, \bar{\theta}}^* = \left[\int_0^{2\pi} t_m^{\frac{\theta_m}{p_m}-1} \left[\dots \left[\int_0^{2\pi} (f^{*1, \dots, *m}(t_1, \dots, t_m))^{\theta_1} t_1^{\frac{\theta_1}{p_1}-1} dt_1 \right]^{\frac{\theta_2}{p_2}} \dots \right]^{\frac{\theta_m}{p_m}} dt_m \right]^{\frac{1}{\theta_m}}$$

is finite, where $f^{*1, \dots, *m}(t_1, \dots, t_m)$ is a nonincreasing permutation of the function $|f(\bar{x})|$ in each variable x_j with fixed remaining variables (see [1]).

In the case $p_1 = \dots = p_m = \theta_1 = \dots = \theta_m = p$, the Lorentz space $L_{\bar{p}, \bar{\theta}}^*(I^m)$ coincides with the Lebesgue space $L_p(I^m)$ with norm $\|f\|_p$ (see [2, Ch. I, Sect. 1.1]).

Denote by $\dot{L}_{\bar{p}, \bar{\theta}}^*(I^m)$ the set of all functions $f \in L_{\bar{p}, \bar{\theta}}^*(I^m)$ such that

$$\int_0^{2\pi} f(\bar{x}) dx_j = 0 \quad \forall j = 1, \dots, m.$$

For a function $f \in L_1(I^m) = L(I^m)$, we consider its Fourier series $\sum_{\bar{n} \in \mathbb{Z}^m} a_{\bar{n}}(f) e^{i\langle \bar{n}, \bar{x} \rangle}$, where $a_{\bar{n}}(f)$ are the Fourier coefficients of f in the multiple trigonometric system $\{e^{i\langle \bar{n}, \bar{x} \rangle}\}_{\bar{n} \in \mathbb{Z}^m}$ and \mathbb{Z}^m is the integer lattice in \mathbb{R}^m .

¹Karaganda State University, Karaganda, 470074 Kazakhstan; Ural Federal University, Yekaterinburg, 620000 Russia
 e-mail: akishev_g@mail.ru

Define $\delta_{\bar{s}}(f, \bar{x}) = \sum_{\bar{n} \in \rho(\bar{s})} a_{\bar{n}}(f) e^{i\langle \bar{n}, \bar{x} \rangle}$, where $\langle \bar{y}, \bar{x} \rangle = \sum_{j=1}^m y_j x_j$, $s_j = 1, 2, \dots$, and $\rho(\bar{s}) = \{\bar{k} = (k_1, \dots, k_m) \in \mathbb{Z}^m : 2^{s_j-1} \leq |k_j| < 2^{s_j}, j = 1, \dots, m\}$.

The numerical sequence $\{a_{\bar{n}}\}_{\bar{n} \in \mathbb{Z}^m}$ belongs to $l_{\bar{p}}$ if

$$\|\{a_{\bar{n}}\}_{\bar{n} \in \mathbb{Z}^m}\|_{l_{\bar{p}}} = \left\{ \sum_{n_m=-\infty}^{\infty} \left[\dots \left[\sum_{n_1=-\infty}^{\infty} |a_{\bar{n}}|^{p_1} \right]^{\frac{p_2}{p_1}} \dots \right]^{\frac{p_m}{p_{m-1}}} \right]^{\frac{1}{p_m}} < +\infty,$$

where $\bar{p} = (p_1, \dots, p_m)$ with $1 \leq p_j < +\infty$ for $j = 1, 2, \dots, m$.

The spaces of functions with dominating mixed derivative $S_{\bar{p}}^{\bar{\tau}} H$ and $S_{\bar{p}, \theta}^{\bar{\tau}} B$ were defined by Nikol'skii [3] and Amanov [4], respectively.

Consider a similar space. Denote by $\overset{\circ}{S}_{\bar{p}, \bar{\theta}, \bar{\tau}}^{\bar{\tau}} B$ the space of all functions $f \in L_{\bar{p}, \bar{\theta}}^*(I^m)$ such that $\|f\|_{\overset{\circ}{S}_{\bar{p}, \bar{\theta}, \bar{\tau}}^{\bar{\tau}} B} = \|\{2^{\langle \bar{s}, \bar{\tau} \rangle} \|\delta_{\bar{s}}(f)\|_{\bar{p}, \bar{\theta}}^*\}_{\bar{s} \in \mathbb{Z}_+^m}\|_{l_{\bar{\tau}}} < \infty$, where $\bar{p} = (p_1, \dots, p_m)$, $\bar{\theta} = (\theta_1, \dots, \theta_m)$, and $\bar{\tau} = (\tau_1, \dots, \tau_m)$ with $1 < p_j < \infty$, $1 \leq \theta_j < \infty$, $1 \leq \tau_j \leq +\infty$, and $r_j > 0$ for $j = 1, \dots, m$. In this space, consider the class (with the notation preserved)

$$\overset{\circ}{S}_{\bar{p}, \bar{\theta}, \bar{\tau}}^{\bar{\tau}} B = \{f \in L_{\bar{p}, \bar{\theta}}^*(I^m) : \|f\|_{\overset{\circ}{S}_{\bar{p}, \bar{\theta}, \bar{\tau}}^{\bar{\tau}} B} = \|\{2^{\langle \bar{s}, \bar{\tau} \rangle} \|\delta_{\bar{s}}(f)\|_{\bar{p}, \bar{\theta}}^*\}_{\bar{s} \in \mathbb{Z}_+^m}\|_{l_{\bar{\tau}}} \leq 1\}.$$

Assume that a vector $\bar{\gamma} = (\gamma_1, \dots, \gamma_m)$ with $\gamma_j > 0$ for $j = 1, \dots, m$ is given. Define

$$Q_n^{\bar{\gamma}} = \bigcup_{(\bar{s}, \bar{\gamma}) < n} \rho(\bar{s}), \quad T(Q_n^{\bar{\gamma}}) = \left\{ t(\bar{x}) = \sum_{\bar{k} \in Q_n^{\bar{\gamma}}} b_{\bar{k}} e^{i\langle \bar{k}, \bar{x} \rangle} \right\},$$

$$Y^m(\bar{\gamma}, n) = \left\{ \bar{s} = (s_1, \dots, s_m) \in \mathbb{Z}_+^m : \sum_{j=1}^m s_j \gamma_j \geq n \right\}.$$

We denote by $E_n^{(\bar{\gamma})}(f)_{\bar{p}, \bar{\theta}}$ the best approximation of a function $f \in L_{\bar{p}, \bar{\theta}}^*(I^m)$ by polynomials from the set $T(Q_n^{\bar{\gamma}})$ and by $S_n^{\bar{\gamma}}(f, \bar{x}) = \sum_{\bar{k} \in Q_n^{\bar{\gamma}}} a_{\bar{k}}(f) e^{i\langle \bar{k}, \bar{x} \rangle}$ the partial sum of the Fourier series of f .

A method for the approximation of multivariate functions by trigonometric polynomials with harmonics from hyperbolic crosses was first proposed by Babenko [5].

Later, the application of this method to the approximation of various classes of smooth functions was studied by S. A. Telyakovskii, B. S. Mityagin, Ya. S. Bugrov, N. S. Nikol'skaya, E. M. Galeev, V. N. Temlyakov, Dinh Dung, N. N. Pustovoitov, E. S. Belinskii, B. S. Kashin, V. N. Temlyakov, A. S. Romanyuk, H.-J. Schmeisser, W. Sickel, and others (see [6, 7]).

The following properties of the Lorentz spaces are known: $L_{\bar{q}, \bar{\theta}}^*(I^m) \subset L_{\bar{p}, \bar{\theta}}^*(I^m)$ in the case $p_j < q_j$, $j = 1, \dots, m$, and $L_{\bar{p}, \bar{\theta}}^*(I^m) \subset L_{\bar{p}, \bar{q}}^*(I^m)$ in the case $\theta_j < q_j$, $j = 1, \dots, m$.

Exact bounds for the order of approximation of the Nikol'skii-Besov classes $\overset{\circ}{S}_{\bar{p}, \bar{\theta}, \bar{\tau}}^{\bar{\tau}} B$ in $L_{\bar{q}, \bar{\theta}}^*(I^m)$ in the case $p_j < q_j$, $j = 1, \dots, m$, were established in [8, 9].

The main aim of the present paper is to derive bounds for the value

$$E_n^{(\bar{\gamma})}(\overset{\circ}{S}_{\bar{p}, \bar{q}, \bar{\tau}}^{\bar{\tau}} B)_{\bar{p}, \theta} = \sup \left\{ E_n^{(\bar{\gamma})}(f)_{\bar{p}, \theta} : f \in \overset{\circ}{S}_{\bar{p}, \bar{q}, \bar{\tau}}^{\bar{\tau}} B \right\}$$

in the case $\theta < q_j$, $j = 1, 2, \dots, m$.

We define $a_+ = \max\{a, 0\}$, and $A(y) \asymp B(y)$ means that there exist positive numbers C_1 and C_2 independent of $n \in \mathbb{N}$ and such that $C_1A(y) \leq B(y) \leq C_2A(y)$. If $B \geq C_1A$ or $B \leq C_2A$, we often write for brevity $B \gg A$ or $B \ll A$, respectively. In addition, $C_j(p, q, \dots)$ for $j \in \mathbb{N}$ denotes a positive constant depending only on the parameters specified in the brackets.

1. AUXILIARY STATEMENTS

Before proving the main results, we introduce additional notation and formulate auxiliary statements.

Denote the index set $\{1, \dots, m\}$ by e_m , its arbitrary subset by e , and its cardinality by $|e|$.

For an element $\bar{r} = (r_1, \dots, r_m)$ of the m -dimensional space with nonnegative coordinates, we denote by $\bar{r}^e = (r_1^e, \dots, r_m^e)$ the vector with components $r_j^e = r_j$ for $j \in e$ and $r_j^e = 0$ for $j \notin e$.

Let $\bar{l} = (l_1, \dots, l_m)$ be an element of the m -dimensional space with positive integer coordinates. For nonempty $e \subset e_m$, define $G_{\bar{l}}(e) = \{\bar{k} = (k_1, \dots, k_m) \in \mathbb{Z}^m : |k_j| \leq l_j, j \in e, |k_j| > l_j, j \notin e\}$.

For given numbers $b_{\bar{n}}$, a mixed difference is defined by the formula

$$\Delta b_{\bar{n}} = \sum_{\bar{0} \leq \bar{\varepsilon} \leq \bar{1}} (-1)^{m - \sum_{j=1}^m \varepsilon_j} b_{\bar{n} - \bar{1} + \bar{\varepsilon}},$$

where $\bar{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_m)$, $\varepsilon_j = 0$ or 1 , and $\bar{n} - \bar{1} + \bar{\varepsilon} = (n_1 - 1 + \varepsilon_1, \dots, n_m - 1 + \varepsilon_m)$.

We consider partial sums over different variables:

$$S_{\bar{l}}(f, \bar{x}) = S_{l_1, \dots, l_m}(f, \bar{x}) = \sum_{|k_1| \leq l_1} \dots \sum_{|k_m| \leq l_m} a_{\bar{k}}(f) e^{i\langle \bar{k}, \bar{x} \rangle}$$

is a partial sum over all variables, and

$$S_{l_1, \infty}(f, \bar{x}) = \sum_{|k_1| \leq l_1} \sum_{k_2 = -\infty}^{+\infty} \dots \sum_{k_m = -\infty}^{+\infty} a_{\bar{k}}(f) e^{i\langle \bar{k}, \bar{x} \rangle}$$

is a partial sum over the variable x_1 . In a more general case,

$$S_{\bar{l}^e, \infty}(f, \bar{x}) = \sum_{\bar{k} \in \prod_{j \in e} [-l_j, l_j] \times \mathbb{R}^{m-|e|}} a_{\bar{k}}(f) e^{i\langle \bar{k}, \bar{x} \rangle}$$

is a partial sum over the variables x_j for $j \in e$.

For a given subset $e \subset e_m$, define

$$U_{\bar{l}}(f, \bar{x}) = \sum_{e \subset e_m, e \neq \emptyset} \sum_{\bar{k} \in G_{\bar{l}}(e)} a_{\bar{k}}(f) e^{i\langle \bar{k}, \bar{x} \rangle}.$$

In particular, for $m = 2$, we have (see [10]) $U_{l_1, l_2}(f, \bar{x}) = S_{l_1, \infty}(f, \bar{x}) + S_{\infty, l_2}(f, \bar{x}) - S_{l_1, l_2}(f, \bar{x})$.

Let $e \subset e_m$. We consider the known Vallée-Poussin multiple sum

$$V_{\bar{l}^e, \infty}(f, \bar{x}) = \prod_{j \in e} \frac{1}{l_j} \sum_{\bar{k} \in \mathbb{R}^{m-|e|} \times \prod_{j \in e} [-l_j, l_j]} S_{\bar{k}^e, \infty}(f, \bar{x}).$$

In particular, for $e = e_m$,

$$V_{\bar{e}e_m}(f, \bar{x}) = V_{l_1, \dots, l_m}(f, \bar{x}) = \prod_{j=1}^m \frac{1}{l_j} \sum_{k_1=l_1}^{2l_1-1} \dots \sum_{k_m=l_m}^{2l_m-1} S_{\bar{k}}(f, \bar{x}).$$

We consider the Vall'ee-Poussin sum over x_1

$$V_{0, \infty}(f, \bar{x}) = S_{0, \infty}(f, \bar{x}), \quad V_{l_1, \infty}(f, \bar{x}) = \frac{1}{l_1} \sum_{k=l_1}^{2l_1-1} S_{k, \infty}(f, \bar{x}),$$

over x_2

$$V_{\infty, 0}(f, \bar{x}) = S_{\infty, 0}(f, \bar{x}), \quad V_{\infty, l_2}(f, \bar{x}) = \frac{1}{l_2} \sum_{k=l_2}^{2l_2-1} S_{\infty, k}(f, \bar{x}),$$

and over x_1 and x_2

$$V_{l_1, l_2}(f, \bar{x}) = V_{l_1, \infty}(V_{\infty, l_2}(f, \bar{x})).$$

Further, define $W_{\bar{l}}(f, \bar{x}) = \sum_{e \subset e_m, e \neq \emptyset} V_{\bar{l}e, \infty}(f, \bar{x})$ and consider the mixed difference over indices

$$\Delta W_{2^{\nu_1}, \dots, 2^{\nu_m}}(f, \bar{x}) = \Delta W_{2^{\bar{\nu}}}(f, \bar{x}) = \sum_{\bar{0} \leq \bar{\varepsilon} \leq \bar{1}} (-1)^{m - \sum_{j=1}^m \varepsilon_j} W_{2^{\bar{\nu} - \bar{1} + \bar{\varepsilon}}}(f, \bar{x}),$$

where $\bar{1} = (1, \dots, 1)$ and $\bar{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_m)$ is a vector with coordinates $\varepsilon_j = 0$ or 1 . Then the function $\Delta W_{2^{\nu_1}, \dots, 2^{\nu_m}}(f, \bar{x})$ is a trigonometric polynomial of order $2^{\nu_j+1} - 1$ in the variable x_j , $j = 1, \dots, m$ [10]. Finally, define $\varphi_{\bar{\nu}}(\bar{x}) = -\Delta W_{2^{\nu_1}, \dots, 2^{\nu_m}}(f, \bar{x})$ and

$$\begin{aligned} \Phi_{\bar{k}}(f, \bar{x}) &= \Phi_{k_1, \dots, k_m}(f, \bar{x}) = \sum_{\nu_m=2^{k_m}}^{2^{k_m+1}-1} \dots \sum_{\nu_1=2^{k_1}}^{2^{k_1+1}-1} \varphi_{\nu_1, \dots, \nu_m}(\bar{x}) = \sum_{\bar{\nu} \in \rho(\bar{k} + \bar{1})} \varphi_{\bar{\nu}}(\bar{x}), \\ &\sum_{k_m=l_m}^{n_m} \dots \sum_{k_1=l_1}^{n_1} \Phi_{\bar{k}}(f, \bar{x}) = \sum_{\bar{l} \leq \bar{k} \leq \bar{n}} \Phi_{\bar{k}}(f, \bar{x}). \end{aligned}$$

For a subset $e \subset e_m$, define

$$\Phi_{\bar{k}e}(f, \bar{x}) = \sum_{\nu_{j_1}=2^{k_{j_1}}}^{2^{k_{j_1}+1}-1} \dots \sum_{\nu_{j_{|e|}}=2^{k_{j_{|e|}}}}^{2^{k_{j_{|e|}}+1}-1} \varphi_{\nu^e}(\bar{x}).$$

Note that $\varphi_{\bar{\nu}e}(\bar{x})$ is a trigonometric polynomial of order 1 in the variable x_j for $j \notin e$. Hence, $\Phi_{\bar{k}e}(f, \bar{x})$ has the same order in these variables.

The value (see [10])

$$Y_{l_1, \dots, l_m}(f)_{\bar{p}, \bar{\theta}} = \inf_{T_{l_j}} \|f - \sum_{j=1}^m T_{l_j}\|_{\bar{p}, \bar{\theta}}^*, \quad l_j = 0, 1, 2, \dots,$$

is called the *best "angle" approximation of the function $f \in L_{\bar{p}, \bar{\theta}}^*(I^m)$ by trigonometric polynomials*, where $T_{l_j} \in L_{\bar{p}, \bar{\theta}}^*(I^m)$ is a trigonometric polynomial of order l_j in the variable x_j , $j = 1, \dots, m$.

The best “angle” approximation in Lebesgue spaces was introduced by Potapov [10].

Now, we present some auxiliary statements.

In what follows, we will use the notation $\varkappa(n) = \{\bar{s} = (s_1, \dots, s_m) \in \mathbb{Z}_+^m : \langle \bar{s}, \bar{\gamma} \rangle = n\}$.

Lemma 1. *Suppose that $\bar{\gamma} = (\gamma_1, \dots, \gamma_m)$, $\bar{\tau} = (\tau_1, \dots, \tau_m)$, $1 = \gamma_1 = \dots = \gamma_\nu < \gamma_{\nu+1} \leq \dots \leq \gamma_m$, and $\tau_j \in [1, +\infty)$ and $\beta_j \geq 0$ for $j = 1, \dots, m$. Then*

$$\left\| \left\{ \prod_{j=1}^m s_j^{\beta_j} \right\}_{\bar{s} \in \varkappa(n)} \right\|_{l_{\bar{\tau}}} \asymp n^{\sum_{j=2}^m 1/\tau_j + \sum_{j=1}^m \beta_j}.$$

Proof. By the definition of the set \varkappa_n , we have

$$\left\| \left\{ \prod_{j=1}^m s_j^{\beta_j} \right\}_{\bar{s} \in \varkappa_n} \right\|_{l_{\bar{\tau}}} = \left\{ \sum_{s_m < \frac{n}{\gamma_m}} \left[\dots \left[\sum_{s_2 < \frac{1}{\gamma_2} (n - \sum_{j=3}^m \gamma_j s_j)} \left[\sum_{s_1 = \frac{1}{\gamma_1} (n - \sum_{j=2}^m \gamma_j s_j)} \prod_{j=2}^m s_j^{-d_j \tau_j} \right]^{\frac{\tau_2}{\tau_1}} \right]^{\frac{\tau_3}{\tau_2}} \dots \right]^{\frac{\tau_m}{\tau_{m-1}}} \right\}^{\frac{1}{\tau_m}}.$$

Applying the relation $\sum_{0 < s < l} s^\alpha (l - s)^\beta \asymp l^{\alpha + \beta + 1}$, $\alpha, \beta > 0$, $l \in \mathbb{N}$, several times, we obtain the assertion of the lemma. \square

Note that, in the case $\beta_j = 0$, $j = 1, \dots, m$, Lemma 1 was proved in [8].

The proof of Theorem 1 is based on the following statement.

Lemma 2. *Suppose that $\bar{p} = (p_1, \dots, p_m)$, $\bar{q} = (q_1, \dots, q_m)$, and $1 \leq p_j < +\infty$ and $1 \leq \theta < q_j < +\infty$ for $j = 1, \dots, m$. Then*

$$I_1 = \left\| \sum_{\bar{l} \leq \bar{k} \leq \bar{n}} \Phi_{\bar{k}}(f) \right\|_{\bar{p}, \theta}^* \ll \left[\sum_{k_m = l_m}^{n_m} \dots \sum_{k_1 = l_1}^{n_1} \prod_{j=1}^m 2^{k_j \theta (\frac{1}{\theta} - \frac{1}{q_j})} (\|\Phi_{\bar{k}}(f)\|_{\bar{p}, \bar{q}})^\theta \right]^{\frac{1}{\theta}}.$$

Proof. Let $s = [\theta] + 1$. Then, by the property of the norm and of a nonincreasing permutation of a function,

$$I_1 \leq \left\| \frac{1}{t_1 \dots t_m} \int_0^{t_m} \dots \int_0^{t_1} \left(\sum_{\bar{l} \leq \bar{k} \leq \bar{n}} \Phi_{\bar{k}}(f) \right)^{*1, \dots, *m}(\bar{y}) dy_1 \dots dy_m \right\|_{\bar{p}, \theta}^* \tag{1.1}$$

From the equality (see [1])

$$\int_0^{t_m} \dots \int_0^{t_1} g^{*1, \dots, *m}(u_1, \dots, u_m) du_1 \dots du_m = \sup_{e_m : \text{mes}(e_m) = t_m} \int_{e_m} \dots \sup_{e_1 : \text{mes}(e_1) = t_1} \int_{e_1} |g(x_1, \dots, x_m)| dx_1 \dots dx_m$$

we have

$$\frac{1}{t_1 \dots t_m} \int_0^{t_m} \dots \int_0^{t_1} \left(\sum_{\bar{l} \leq \bar{k} \leq \bar{n}} \Phi_{\bar{k}}(f) \right)^{*1, \dots, *m}(\bar{y}) dy_1 \dots dy_m \leq \sum_{\bar{l} \leq \bar{k} \leq \bar{n}} \frac{1}{t_1 \dots t_m} \int_0^{t_m} \dots \int_0^{t_1} \Phi_{\bar{k}}^{*1, \dots, *m}(f, \bar{y}) dy_1 \dots dy_m.$$

Therefore, it follows from (1.1) that

$$I_1^\theta \leq \left\| \sum_{\bar{l} \leq \bar{k} \leq \bar{n}} \frac{1}{t_1 \dots t_m} \int_0^{t_m} \dots \int_0^{t_1} \Phi_{\bar{k}}^{*1, \dots, *m}(f, \bar{y}) dy_1 \dots dy_m \right\|_{\bar{p}, \theta}^{*\theta}$$

$$= \int_0^{2\pi} \dots \int_0^{2\pi} \left[\sum_{\bar{l} \leq \bar{k} \leq \bar{n}} \frac{1}{t_1 \dots t_m} \int_0^{t_m} \dots \int_0^{t_1} \Phi_{\bar{k}}^{*1, \dots, *m}(f, \bar{y}) dy_1 \dots dy_m \right]^{\frac{\theta}{s}} \prod_{j=1}^m t_j^{\frac{\theta}{s}-1} dt_1 \dots dt_m. \tag{1.2}$$

Further, we apply Jensen’s inequality [2, p. 125] $\theta/s < 1$ to the sum in (1.2) and Hölder’s inequality for several factors with exponents $s > 1$ [11] to the integral. We obtain

$$\begin{aligned} I_1^\theta &\leq \int_0^{2\pi} \dots \int_0^{2\pi} \left\{ \sum_{\bar{l} \leq \bar{k} \leq \bar{n}} \left[\frac{1}{t_m} \int_0^{t_m} \dots \frac{1}{t_1} \int_0^{t_1} \Phi_{\bar{k}}^{*1, \dots, *m}(f, \bar{y}) dy_1 \dots dy_m \right]^{\frac{\theta}{s}} \right\}^s \prod_{j=1}^m t_j^{\frac{\theta}{s}-1} dt_1 \dots dt_m \\ &\leq \sum_{\bar{l} \leq \bar{k}(1) \leq \bar{n}} \dots \sum_{\bar{l} \leq \bar{k}(s) \leq \bar{n}} \prod_{j=1}^s \left\{ \int_0^{2\pi} \dots \int_0^{2\pi} \left[\frac{1}{t_1 \dots t_m} \int_0^{t_m} \dots \int_0^{t_1} \Phi_{\bar{k}(j)}^{*1, \dots, *m}(f, \bar{y}) dy_1 \dots dy_m \right]^{\frac{\theta}{s}} \prod_{j=1}^m t_j^{\frac{\theta}{s}-1} dt_1 \dots dt_m \right\}^{\frac{1}{s}}. \end{aligned} \tag{1.3}$$

Define

$$\delta_{\bar{k}(j)}(f, \bar{t}) = \left\{ \prod_{j=1}^m t_j^{\frac{1}{s}} \left[\prod_{j=1}^m \frac{1}{t_j} \int_0^{t_m} \dots \int_0^{t_1} \Phi_{\bar{k}(j)}^{*1, \dots, *m}(f, \bar{y}) dy_1 \dots dy_m \right]^{\frac{\theta}{s}} \right\}.$$

Then, using the equality

$$\prod_{j=1}^s \delta_{\nu_j} = \left[\prod_{1 \leq i < j \leq s} \delta_{\nu_i} \delta_{\nu_j} \right]^{\frac{1}{s-1}} \tag{1.4}$$

and Hölder’s inequality with exponents $\tau = \frac{s(s-1)}{2}$ and then with exponent $\beta > 1, \frac{1}{\beta} + \frac{1}{\beta'} = 1$, we obtain

$$\begin{aligned} I_2 &= \int_0^{2\pi} \dots \int_0^{2\pi} \prod_{j=1}^s \left[\frac{1}{t_1 \dots t_m} \int_0^{t_m} \dots \int_0^{t_1} \Phi_{\bar{k}(j)}^{*1, \dots, *m}(f, \bar{y}) dy_1 \dots dy_m \right]^{\frac{\theta}{s}} \prod_{j=1}^m t_j^{\frac{\theta}{s}-1} dt_1 \dots dt_m \\ &= \int_0^{2\pi} \dots \int_0^{2\pi} \prod_{j=1}^s \delta_{\bar{k}(j)}(f, \bar{t}) \frac{dt_1 \dots dt_m}{t_1 \dots t_m} = \int_0^{2\pi} \dots \int_0^{2\pi} \left[\prod_{1 \leq i < j \leq s} \delta_{\bar{k}(i)}(f, \bar{t}) \delta_{\bar{k}(j)}(f, \bar{t}) \right]^{\frac{1}{s-1}} \frac{dt_1 \dots dt_m}{t_1 \dots t_m} \\ &\leq \prod_{1 \leq i < j \leq s} \left[\int_0^{2\pi} \dots \int_0^{2\pi} (\delta_{\bar{k}(i)}(f, \bar{t}) \delta_{\bar{k}(j)}(f, \bar{t}))^{\frac{s}{2}} \frac{dt_1 \dots dt_m}{t_1 \dots t_m} \right]^{\frac{1}{\tau}}. \end{aligned}$$

Therefore, from inequality (1.3), we obtain

$$I_1^\theta \leq \sum_{\bar{l} \leq \bar{k}(1) \leq \bar{n}} \dots \sum_{\bar{l} \leq \bar{k}(s) \leq \bar{n}} \prod_{1 \leq i < j \leq s} \left[\int_0^{2\pi} \dots \int_0^{2\pi} (\delta_{\bar{k}(i)}(f, \bar{t}) \delta_{\bar{k}(j)}(f, \bar{t}))^{\frac{s}{2}} \frac{dt_1 \dots dt_m}{t_1 \dots t_m} \right]^{\frac{1}{\tau}}. \tag{1.5}$$

Define $\rho_{\bar{\mu}, \bar{\nu}} = \int_0^{2\pi} \dots \int_0^{2\pi} (\delta_{\bar{\mu}}(f, \bar{t}) \delta_{\bar{\nu}}(f, \bar{t}))^{\frac{s}{2}} \frac{dt_1 \dots dt_m}{t_1 \dots t_m}$ and $\alpha_j = \frac{\theta + q_j}{\theta}$. Then $\alpha'_j = \frac{\alpha_j}{\alpha_j - 1} = \frac{\theta + q_j}{q_j}, \frac{\alpha_j \theta}{2} = \frac{\theta + q_j}{2} < q_j$, and $\frac{\alpha'_j \theta}{2} < q_j, j = 1, 2, \dots, m$. Applying to this integral Hölder’s

inequality $\left(\frac{1}{\alpha_j} + \frac{1}{\alpha'_j} = 1, j = 1, \dots, m\right)$ and then the inequality of different metrics for trigonometric polynomials (see [12, Lemma B; 13, Theorem 10]), we get

$$\begin{aligned}
 \rho_{\bar{\mu}, \bar{\nu}} &= \int_0^{2\pi} \dots \int_0^{2\pi} \left(\delta_{\bar{\mu}}(f, \bar{t})\right)^{\frac{s}{2}} \prod_{j=1}^m t_j^{-\frac{1}{\alpha}} \left(\delta_{\bar{\nu}}(f, \bar{t})\right)^{\frac{s}{2}} \prod_{j=1}^m t_j^{-\frac{1}{\alpha'}} dt_1 \dots dt_m \\
 &\leq \left[\int_0^{2\pi} \dots \int_0^{2\pi} \left(\delta_{\bar{\mu}}(f, \bar{t})\right)^{\frac{s\alpha}{2}} \frac{dt_1}{t_1} \dots \frac{dt_m}{t_m} \right]^{\frac{1}{\alpha}} \left[\int_0^{2\pi} \dots \int_0^{2\pi} \left(\delta_{\bar{\nu}}(f, \bar{t})\right)^{\frac{s\alpha'}{2}} \frac{dt_1}{t_1} \dots \frac{dt_m}{t_m} \right]^{\frac{1}{\alpha'}} \\
 &= \left[\int_0^{2\pi} \dots \int_0^{2\pi} \left(\prod_{j=1}^m t_j^{-1} \int_0^{t_m} \dots \int_0^{t_1} \Phi_{\bar{\mu}}^{*1, \dots, *m}(f, \bar{y}) dy_1 \dots dy_m\right)^{\frac{\theta\alpha}{2}} t_1^{\frac{\theta\alpha}{2p_1}-1} dt_1 \dots t_m^{\frac{\theta\alpha}{2p_m}-1} dt_m \right]^{\frac{1}{\alpha}} \\
 &\times \left[\int_0^{2\pi} \dots \int_0^{2\pi} \left(\prod_{j=1}^m t_j^{-1} \int_0^{t_m} \dots \int_0^{t_1} \Phi_{\bar{\nu}}^{*1, \dots, *m}(f, \bar{y}) dy_1 \dots dy_m\right)^{\frac{\theta\alpha'}{2}} t_1^{\frac{\theta\alpha'}{2p_1}-1} dt_1 \dots t_m^{\frac{\theta\alpha'}{2p_m}-1} dt_m \right]^{\frac{1}{\alpha'}} \\
 &\leq C \left[\prod_{j=1}^m 2^{\mu_j(\frac{2}{\theta\alpha} - \frac{1}{q_j})} \|\Phi_{\bar{\mu}}(f)\|_{\bar{p}, \bar{q}}^* \right]^{\frac{\theta}{2}} \left[\prod_{j=1}^m 2^{\nu_j(\frac{2}{\theta\alpha'} - \frac{1}{q_j})} \|\Phi_{\bar{\nu}}(f)\|_{\bar{p}, \bar{q}}^* \right]^{\frac{\theta}{2}} \\
 &= C \prod_{j=1}^m 2^{-|\nu_j - \mu_j|(\frac{1}{2} - \frac{1}{\alpha})} \left[\prod_{j=1}^m 2^{\mu_j(\frac{1}{\theta} - \frac{1}{q_j})\theta} (\|\Phi_{\bar{\mu}}(f)\|_{\bar{p}, \bar{q}}^*)^\theta \prod_{j=1}^m 2^{\nu_j(\frac{1}{\theta} - \frac{1}{q_j})\theta} (\|\Phi_{\bar{\nu}}(f)\|_{\bar{p}, \bar{q}}^*)^\theta \right]^{\frac{1}{2}}. \tag{1.6}
 \end{aligned}$$

It follows from (1.5) and (1.6) that

$$\begin{aligned}
 I_1^\theta &\ll \sum_{\bar{l} \leq \bar{k}(1) \leq \bar{n}} \dots \sum_{\bar{l} \leq \bar{k}(s) \leq \bar{n}} \prod_{1 \leq i < j \leq s} \left\{ \left[\prod_{\nu=1}^m 2^{k_\nu(i)(\frac{1}{\theta} - \frac{1}{q_\nu})\theta} (\|\Phi_{\bar{k}(i)}(f)\|_{\bar{p}, \bar{q}}^*)^\theta \prod_{\nu=1}^m 2^{k_\nu(j)(\frac{1}{\theta} - \frac{1}{q_\nu})\theta} (\|\Phi_{\bar{k}(j)}(f)\|_{\bar{p}, \bar{q}}^*)^\theta \right]^{\frac{1}{2}} \right. \\
 &\quad \times \left. \prod_{\nu=1}^m 2^{-|k_\nu(i) - k_\nu(j)|(\frac{1}{2} - \frac{1}{\alpha_\nu})} \right\}^{\frac{1}{\tau}}. \tag{1.7}
 \end{aligned}$$

Now, using formula (1.4), we obtain

$$\begin{aligned}
 &\prod_{1 \leq i < j \leq s} \left\{ \left[\prod_{\nu=1}^m 2^{k_\nu(i)(\frac{1}{\theta} - \frac{1}{q_\nu})\theta} (\|\Phi_{\bar{k}(i)}(f)\|_{\bar{p}, \bar{q}}^*)^\theta \prod_{\nu=1}^m 2^{k_\nu(j)(\frac{1}{\theta} - \frac{1}{q_\nu})\theta} (\|\Phi_{\bar{k}(j)}(f)\|_{\bar{p}, \bar{q}}^*)^\theta \right]^{\frac{1}{2}} \right. \\
 &\quad \times \left. \prod_{\nu=1}^m 2^{-|k_\nu(i) - k_\nu(j)|(\frac{1}{2} - \frac{1}{\alpha_\nu})} \right\}^{\frac{1}{\tau}} \\
 &= \prod_{i=1}^s \left[\prod_{\nu=1}^m 2^{k_\nu(i)(\frac{1}{\theta} - \frac{1}{q_\nu})\theta} (\|\Phi_{\bar{k}(i)}(f)\|_{\bar{p}, \bar{q}}^*)^\theta \right]^{\frac{1}{s}} \prod_{j=1}^s \prod_{\nu=1}^m 2^{-|k_\nu(i) - k_\nu(j)|(\frac{1}{2} - \frac{1}{\alpha_\nu})\frac{1}{2\tau}}.
 \end{aligned}$$

Therefore, applying Hölder's inequality with exponent $s > 1$, we find from (1.7) that

$$I_1^\theta \ll \sum_{\bar{l} \leq \bar{k}(1) \leq \bar{n}} \dots \sum_{\bar{l} \leq \bar{k}(s) \leq \bar{n}} \prod_{i=1}^s \left[\prod_{\nu=1}^m 2^{k_\nu(i)(\frac{1}{\theta} - \frac{1}{q_\nu})\theta} (\|\Phi_{\bar{k}(i)}(f)\|_{\bar{p}, \bar{q}}^*)^\theta \right]^{\frac{1}{s}}$$

$$\begin{aligned} & \times \prod_{j=1}^s \prod_{\nu=1}^m 2^{-|k_\nu(i)-k_\nu(j)|(\frac{1}{2}-\frac{1}{\alpha_\nu})\frac{1}{2r}} \leq C(p, q, \theta) \\ & \times \prod_{i=1}^s \left[\sum_{\bar{l} \leq \bar{k}(1) \leq \bar{n}} \dots \sum_{\bar{l} \leq \bar{k}(s) \leq \bar{n}} \prod_{\nu=1}^m 2^{k_\nu(i)(\frac{1}{\theta}-\frac{1}{q_\nu})\theta} (\|\Phi_{\bar{k}(i)}(f)\|_{\bar{p},q}^*)^\theta \prod_{j=1}^s \prod_{\nu=1}^m 2^{-|k_\nu(i)-k_\nu(j)|(\frac{1}{2}-\frac{1}{\alpha_\nu})\frac{1}{s-1}} \right]^{\frac{1}{s}} \\ & \ll \left[\sum_{k_1=-\infty}^{+\infty} \dots \sum_{k_m=-\infty}^{+\infty} \prod_{\nu=1}^m 2^{-|k_\nu|(\frac{1}{2}-\frac{1}{\alpha_\nu})} \right]^{\frac{1}{s-1}} \sum_{\bar{l} \leq \bar{k} \leq \bar{n}} \prod_{\nu=1}^m 2^{k_\nu(\frac{1}{\theta}-\frac{1}{q_\nu})\theta} (\|\Phi_{\bar{k}}(f)\|_{\bar{p},q}^*)^\theta \\ & = C \sum_{\bar{l} \leq \bar{k} \leq \bar{n}} \prod_{\nu=1}^m 2^{k_\nu(\frac{1}{\theta}-\frac{1}{q_\nu})\theta} (\|\Phi_{\bar{k}}(f)\|_{\bar{p},q}^*)^\theta. \end{aligned}$$

Lemma 2 is proved. □

Lemma 3. *Suppose that $\bar{p} = (p_1, \dots, p_m)$, $\bar{q} = (q_1, \dots, q_m)$, and $1 \leq p_j < +\infty$ and $1 \leq \theta < q_j < +\infty$ for $j = 1, \dots, m$. Then, for any $f \in L_{\bar{p},\bar{q}}^*(I^m)$ and a subset $e \subset e_m$, we have the inequality*

$$I_1 = \left\| \sum_{\bar{l}^e \leq \bar{k}^e \leq \bar{n}^e} \Phi_{\bar{k}^e}(f) \right\|_{\bar{p},\theta}^* \ll \left[\sum_{\bar{l}^e \leq \bar{k}^e \leq \bar{n}^e} \prod_{j \in e} 2^{k_j^e \theta (\frac{1}{\theta} - \frac{1}{q_j})} (\|\Phi_{\bar{k}^e}(f)\|_{\bar{p},\bar{q}}^*)^\theta \right]^{\frac{1}{\theta}}.$$

To prove this lemma, one should repeat the argument given in the proof of Lemma 2 for the sum $\sum_{\bar{l}^e \leq \bar{k}^e \leq \bar{n}^e} \Phi_{\bar{k}^e}(f, \bar{x})$ and use the fact that $\Phi_{\bar{k}^e}(f, \bar{x})$ is a trigonometric polynomial of order 1 in the variables x_j for which $j \notin e$. □

2. MAIN RESULTS

Theorem 1. *Let $1 \leq \theta < p < +\infty$. If $f \in L_p(I^m)$ and*

$$\sum_{k_m=0}^{\infty} \dots \sum_{k_1=0}^{\infty} \prod_{j=1}^m 2^{k_j \theta (\frac{1}{\theta} - \frac{1}{q_j})} (Y_{2^{2^{k_1}-1}, \dots, 2^{2^{k_m}-1}}(f)_p)^\theta < +\infty,$$

then $f \in L_{p,\theta}^*(I^m)$ and

$$\|f\|_{p,\theta}^* \ll \left[\sum_{k_m=0}^{\infty} \dots \sum_{k_1=0}^{\infty} \prod_{j=1}^m 2^{k_j \theta (\frac{1}{\theta} - \frac{1}{q_j})} (Y_{2^{2^{k_1}-1}, \dots, 2^{2^{k_m}-1}}(f)_p)^\theta \right]^{\frac{1}{\theta}}.$$

Proof. Consider the series

$$\sum_{\bar{\nu} \in \mathbb{Z}^m} \varphi_{\bar{\nu}}(\bar{x}), \tag{2.1}$$

where $\varphi_{\bar{\nu}}(\bar{x}) = -\Delta_{\bar{\nu}}(f, \bar{x})$. Using the equality $W_{0,0}(f, \bar{x}) = 0$ and Lemma 5 from [10], we get

$$\left\| f - \sum_{\nu_m=0}^{n_m} \dots \sum_{\nu_1=0}^{n_1} \varphi_{\bar{\nu}}(\bar{x}) \right\|_p \leq C(p, q, m) \sum_{j=1}^m Y_{2^{n_1-1-1}, \dots, 2^{n_j-1-1}-1, 0, 2^{n_{j+1}-1}, \dots, 2^{n_m-1}}(f)_p \rightarrow 0$$

as $n_j \rightarrow +\infty$, $j = 1, \dots, m$. Thus, series (2.1) converges in $L_p(I^m)$ to the function $f \in L_p(I^m)$.

By the conditions of Theorem 1,

$$\sum_{k_m=0}^{\infty} \dots \sum_{k_1=0}^{\infty} \prod_{j=1}^m 2^{k_j \theta (\frac{1}{\theta} - \frac{1}{q_j})} (Y_{2^{2^{k_1}-1}, \dots, 2^{2^{k_m}-1}}(f)_p)^\theta < +\infty. \tag{2.2}$$

By Lemma 2 for $p_j = q_j = p, j = 1, \dots, m$, we obtain

$$\left\| \sum_{k_1=l_1}^{n_1} \dots \sum_{k_m=l_m}^{n_m} \Phi_{\bar{k}}(f) \right\|_{p,\theta}^* \ll \left[\sum_{k_m=l_m}^{n_m} \dots \sum_{k_1=l_1}^{n_1} \prod_{j=1}^m 2^{k_j \theta (\frac{1}{\theta} - \frac{1}{q_j})} \|\Phi_{\bar{k}}(f)\|_p^\theta \right]^{\frac{1}{\theta}}. \tag{2.3}$$

By the definition of the function $\varphi_{\bar{\nu}}(\bar{x})$, the property of the norm, and the monotonicity of the best approximation,

$$\|\Phi_{\bar{k}}(f)\|_p \leq \sum_{j=1}^m \|f - W_{2^{2^{k_1}-1}, \dots, 2^{2^{k_j}-1}, 2^{2^{k_{j+1}}+1}-1, \dots, 2^{2^{k_{m+1}}-1}}\|_p \ll Y_{2^{2^{k_1}-1}, \dots, 2^{2^{k_m}-1}}(f)_p. \tag{2.4}$$

Thus, (2.2) and (2.3) imply that series (2.1) converges in the metric of the space $L_{p,\theta}^*(I^m)$. Hence, $f \in L_{p,\theta}^*(I^m)$. Setting $l_1 = \dots = l_m = 0$ in (2.3) and using (2.4), we get

$$\left\| \sum_{k_1=0}^{n_1} \dots \sum_{k_m=0}^{n_m} \Phi_{\bar{k}}(f) \right\|_{p,\theta}^* \ll \left[\sum_{k_m=0}^{n_m} \dots \sum_{k_1=0}^{n_1} \prod_{j=1}^m 2^{k_j \theta (\frac{1}{\theta} - \frac{1}{q_j})} (Y_{2^{2^{k_1}-1}, \dots, 2^{2^{k_m}-1}}(f)_p)^\theta \right]^{\frac{1}{\theta}}. \tag{2.5}$$

By the property of the norm, Lemma 3, and inequality (2.5), we obtain

$$\begin{aligned} \left\| \sum_{\nu_2=0}^{2^{n_1+1}-1} \dots \sum_{\nu_1=0}^{2^{n_1+1}-1} \varphi_{\bar{\nu}} \right\|_{p,\theta}^* &\leq \left\| \sum_{\nu_2=1}^{2^{n_1+1}-1} \dots \sum_{\nu_1=1}^{2^{n_1+1}-1} \varphi_{\bar{\nu}} \right\|_{p,\theta}^* + \sum_{e \in e_m} \left\| \sum_{\bar{1}^e \leq \bar{\nu}^e \leq 2^{\bar{n}^e+1}} \varphi_{\bar{\nu}^e} \right\|_{p,\theta}^* \\ &= \left\| \sum_{k_1=0}^{n_1} \dots \sum_{k_m=0}^{n_m} \Phi_{\bar{k}}(f) \right\|_{p,\theta}^* + \sum_{e \in e_m} \left\| \sum_{\bar{0} \leq \bar{k}^e \leq \bar{n}^e} \Phi_{\bar{k}^e}(f) \right\|_{p,\theta}^* \\ &\ll \left\{ \left[\sum_{k_m=0}^{n_m} \dots \sum_{k_1=0}^{n_1} \prod_{j=1}^m 2^{k_j \theta (\frac{1}{\theta} - \frac{1}{q_j})} (Y_{2^{2^{k_1}-1}, \dots, 2^{2^{k_m}-1}}(f)_p)^\theta \right]^{\frac{1}{\theta}} \right. \\ &\quad \left. + \left[\sum_{e \in e_m} \sum_{\bar{0} \leq \bar{k}^e \leq \bar{n}^e} \prod_{j \in e} 2^{k_j \theta (\frac{1}{\theta} - \frac{1}{q_j})} (\|\Phi_{\bar{k}^e}(f)\|_p)^\theta \right]^{\frac{1}{\theta}} \right\}. \end{aligned} \tag{2.6}$$

By the property the best ‘‘angle’’ approximation, we have the inequalities

$$\|\Phi_{\bar{k}^e}(f)\|_p = \left\| \sum_{2^{\bar{k}^e} \leq \bar{\nu}^e \leq 2^{\bar{k}^e+1}} \varphi_{\bar{\nu}^e} \right\|_p \ll Y_{(2^{2^{\bar{k}^e}-1})^e}(f)_p. \tag{2.7}$$

Combining (2.7) and (2.6), we obtain

$$\begin{aligned} \left\| \sum_{\nu_2=0}^{2^{n_1+1}-1} \dots \sum_{\nu_1=0}^{2^{n_1+1}-1} \varphi_{\bar{\nu}} \right\|_{p,\theta}^* &\ll \left[\sum_{k_m=0}^{n_m} \dots \sum_{k_1=0}^{n_1} \prod_{j=1}^m 2^{k_j \theta (\frac{1}{\theta} - \frac{1}{q_j})} (Y_{2^{2^{k_1}-1}, \dots, 2^{2^{k_m}-1}}(f)_p)^\theta \right]^{\frac{1}{\theta}} \\ &\quad + \left[\sum_{e \in e_m} \sum_{\bar{0} \leq \bar{k}^e \leq \bar{n}^e} \prod_{j \in e} 2^{k_j \theta (\frac{1}{\theta} - \frac{1}{q_j})} (Y_{(2^{2^{\bar{k}^e}-1})^e}(f)_{\bar{p},\bar{q}})^\theta \right]^{\frac{1}{\theta}} \end{aligned}$$

$$\ll \left[\sum_{k_m=0}^{\infty} \dots \sum_{k_1=0}^m \prod_{j=1}^m 2^{k_j \theta (\frac{1}{\theta} - \frac{1}{q_j})} (Y_{2^{2^{k_1}-1}, \dots, 2^{2^{k_m}-1}}(f)_p)^\theta \right]^{\frac{1}{\theta}}. \tag{2.8}$$

Since series (2.1) converges to f in the metric of the space $L_{\bar{p}, \theta}^*(I^m)$, we pass to the limit in (2.8) as $n_1 \rightarrow +\infty, \dots, n_m \rightarrow +\infty$ and get

$$\|f\|_{p, \theta}^* \ll \left[\sum_{k_m=0}^{\infty} \dots \sum_{k_1=0}^m \prod_{j=1}^m 2^{k_j \theta (\frac{1}{\theta} - \frac{1}{q_j})} (Y_{2^{2^{k_1}-1}, \dots, 2^{2^{k_m}-1}}(f)_p)^\theta \right]^{\frac{1}{\theta}}.$$

Theorem 1 is proved. □

Theorem 2. *Suppose that $1 \leq \theta \leq \min\{2, p\}$ and $1 < p < +\infty$. If $f \in L_p(I^m)$ and*

$$\sum_{s_m=1}^{\infty} \dots \sum_{s_1=1}^m \prod_{j=1}^m s_j^{1-\frac{\theta}{p}} (\|\delta_{\bar{s}}(f)\|_p^*)^\theta < +\infty,$$

then $f \in L_{p, \theta}^*(I^m)$ and

$$\|f\|_{p, \theta}^* \ll \left[\sum_{s_m=1}^{\infty} \dots \sum_{s_1=1}^m \prod_{j=1}^m s_j^{1-\frac{\theta}{p}} (\|\delta_{\bar{s}}(f)\|_p)^\theta \right]^{\frac{1}{\theta}}.$$

Proof. By the definition of the best “angle” approximation and the property of the norm,

$$Y_{2^{2^{k_1}-1}, \dots, 2^{2^{k_m}-1}}(f)_p \leq \|f - U_{2^{2^{k_1}-1}, \dots, 2^{2^{k_m}-1}}\|_p \leq \sum_{\nu_m=k_m}^{\infty} \dots \sum_{\nu_1=k_1}^{\infty} \left\| \sum_{s_m=2^{\nu_m+1}}^{2^{\nu_m+1}} \dots \sum_{s_1=2^{\nu_1+1}}^{2^{\nu_1+1}} \delta_{\bar{s}}(f) \right\|_p.$$

Hence, applying Lemma 2.2 from [14], we obtain

$$\begin{aligned} & \sum_{k_m=0}^{\infty} \dots \sum_{k_1=0}^m \prod_{j=1}^m 2^{k_j \theta (\frac{1}{\theta} - \frac{1}{p})} (Y_{2^{2^{k_1}-1}, \dots, 2^{2^{k_m}-1}}(f)_p)^\theta \\ & \ll \sum_{k_m=0}^{\infty} \dots \sum_{k_1=0}^m \prod_{j=1}^m 2^{k_j \theta (\frac{1}{\theta} - \frac{1}{p})} \left(\left\| \sum_{s_m=2^{k_m+1}}^{2^{k_m+1}} \dots \sum_{s_1=2^{k_1+1}}^{2^{k_1+1}} \delta_{\bar{s}}(f) \right\|_p \right)^\theta. \end{aligned} \tag{2.9}$$

By the Littlewood–Paley theorem (see [2, p. 55]), for $1 < p < +\infty$, we have

$$\left\| \sum_{s_m=2^{k_m+1}}^{2^{k_m+1}} \dots \sum_{s_1=2^{k_1+1}}^{2^{k_1+1}} \delta_{\bar{s}}(f) \right\|_p \asymp \left\| \left(\sum_{s_m=2^{k_m+1}}^{2^{k_m+1}} \dots \sum_{s_1=2^{k_1+1}}^{2^{k_1+1}} |\delta_{\bar{s}}(f)|^2 \right)^{\frac{1}{2}} \right\|_p.$$

Therefore, if $1 < p \leq 2$, by Jensen’s inequality (see [2, p. 125]), we have

$$\left\| \sum_{s_m=2^{k_m+1}}^{2^{k_m+1}} \dots \sum_{s_1=2^{k_1+1}}^{2^{k_1+1}} \delta_{\bar{s}}(f) \right\|_p \ll \left(\sum_{s_m=2^{k_m+1}}^{2^{k_m+1}} \dots \sum_{s_1=2^{k_1+1}}^{2^{k_1+1}} \|\delta_{\bar{s}}(f)\|_p^p \right)^{\frac{1}{p}}.$$

Consequently,

$$\sum_{k_m=0}^{\infty} \dots \sum_{k_1=0}^m \prod_{j=1}^m 2^{k_j \theta (\frac{1}{\theta} - \frac{1}{p})} \left\| \sum_{s_m=2^{k_m+1}}^{2^{k_m+1}} \dots \sum_{s_1=2^{k_1+1}}^{2^{k_1+1}} \delta_{\bar{s}}(f) \right\|_p^\theta$$

$$\ll \sum_{k_m=0}^{\infty} \dots \sum_{k_1=0}^{\infty} \prod_{j=1}^m 2^{k_j \theta (\frac{1}{\theta} - \frac{1}{p})} \left(\sum_{s_m=2^{k_m+1}}^{2^{k_m+1}} \dots \sum_{s_1=2^{k_1+1}}^{2^{k_1+1}} \|\delta_{\bar{s}}(f)\|_p^p \right)^{\frac{\theta}{p}}.$$

By the hypothesis of the theorem, we have $\theta < p$. Then, by Jensen's inequality (see [2, p. 125]),

$$\begin{aligned} & \sum_{k_m=0}^{\infty} \dots \sum_{k_1=0}^{\infty} \prod_{j=1}^m 2^{k_j \theta (\frac{1}{\theta} - \frac{1}{p})} \left(\sum_{s_m=2^{k_m+1}}^{2^{k_m+1}} \dots \sum_{s_1=2^{k_1+1}}^{2^{k_1+1}} \|\delta_{\bar{s}}(f)\|_p^p \right)^{\frac{\theta}{p}} \\ & \leq \sum_{k_m=0}^{\infty} \dots \sum_{k_1=0}^{\infty} \prod_{j=1}^m 2^{k_j \theta (\frac{1}{\theta} - \frac{1}{p})} \sum_{s_m=2^{k_m+1}}^{2^{k_m+1}} \dots \sum_{s_1=2^{k_1+1}}^{2^{k_1+1}} \|\delta_{\bar{s}}(f)\|_p^{\theta} \leq \sum_{s_m=1}^{\infty} \dots \sum_{s_1=1}^{\infty} \prod_{j=1}^m s_j^{\theta (\frac{1}{\theta} - \frac{1}{p})} \|\delta_{\bar{s}}(f)\|_p^{\theta}. \end{aligned}$$

Thus, for $1 < p \leq 2$,

$$\sum_{k_m=0}^{\infty} \dots \sum_{k_1=0}^{\infty} \prod_{j=1}^m 2^{k_j \theta (\frac{1}{\theta} - \frac{1}{p})} \left\| \sum_{s_m=2^{k_m+1}}^{2^{k_m+1}} \dots \sum_{s_1=2^{k_1+1}}^{2^{k_1+1}} \delta_{\bar{s}}(f) \right\|_p^{\theta} \ll \sum_{s_m=1}^{\infty} \dots \sum_{s_1=1}^{\infty} \prod_{j=1}^m s_j^{\theta (\frac{1}{\theta} - \frac{1}{p})} \|\delta_{\bar{s}}(f)\|_p^{\theta}. \tag{2.10}$$

Now, if $2 < p < +\infty$, we find by the property of the norm that

$$\left\| \left(\sum_{s_m=2^{k_m+1}}^{2^{k_m+1}} \dots \sum_{s_1=2^{k_1+1}}^{2^{k_1+1}} |\delta_{\bar{s}}(f)|^2 \right)^{\frac{1}{2}} \right\|_p \leq \left(\sum_{s_m=2^{k_m+1}}^{2^{k_m+1}} \dots \sum_{s_1=2^{k_1+1}}^{2^{k_1+1}} \|\delta_{\bar{s}}(f)\|_p^2 \right)^{\frac{1}{2}}.$$

Therefore, if $\theta \leq 2$, using Jensen's inequality (see [2, p. 125]), we obtain

$$\begin{aligned} & \sum_{k_m=0}^{\infty} \dots \sum_{k_1=0}^{\infty} \prod_{j=1}^m 2^{k_j \theta (\frac{1}{\theta} - \frac{1}{p})} \left(\left\| \sum_{s_m=2^{k_m+1}}^{2^{k_m+1}} \dots \sum_{s_1=2^{k_1+1}}^{2^{k_1+1}} \delta_{\bar{s}}(f) \right\|_p \right)^{\theta} \\ & \ll \sum_{k_m=0}^{\infty} \dots \sum_{k_1=0}^{\infty} \prod_{j=1}^m 2^{k_j \theta (\frac{1}{\theta} - \frac{1}{p})} \left(\sum_{s_m=2^{k_m+1}}^{2^{k_m+1}} \dots \sum_{s_1=2^{k_1+1}}^{2^{k_1+1}} \|\delta_{\bar{s}}(f)\|_p^2 \right)^{\frac{\theta}{2}} \\ & \ll \sum_{k_m=0}^{\infty} \dots \sum_{k_1=0}^{\infty} \prod_{j=1}^m 2^{k_j \theta (\frac{1}{\theta} - \frac{1}{p})} \sum_{s_m=2^{k_m+1}}^{2^{k_m+1}} \dots \sum_{s_1=2^{k_1+1}}^{2^{k_1+1}} \|\delta_{\bar{s}}(f)\|_p^{\theta} \ll \sum_{s_m=1}^{\infty} \dots \sum_{s_1=1}^{\infty} \prod_{j=1}^m s_j^{\theta (\frac{1}{\theta} - \frac{1}{p})} \|\delta_{\bar{s}}(f)\|_p^{\theta}. \end{aligned} \tag{2.11}$$

The assertion of Theorem 2 follows from inequalities (2.9)–(2.11) and Theorem 1. \square

Corollary. Assume that $1 < \theta \leq \min\{2, p\}$ and $p \leq q_j$ for $j = 1, \dots, m$. If $f \in L_{p, \bar{q}}^*(I^m)$ and

$$\sum_{s_m=1}^{\infty} \dots \sum_{s_1=1}^{\infty} \prod_{j=1}^m s_j^{\theta (\frac{1}{\theta} - \frac{1}{q_j})} (\|\delta_{\bar{s}}(f)\|_{p, \bar{q}}^*)^{\theta} < \infty,$$

then $f \in L_{p, \theta}^*(I^m)$ and

$$\|f\|_{p, \theta}^* \ll \left[\sum_{s_m=1}^{\infty} \dots \sum_{s_1=1}^{\infty} \prod_{j=1}^m s_j^{\theta (\frac{1}{\theta} - \frac{1}{q_j})} (\|\delta_{\bar{s}}(f)\|_{p, \bar{q}}^*)^{\theta} \right]^{\frac{1}{\theta}}.$$

Proof. Since $p \leq q_j$ for $j = 1, \dots, m$, we have $L_{p,p}^*(I^m) = L_p(I^m) \subset L_{p,q}^*(I^m)$. Using the inequality of different metrics for trigonometric polynomials (see [12, Lemma B; 13, Theorem 10]), we get

$$\sum_{s_m=1}^{\infty} \dots \sum_{s_1=1}^{\infty} \prod_{j=1}^m s_j^{\theta(\frac{1}{\theta}-\frac{1}{p})} \|\delta_{\bar{s}}(f)\|_p^{\theta} \ll \sum_{s_m=1}^{\infty} \dots \sum_{s_1=1}^{\infty} \prod_{j=1}^m s_j^{\theta(\frac{1}{\theta}-\frac{1}{q_j})} (\|\delta_{\bar{s}}(f)\|_{p,\bar{q}}^*)^{\theta}.$$

Hence, $f \in L_p(I^m)$ and, applying Theorem 2 to this function, we obtain the assertion of the corollary. \square

Theorem 3. Suppose that $\bar{\lambda} = (\lambda_1, \dots, \lambda_m)$, $1 < \lambda_j < \theta < +\infty$ for $j = 1, \dots, m$, and $\max\{p, 2\} \leq \theta < +\infty$. If $f_0 \in L_{\bar{p},\theta}^*(I^m)$ and

$$f_0(\bar{x}) \sim \sum_{\bar{s} \in \mathbb{Z}_+^m} a_{\bar{s}} \sum_{\bar{k} \in \rho(\bar{s})} \prod_{j=1}^m (n_j - 2^{s_j-1} + 1)^{\frac{1}{p}-1} e^{i\langle \bar{k}, \bar{x} \rangle},$$

then

$$\|f_0\|_{p,\theta}^* \gg \left[\sum_{s_m=1}^{\infty} \dots \sum_{s_1=1}^{\infty} \prod_{j=1}^m s_j^{(\frac{1}{\theta}-\frac{1}{\lambda_j})\theta} (\|\delta_{\bar{s}}(f_0)\|_{p,\bar{\lambda}}^*)^{\theta} \right]^{\frac{1}{\theta}}.$$

Proof. Blozinski [1] proved that

$$\|f\|_{p,\theta}^* \gg \sup_{\substack{g \in L_{p',\theta'}^* \\ \|\theta\|_{p',\theta'}^* \leq 1}} \left| \int_{I^m} f(\bar{x}) g(\bar{x}) d\bar{x} \right|, \tag{2.12}$$

where $p^{-1} + (p')^{-1} = 1$, $\theta^{-1} + (\theta')^{-1} = 1$, and $1 < p, \theta < +\infty$.

Consider the function $g_{\nu}(\bar{x}) = \sum_{s_m=1}^{\nu-1} \dots \sum_{s_1=1}^{\nu-1} b_{\bar{s},\nu} \sum_{\bar{k} \in \rho(\bar{s})} (n_j - 2^{s_j-1} + 1)^{\frac{1}{p'}-1} e^{i\langle \bar{k}, \bar{x} \rangle}$, where

$$b_{\bar{s},\nu} = \left\| \left\{ \prod_{j=1}^m s_j^{\frac{1}{\theta'}-\frac{1}{p}} \|\delta_{\bar{s}}(f_0)\|_{p,\bar{\lambda}} \right\}_{\bar{s}=\bar{1}}^{\bar{\nu}-\bar{1}} \right\|_{l_{\theta}}^{-\frac{\theta}{\theta'}} |a_{\bar{s}}|^{\theta-1} \operatorname{sgn} a_{\bar{s}}.$$

By continuity, we have $g_{\nu} \in L_{p',\theta'}^*(I^m)$. Now, let us show that $\|g_{\nu}\|_{p',\theta'}^* \leq C_0$, where C_0 is some positive number. By the definition of g_{ν} , in view of the relation

$$\left\| \sum_{\bar{k} \in \rho(\bar{s})} \prod_{j=1}^m (n_j - 2^{s_j-1} + 1)^{\frac{1}{p}-1} e^{i\langle \bar{k}, \bar{x} \rangle} \right\|_{p,\bar{\lambda}} \asymp \prod_{j=1}^m s_j^{\frac{1}{\lambda_j}} \tag{2.13}$$

for $1 < p, \lambda_j < +\infty$, $j = 1, \dots, m$, we have

$$\begin{aligned} & \left\| \left\{ \prod_{j=1}^m s_j^{\frac{1}{\theta'}-\frac{1}{\lambda_j}} \|\delta_{\bar{s}}(g_{\nu})\|_{p',\bar{\lambda}'}^* \right\}_{\bar{s}=\bar{1}}^{\bar{\nu}-\bar{1}} \right\|_{l_{\theta'}} \ll \left\| \left\{ \prod_{j=1}^m s_j^{\frac{1}{\theta'}-\frac{1}{p}} \|\delta_{\bar{s}}(f_0)\|_{p,\bar{\lambda}}^* \right\}_{\bar{s}=\bar{1}}^{\bar{\nu}-\bar{1}} \right\|_{l_{\theta}}^{-\frac{\theta}{\theta'}} \\ & \times \left[\sum_{s_m=1}^{\nu-1} \dots \sum_{s_1=1}^{\nu-1} \prod_{j=1}^m s_j^{1-\frac{\theta}{\lambda_j}} (\|\delta_{\bar{s}}(f_0)\|_{p,\bar{\lambda}}^*)^{\theta} \right]^{\frac{1}{\theta'}} \leq C_0, \end{aligned}$$

where $\lambda_j' = \lambda_j/(\lambda_j - 1)$ for $j = 1, \dots, m$.

Thus, $\left\| \left\{ \prod_{j=1}^m s_j^{\frac{1}{\theta'} - \frac{1}{\lambda_j}} \|\delta_{\bar{s}}(g_\nu)\|_{p', \bar{\lambda}} \right\}_{\bar{s}=1}^{\bar{\nu}-1} \right\|_{l_{\theta'}} \leq C_0$. Hence, by the corollary of Theorem 2, we have

$$G_\nu(\bar{x}) = C_0^{-1} g_\nu(\bar{x}) \in L_{p', \theta'}^*(I^m) \text{ and } \|G_\nu\|_{p', \theta'}^* \leq 1.$$

By (2.12) and the orthogonality of the system $\{e^{i\langle \bar{n}, \bar{x} \rangle}\}$,

$$\|f_0\|_{p, \theta}^* \gg \int_{I^m} f_0(\bar{x}) g_\nu(\bar{x}) d\bar{x} = \left\| \left\{ \prod_{j=1}^m s_j^{\frac{1}{\theta} - \frac{1}{\lambda_j}} \|\delta_{\bar{s}}(f_0)\|_{p, \bar{\lambda}}^* \right\}_{\bar{s}=1}^{\bar{\nu}-1} \right\|_{l_\theta}^{-\frac{\theta}{\theta'}} \times \sum_{s_m=1}^{\nu-1} \dots \sum_{s_1=1}^{\nu-1} |a_{\bar{s}}|^\theta \prod_{j=1}^m s_j. \quad (2.14)$$

Further, by (2.13), $|a_{\bar{s}}(f_0)|^\theta \prod_{j=1}^m s_j \asymp \prod_{j=1}^m s_j^{1 - \frac{\theta}{\lambda_j}} (\|\delta_{\bar{s}}(f)\|_{p, \bar{\lambda}}^*)^\theta$. Therefore, by (2.14),

$$\|f_0\|_{p, \theta}^* \gg \left\| \left\{ \prod_{j=1}^m s_j^{\frac{1}{\theta} - \frac{1}{\lambda_j}} \|\delta_{\bar{s}}(f_0)\|_{p, \bar{\lambda}}^* \right\}_{\bar{s}=1}^{\bar{\nu}-1} \right\|_{l_\theta}^{-\frac{\theta}{\theta'}} \times \sum_{s_m=1}^{\nu-1} \dots \sum_{s_1=1}^{\nu-1} \prod_{j=1}^m s_j^{1 - \frac{\theta}{\lambda_j}} \|\delta_{\bar{s}}(f)\|_{p, \bar{\lambda}}^\theta.$$

Passing to the limit as $\nu \rightarrow +\infty$ in this inequality, we obtain

$$\|f_0\|_{p, \theta}^* \gg \left\| \left\{ \prod_{j=1}^m s_j^{\frac{1}{\theta} - \frac{1}{\lambda_j}} \|\delta_{\bar{s}}(f_0)\|_{p, \bar{\lambda}}^* \right\}_{\bar{s} \in \mathbb{Z}_+^m} \right\|_{l_\theta}.$$

Theorem 3 is proved. □

Theorem 4. Suppose that $1 < p < +\infty$, $0 < r_1 = r_2 = \dots = r_\nu < r_{\nu+1} \leq \dots \leq r_m$, and $\gamma_j = r_j/r_1$ for $j = 1, 2, \dots, m$.

1. Let $\theta \leq \tau_j$ for $j = 1, \dots, m$. If $1 < \theta \leq \min\{p, 2\}$ and $p \leq q_j$ for $j = 1, \dots, m$, then

$$E_n^{(\bar{\gamma})}(\overset{\circ}{S}_{p, \bar{q}, \bar{\tau}} B)_{p, \theta} \ll 2^{-nr_1} n^{\sum_{j=1}^m (1/\theta - 1/q_j) + \sum_{j=2}^m (1/\theta - 1/\tau_j)}.$$

If $1 < p \leq q_j < \infty$ for $j = 1, \dots, m$ and $\max\{p, 2\} \leq \theta < \infty$, then

$$E_n^{(\bar{\gamma})}(\overset{\circ}{S}_{p, \bar{q}, \bar{\tau}} B)_{p, \theta} \gg 2^{-nr_1} n^{\sum_{j=1}^m (1/\theta - 1/q_j) + \sum_{j=2}^m (1/\theta - 1/\tau_j)}.$$

2. Let $1 \leq \tau_j \leq \theta$ for $j = 1, \dots, m$. If $1 < \theta \leq \min\{p, 2\}$ and $p \leq q_j$ for $j = 1, \dots, m$, then

$$E_n^{(\bar{\gamma})}(\overset{\circ}{S}_{p, \bar{q}, \bar{\tau}} B)_{p, \theta} \ll \sup_{\bar{s} \in Y^m(\bar{\gamma}, n)} \prod_{j=1}^m 2^{-s_j r_j} s_j^{\frac{1}{\theta} - \frac{1}{q_j}}.$$

If $1 < p_j < \infty$ and $1 < \theta_j < q_j < \infty$ for $j = 1, \dots, m$, then

$$E_n^{(\bar{\gamma})}(\overset{\circ}{S}_{p, \bar{q}, \bar{\tau}} B)_{\bar{p}, \bar{\theta}} \gg \sup_{\bar{s} \in Y^m(\bar{\gamma}, n)} \prod_{j=1}^m 2^{-s_j r_j} s_j^{\frac{1}{\theta_j} - \frac{1}{q_j}}.$$

Proof. Let $\theta < \tau_j$ for $j = 1, \dots, m$. Applying Hölder's inequality, we see that the assumptions of the corollary hold for $f \in \overset{\circ}{S}_{p, \bar{q}, \bar{\tau}} B$. Consequently, this function belongs to the space $L_{p, \theta}^*(I^m)$. Therefore, applying the corollary of Theorem 1 to the function $f - S_n^{(\bar{\gamma})}(f) \in L_{p, \theta}^*(I^m)$, we obtain

$$\|f - S_n^{(\bar{\gamma})}(f)\|_{p, \theta}^* \ll \left\| \left\{ \prod_{j=1}^m s_j^{\frac{1}{\theta} - \frac{1}{p}} \|\delta_{\bar{s}}(f)\|_{p, \bar{q}}^* \right\}_{\bar{s} \in Y^m(\bar{\gamma}, n)} \right\|_{l_\theta}. \quad (2.15)$$

Further, applying Hölder's inequality ($\beta_j = \tau_j/\theta$, $\beta_j^{-1} + (\beta'_j)^{-1} = 1$) and Lemma 1 from [12], for $\beta_j = \theta^{-1} - p^{-1}L$, we have

$$\begin{aligned} \left\| \left\{ \prod_{j=1}^m s_j^{\frac{1}{\theta} - \frac{1}{p}} \|\delta_{\bar{s}}(f)\|_{p,\bar{q}}^* \right\}_{\bar{s} \in Y^m(\bar{\gamma}, n)} \right\|_{l_\theta} &\leq \left\| \left\{ \prod_{j=1}^m 2^{s_j r_j} \|\delta_{\bar{s}}(f)\|_{p,\bar{q}}^* \right\}_{\bar{s} \in \mathbb{Z}_+^m} \right\|_{l_{\bar{\tau}}} \times \left\| \left\{ \prod_{j=1}^m 2^{-s_j r_j} s_j^{\frac{1}{\theta} - \frac{1}{p}} \right\}_{\bar{s} \in Y^m(\bar{\gamma}, n)} \right\|_{l_{\bar{\tau}}} \\ &\ll \left\| \left\{ \prod_{j=1}^m 2^{s_j r_j} \|\delta_{\bar{s}}(f)\|_{p,\bar{q}}^* \right\}_{\bar{s} \in \mathbb{Z}_+^m} \right\|_{l_{\bar{\tau}}} \times 2^{-nr_1} n^{\sum_{j=1}^m (1/\theta - 1/q_j) + \sum_{j=2}^m 1/\epsilon_j}, \end{aligned} \quad (2.16)$$

where $\epsilon_j = \theta\beta'_j$ and $\epsilon_j^{-1} = \beta'_j/\theta = \theta^{-1} - \tau_j^{-1}$ for $j = 1, \dots, m$.

Now, it follows from (2.15) and (2.16) that $\|f - S_n^{(\bar{\gamma})}(f)\|_{p,\theta}^* \ll 2^{-nr_1} n^{\sum_{j=1}^m (1/\theta - 1/q_j) + \sum_{j=2}^m (1/\theta - 1/\tau_j)}$ in the case $\theta < \tau_j$, $j = 1, 2, \dots, m$, for $f \in \overset{\circ}{S}_{p,\bar{q},\bar{\tau}}^{\bar{\gamma}} B$. Hence, in the case $\theta < \tau_j$, $j = 1, 2, \dots, m$,

$$E_n^{(\bar{\gamma})}(\overset{\circ}{S}_{p,\bar{q},\bar{\tau}}^{\bar{\gamma}} B)_{p,\theta} \ll 2^{-nr_1} n^{\sum_{j=1}^m (1/\theta - 1/q_j) + \sum_{j=2}^m (1/\theta - 1/\tau_j)}.$$

Let us prove the lower bound. Consider the function

$$f_0(\bar{x}) = n^{-\sum_{j=2}^m \frac{1}{\tau_j}} \sum_{\bar{s} \in \mathcal{X}(n)} \prod_{j=1}^m 2^{-s_j r_j} s_j^{-\frac{1}{q_j}} \sum_{\bar{k} \in \rho(\bar{s})} \prod_{j=1}^m (n_j - 2^{s_j - 1} + 1)^{\frac{1}{p} - 1} e^{i\langle \bar{k}, \bar{x} \rangle}.$$

By continuity, $f_0 \in L_{p,\bar{q}}^*(I^m)$. Further, in view of relation (2.13) and [8, Lemma 2],

$$\left\| \left\{ 2^{\langle \bar{s}, \bar{\tau} \rangle} \|\delta_{\bar{s}}(f_0)\|_{p,\bar{q}}^* \right\}_{\bar{s} \in \mathcal{X}(n)} \right\|_{l_{\bar{\tau}}} \ll n^{-\sum_{j=2}^m \frac{1}{\tau_j}} \left\| \left\{ \chi_{\mathcal{X}(n)}(\bar{s}) \right\}_{\bar{s} \in \mathcal{X}(n)} \right\|_{l_{\bar{\tau}}} \ll n^{-\sum_{j=2}^m 1/\tau_j} n^{\sum_{j=2}^m 1/\tau_j} = C_0.$$

Thus, $F_0 = C_0^{-1} f_0 \in \overset{\circ}{S}_{p,\bar{q},\bar{\tau}}^{\bar{\gamma}} B$. By the definition of the function f_0 and of the best approximation, $E_n^{(\bar{\gamma})}(f)_{p,\theta} = \|f_0\|_{p,\theta}^*$. Choose numbers $\lambda_j \in [1, \theta]$ for $j = 1, 2, \dots, m$. Now, applying Theorem 3 and Lemma 1, we get

$$\begin{aligned} E_n^{(\bar{\gamma})}(f)_{p,\theta} = \|f_0\|_{p,\theta}^* &>> \left\| \left\{ \prod_{j=1}^m s_j^{\frac{1}{\theta} - \frac{1}{\lambda_j}} \|\delta_{\bar{s}}(f_0)\|_{p,\bar{\lambda}}^* \right\}_{\bar{s} \in \mathbb{Z}_+^m} \right\|_{l_\theta} \\ &>> n^{-\sum_{j=2}^m 1/\tau_j} \left\| \left\{ \prod_{j=1}^m 2^{-s_j r_j} s_j^{\frac{1}{\theta} - \frac{1}{q_j}} \right\}_{\bar{s} \in \mathcal{X}(n)} \right\|_{l_\theta} >> 2^{-nr_1} n^{\sum_{j=1}^m (1/\theta - 1/q_j) + \sum_{j=2}^m (1/\theta - 1/\tau_j)}. \end{aligned}$$

This proves the lower bound in the case $\theta < \tau_j$, $j = 1, 2, \dots, m$.

Let us prove the second assertion. Let $\tau_j < \theta$, $j = 1, 2, \dots, m$. Then, by Jensen's inequality,

$$\begin{aligned} \left\| \left\{ \prod_{j=1}^m s_j^{\frac{1}{\theta} - \frac{1}{p}} \|\delta_{\bar{s}}(f)\|_{p,\bar{q}}^* \right\}_{\bar{s} \in Y^m(\bar{\gamma}, n)} \right\|_{l_\theta} &\leq \left\| \left\{ \prod_{j=1}^m s_j^{\frac{1}{\theta} - \frac{1}{q_j}} \|\delta_{\bar{s}}(f)\|_{p,\bar{q}} \right\}_{\bar{s} \in Y^m(\bar{\gamma}, n)} \right\|_{l_{\bar{\tau}}} \\ &\leq \left\| \left\{ \prod_{j=1}^m 2^{s_j r_j} \|\delta_{\bar{s}}(f)\|_{p,\bar{q}} \right\}_{\bar{s} \in \mathbb{Z}_+^m} \right\|_{l_{\bar{\tau}}} \sup_{\bar{s} \in Y^m(\bar{\gamma}, n)} \prod_{j=1}^m 2^{-s_j r_j} s_j^{\frac{1}{\theta} - \frac{1}{q_j}}. \end{aligned}$$

Therefore, it follows from inequality (2.16) that

$$E_n^{(\bar{\gamma})}(\overset{\circ}{S}_{p,\bar{q},\bar{\tau}}^{\bar{\tau}}B)_{p,\theta} \ll \sup_{\bar{s} \in Y^m(\bar{\gamma},n)} \prod_{j=1}^m 2^{-s_j r_j} s_j^{\frac{1}{\theta} - \frac{1}{p}}$$

in the case $\tau_j \leq \theta$, $j = 1, 2, \dots, m$. The upper bound is proved.

Let us prove the lower bound. Let $\tau_j \leq \theta$, $j = 1, 2, \dots, m$, and $\tilde{s} \in Y^m(\bar{\gamma}, n)$. Define

$$f_1(\bar{x}) = \prod_{j=1}^m 2^{-\tilde{s}_j r_j} \tilde{s}_j^{-\frac{1}{q_j}} \sum_{\bar{k} \in \rho(\tilde{s})} \prod_{j=1}^m (n_j - 2^{\tilde{s}_j - 1} + 1)^{\frac{1}{p_j} - 1} e^{i\langle \bar{k}, \bar{x} \rangle}.$$

By relation (2.13), we have $\|\delta_{\tilde{s}}(f_1)\|_{\bar{p},\bar{q}}^* \asymp \prod_{j=1}^m 2^{-\tilde{s}_j r_j}$. If $\bar{s} \neq \tilde{s}$, then $\|\delta_{\bar{s}}(f_1)\|_{\bar{p},\bar{q}}^* = 0$. Therefore, $\|\{2^{\langle \bar{s}, \bar{\tau} \rangle} \|\delta_{\bar{s}}(f_1)\|_{\bar{p},\bar{q}}^*\}_{\bar{s} \in \mathbb{Z}_+^m}\|_{l_{\bar{\tau}}} \leq C_1$ and $C_1^{-1} f_1 \in S_{\bar{p},\bar{q},\bar{\tau}}^{\bar{\tau}}B$. Since $\langle \tilde{s}, \bar{\gamma} \rangle \geq n$, $E_n^{(\bar{\gamma})}(f_1)_{\bar{p},\theta} = \|f_1\|_{\bar{p},\theta}^*$.

Therefore, in view of relation (2.13),

$$E_n^{(\bar{\gamma})}(f_1)_{\bar{p},\theta} = \|f_1\|_{\bar{p},\theta}^* \gg \prod_{j=1}^m 2^{-\tilde{s}_j r_j} \tilde{s}_j^{\frac{1}{\theta} - \frac{1}{q_j}} \quad \forall \tilde{s} \in Y^m(\bar{\gamma}, n).$$

Hence,

$$E_n^{(\bar{\gamma})}(\overset{\circ}{S}_{\bar{p},\bar{q},\bar{\tau}}^{\bar{\tau}}B)_{\bar{p},\theta} \gg \sup_{\bar{s} \in Y^m(\bar{\gamma},n)} \prod_{j=1}^m 2^{-s_j r_j} s_j^{\frac{1}{\theta} - \frac{1}{q_j}}.$$

Theorem 4 is proved. □

Remark. If $p = \theta = 2$, then $\min\{p, 2\} = \max\{p, 2\} = 2$ and $L_{p,\theta}^*(I^m) = L_2(I^m)$. Therefore, by the first assertion of Theorem 4,

$$E_n^{(\bar{\gamma})}(\overset{\circ}{S}_{2,\bar{q},\bar{\tau}}^{\bar{\tau}}B)_2 \asymp 2^{-nr_1} \prod_{j=1}^m (1/2 - 1/q_j) + \sum_{j=2}^m (1/2 - 1/\tau_j)$$

for $2 < q_j < \infty$ and $2 < \tau_j \leq \infty$, $j = 1, \dots, m$.

3. CONCLUSIONS

In [12, inequality (32)], the following relation was proved:

$$\begin{aligned} & \sum_{k_2=0}^{\infty} 2^{k_2 \theta_2 (\frac{1}{\theta_2} - \frac{1}{q_2})} \left\{ \sum_{s_2=2^{k_2+1}}^{2^{k_2+1}} \left[\sum_{k_1=0}^{\infty} 2^{k_1 \theta_1 (\frac{1}{\theta_1} - \frac{1}{q_1})} \left(\sum_{s_1=2^{k_1+1}}^{2^{k_1+1}} \|\delta_{\bar{s}}(f)\|_{\bar{p},\bar{q}}^* \right)^{\theta_1} \right]^{\frac{1}{\theta_1}} \right\}^{\theta_2} \\ & \ll \sum_{s_2=2}^{\infty} s_2^{1 - \frac{\theta_2}{q_2}} \left[\sum_{s_1=2}^{\infty} s_1^{1 - \frac{\theta_1}{q_1}} \left(\|\delta_{\bar{s}}(f)\|_{\bar{p},\bar{q}}^* \right)^{\theta_1} \right]^{\frac{\theta_2}{\theta_1}}. \end{aligned} \tag{3.1}$$

As shown in [12], applying Hölder's inequality to prove (3.1) with exponents $\theta_j^{-1} + (\theta_j')^{-1} = 1$, $j = 1, 2$, we obtain

$$2^{k_1 \theta_1 (\frac{1}{\theta_1} - \frac{1}{q_1})} \left(\sum_{s_1=2^{k_1+1}}^{2^{k_1+1}} \|\delta_{\bar{s}}(f)\|_{\bar{p},\bar{q}}^* \right)^{\theta_1} \ll \sum_{s_1=2^{k_1+1}}^{2^{k_1+1}} s_1^{\theta_1 (\frac{1}{\theta_1} - \frac{1}{q_1})} \left(\|\delta_{\bar{s}}(f)\|_{\bar{p},\bar{q}}^* \right)^{\theta_1}.$$

Unfortunately, there was an inaccuracy in the proof of this inequality. The correct relation is

$$2^{k_1\theta_1(\frac{1}{\theta_1}-\frac{1}{q_1})} \left(\sum_{s_1=2^{k_1+1}}^{2^{k_1+1}} \|\delta_{\bar{s}}(f)\|_{\bar{p},\bar{q}}^* \right)^{\theta_1} \\ \leq 2^{k_1\theta_1(\frac{1}{\theta_1}-\frac{1}{q_1})} 2^{k_1(\theta_1-1)} \sum_{s_1=2^{k_1+1}}^{2^{k_1+1}} (\|\delta_{\bar{s}}(f)\|_{\bar{p},\bar{q}}^*)^{\theta_1} \ll \sum_{s_1=2^{k_1+1}}^{2^{k_1+1}} s_1^{\theta_1(1-\frac{1}{q_1})} (\|\delta_{\bar{s}}(f)\|_{\bar{p},\bar{q}}^*)^{\theta_1}.$$

Theorem 2 rectifies this inaccuracy to some extent.

Note that, in the case $\bar{\theta} = \bar{q}$, there is no known estimate for the value

$$E_n^{(\bar{\gamma})}(\overset{\circ}{S}_{\bar{p},\bar{q},\bar{\tau}}B)_{\bar{p},\bar{\theta}} = \sup \left\{ E_n^{(\bar{\gamma})}(f)_{\bar{p},\bar{\theta}} : f \in \overset{\circ}{S}_{\bar{p},\bar{q},\bar{\tau}}B \right\}.$$

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