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Well-posedness of the initial-boundary value problems for the time-fractional degenerate diffusion equations

This paper deals with the solving of initial-boundary value problems for the one-dimensional linear time-fractional diffusion equations with time-degenerate diffusive coefficients t^β with $\beta > 1 - \alpha$. The solutions to initial-boundary value problems for the one-dimensional time-fractional degenerate diffusion equations with Riemann-Liouville fractional integral $I_{0+,t}^{1-\alpha}$ of order $\alpha \in (0, 1)$ and with Riemann-Liouville fractional derivative $D_{0+,t}^\alpha$ of order $\alpha \in (0, 1)$ in the variable, are shown. The solutions to these fractional diffusive equations are presented using the Kilbas-Saigo function $E_{\alpha,m,l}(z)$. The solution to the problems is discovered by the method of separation of variables, through finding two problems with one variable. Rather, through finding a solution to the fractional problem depending on the parameter t , with the Dirichlet or Neumann boundary conditions. The solution to the Sturm-Liouville problem depends on the variable x with the initial fractional-integral Riemann-Liouville condition. The existence and uniqueness of the solution to the problem are confirmed. The convergence of the solution was evidenced using the estimate for the Kilbas-Saigo function $E_{\alpha,m,l}(z)$ from and by Parseval's identity.

Keywords: time-fractional diffusion equation, method of separation variables, Kilbas-Saigo function.

Introduction

Many mathematicians have attracted most interest to the fractional diffusion equations. Inverse source problems for degenerate time-fractional PDE were studied in [1]. In [2, 3], Al-Refai and Luchko analyzed the initial-boundary value problems for the linear and non-linear fractional diffusion equations with the Riemann-Liouville time-fractional derivative. Various types of fractional derivatives and their properties were investigated in the monograph [4–8]. Fractional calculus can be applied in mechanics, physics, mathematics, etc. [8–12]. Note that degenerate fractional evolutionary equations were investigated in [13, 14]. In [15], maximum and minimum principles for time-fractional diffusion equations involving fractional derivatives were proposed. Luchko studied initial-boundary value problems for a generalized diffusion equation with a distributed order [16].

In our previous work [17], we studied the Cauchy-Dirichlet and Cauchy-Neumann problems for the Caputo time-fractional diffusion equation. This paper considers the Cauchy-Dirichlet and Cauchy-Neumann problems for the diffusion equation with Riemann-Liouville time-fractional derivative. A solution is discovered by using the Kilbas-Saigo function and by the method of separation of variables. The existence, convergence, and uniqueness of the solution are proved.

1 Dirichlet problem

Let us consider the one-dimensional case of the time-fractional diffusion equation

$$D_{0+,t}^\alpha u(t, x) - t^\beta u_{xx}(t, x) = 0, \quad (t, x) \in (0, \infty) \times (0, 1), \quad (1)$$

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with the Dirichlet boundary condition

$$u(t, 0) = u(t, 1) = 0, \quad t > 0, \quad x \in [0, 1], \tag{2}$$

and the Cauchy initial condition

$$I_{0+,t}^{1-\alpha} u(0, x) = \phi(x), \quad x \in [0, 1], \tag{3}$$

where $\beta > 1 - \alpha$, $D_{0+,t}^\alpha$ is the Riemann-Liouville fractional derivative of order $\alpha \in (0, 1)$ defined by [5; 79]

$$D_{0+,t}^\alpha f(t) = \frac{d}{dt} I_{0+,t}^{1-\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{f(s) ds}{(t-s)^\alpha}.$$

Here $I_{0+,t}^{1-\alpha}$ is the Riemann-Liouville fractional integral given by [5; 80]

$$I_{0+,t}^{1-\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f(s) ds}{(t-s)^\alpha}.$$

Let $H^2(0, 1)$ is a Hilbert space, defined by

$$H^2(0, 1) = \{u : u \in L^2(0, 1); u_{xx} \in L^2(0, 1)\},$$

where the norm is

$$\|u\|_{H^2(0,1)}^2 = \sum_{k=1}^{\infty} \lambda_k^2 |(u, e_k)|^2 < \infty.$$

Definition 1. The solution to problem (1)–(3) is $t^{1-\alpha} u \in C((0, \infty); L^2(0, 1))$, which satisfies $t^{1-\alpha-\beta} D_{0+,t}^\alpha u, t^{1-\alpha} u_{xx} \in C((0, \infty); L^2(0, 1))$.

Theorem 1. Let $\phi(x) \in H^2(0, 1)$, then there exists a unique solution u to problem (1)–(3), which has the form

$$u(t, x) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} \sum_{k=1}^{\infty} \phi_k E_{\alpha, 1+\frac{\beta}{\alpha}, 1+\frac{\beta-1}{\alpha}}(-\pi^2 k^2 t^{\beta+\alpha}) \sin \pi k x, \quad (t, x) \in (0, \infty) \times (0, 1),$$

where $\phi_k = 2 \int_0^1 \phi(x) \sin \pi k x dx$, $k \in \mathbb{N}$ and $E_{\alpha, m, l}(z)$ is the Kilbas-Saigo function defined as [8, Remark 5.1]

$$E_{\alpha, m, l}(z) = \sum_{i=0}^{\infty} c_i z^i, \quad c_0 = 1, \quad c_i = \prod_{j=0}^{i-1} \frac{\Gamma(\alpha(jm+l)+1)}{\Gamma(\alpha(jm+l+1)+1)}, \quad i \geq 1. \tag{4}$$

For the function $E_{\alpha, m, m-\frac{1}{\alpha}}(-\lambda_k t^{\beta+\alpha})$ the following estimate holds [4, Proposition 3.6]

$$E_{\alpha, m, m-\frac{1}{\alpha}}(-\lambda_k t^{\beta+\alpha}) \leq \frac{1}{\left(1 + \frac{\Gamma(1+\alpha m)}{\Gamma(1+\alpha(m+1))} \lambda_k t^{\beta+\alpha}\right)^{1+\frac{1}{m}}}, \quad m = \frac{\beta+\alpha}{\alpha}, \quad t > 0. \tag{5}$$

Proof Theorem 1.

Existence of solution. Since the Sturm–Liouville operator has eigenvalues $\{\lambda_k > 0, k \in \mathbb{N}\}$ on $L^2(0, 1)$ and the corresponding orthonormal eigenfunctions $\{X_k(x), k \in \mathbb{N}\}$ in $L^2(0, 1)$ and $\phi(x) \in H^2(0, 1)$, then we can write the solution to problem (1)–(3) as follows

$$u(t, x) = \sum_{k=1}^{\infty} T_k(t) X_k(x), \quad (t, x) \in (0, \infty) \times (0, 1), \tag{6}$$

$$\phi(x) = \sum_{k=1}^{\infty} \phi_k X_k(x), \quad x \in (0, 1),$$

where

$$\phi_k = 2 \int_0^1 \phi(x) X_k(x) dx.$$

Substituting (6) to diffusion equation (1)–(3), we gain the next problem

$$D_{0+,t}^{\alpha} T_k(t) + \lambda_k t^{\beta} T_k(t) = 0, \quad t > 0, \quad (7)$$

$$I_{0+,t}^{1-\alpha} T_k(0) = \phi_k, \quad (8)$$

$$X_k''(x) + \lambda_k X_k(x) = 0, \quad (9)$$

$$X_k(0) = X_k(1) = 0. \quad (10)$$

The orthonormal eigenfunctions and the corresponding eigenvalues of the Dirichlet problem (9)–(10) are $X_k(x) = \sin \pi k x$ and $\lambda_k = (\pi k)^2$, respectively. It is known that a unique solution to problem (7)–(8) is [5; 227]

$$T_k(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} \phi_k E_{\alpha, 1+\frac{\beta}{\alpha}, 1+\frac{\beta-1}{\alpha}}(-\pi^2 k^2 t^{\beta+\alpha}). \quad (11)$$

Substituting (11) and the orthonormal eigenfunctions $X_k(x) = \sin \pi k x$ to (6), we can get the solution to problem (1)–(3) in the next form

$$u(t, x) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} \sum_{k=1}^{\infty} \phi_k E_{\alpha, 1+\frac{\beta}{\alpha}, 1+\frac{\beta-1}{\alpha}}(-\pi^2 k^2 t^{\beta+\alpha}) \sin \pi k x, \quad (t, x) \in (0, \infty) \times (0, 1). \quad (12)$$

Convergence of solution. Applying (5) to (11), we get

$$T_k(t) \leq \frac{|\phi_k| |t^{\alpha-1}|}{\Gamma(\alpha) \left(1 + \frac{\Gamma(1+\alpha m)}{\Gamma(1+\alpha(m+1))} \pi^2 k^2 t^{\beta+\alpha}\right)^{1+\frac{1}{m}}}, \quad m = \frac{\beta + \alpha}{\alpha}.$$

By Parseval's identity, it follows from (12) that

$$\begin{aligned} \sup_{t \geq 0} \|t^{1-\alpha} u(t, \cdot)\|_{L^2(0,1)}^2 &= \sup_{t \geq 0} \frac{1}{|\Gamma(\alpha)|^2} \sum_{k=1}^{\infty} |\phi_k|^2 \left| E_{\alpha, 1+\frac{\beta}{\alpha}, 1+\frac{\beta-1}{\alpha}}(-\pi^2 k^2 t^{\beta+\alpha}) \right|^2 \leq \\ &\leq \sup_{t \geq 0} \frac{1}{|\Gamma(\alpha)|^2} \sum_{k=1}^{\infty} \frac{|\phi_k|^2}{\left(1 + \frac{\Gamma(1+\alpha m)}{\Gamma(1+\alpha(m+1))} \pi^2 k^2 t^{\beta+\alpha}\right)^{2(1+\frac{1}{m})}} \leq \\ &\leq \sup_{t \geq 0} \frac{1}{|\Gamma(\alpha)|^2 \left(1 + \frac{\Gamma(1+\alpha m)}{\Gamma(1+\alpha(m+1))} \pi^2 t^{\beta+\alpha}\right)^{2(1+\frac{1}{m})}} \sum_{k=1}^{\infty} |\phi_k|^2 \leq \sum_{k=1}^{\infty} |\phi_k|^2 = \|\phi(\cdot)\|_{L^2(0,1)}^2. \end{aligned} \quad (13)$$

Solving $D_{0+,t}^{\alpha} u$ and u_{xx} we get

$$D_{0+,t}^{\alpha} u(t, x) = \frac{1}{\Gamma(\alpha)} \sum_{k=1}^{\infty} \phi_k D_{0+,t}^{\alpha} t^{\alpha-1} E_{\alpha, 1+\frac{\beta}{\alpha}, 1+\frac{\beta-1}{\alpha}}(-\pi^2 k^2 t^{\beta+\alpha}) \sin \pi k x =$$

$$= -\frac{t^{\alpha+\beta-1}}{\Gamma(\alpha)} \sum_{k=1}^{\infty} \pi^2 k^2 \phi_k E_{\alpha, 1+\frac{\beta}{\alpha}, 1+\frac{\beta-1}{\alpha}}(-\pi^2 k^2 t^{\beta+\alpha}) \sin \pi k x, \tag{14}$$

and

$$\begin{aligned} u_{xx}(t, x) &= \frac{t^{\alpha-1}}{\Gamma(\alpha)} \sum_{k=1}^{\infty} \phi_k E_{\alpha, 1+\frac{\beta}{\alpha}, 1+\frac{\beta-1}{\alpha}}(-\pi^2 k^2 t^{\beta+\alpha}) \sin'' \pi k x = \\ &= -\frac{t^{\alpha-1}}{\Gamma(\alpha)} \sum_{k=1}^{\infty} \pi^2 k^2 \phi_k E_{\alpha, 1+\frac{\beta}{\alpha}, 1+\frac{\beta-1}{\alpha}}(-\pi^2 k^2 t^{\beta+\alpha}) \sin \pi k x. \end{aligned} \tag{15}$$

Applying (13)–(15) we get

$$\sup_{t \geq 0} \|t^{1-\alpha-\beta} \mathcal{D}_t^\alpha u(t, \cdot)\|_{L^2(0,1)}^2 \leq \sum_{k=1}^{\infty} \pi^4 k^4 |\phi_k|^2 = \|\phi(\cdot)\|_{H^2(0,1)}^2 < \infty,$$

and

$$\sup_{t \geq 0} \|t^{1-\alpha} u_{xx}(t, \cdot)\|_{L^2(0,1)}^2 \leq \sum_{k=1}^{\infty} \pi^4 k^4 |\phi_k|^2 = \|\phi(\cdot)\|_{H^2(0,1)}^2 < \infty.$$

Uniqueness of the solution. Suppose that u_1 and u_2 are solutions to problem (1)–(3) and we choose $u = u_1 - u_2$ in such a way, that u satisfies the diffusion equation (1) and boundaries, initial conditions (2), (3). Define

$$T_k(t) = \int_0^1 u(t, x) \sin \pi k x dx, \quad k \in N, t > 0. \tag{16}$$

Applying $D_{0+,t}^\alpha$ to left-side (16) equation by using (1) we obtain

$$\begin{aligned} D_{0+,t}^\alpha T_k(t) &= \int_0^1 D_{0+,t}^\alpha u(t, x) \sin \pi k x dx \\ &= t^\beta \int_0^1 u_{xx}(t, x) \sin \pi k x dx \\ &= t^\beta \int_0^1 u(t, x) \sin'' \pi k x dx \\ &= -t^\beta \pi^2 k^2 \int_0^1 u(t, x) \sin \pi k x dx \\ &= -t^\beta \pi^2 k^2 T_k(t), \quad k \in N, t > 0. \end{aligned}$$

From (2), (3) we have

$$I_{0+,t}^{1-\alpha} T_k(0) = 0,$$

which means that $u(t, x) \equiv 0$. Hence $u_1(t, x) = u_2(t, x)$, therefore the problem (1)–(3) has a unique solution.

2 Cauchy-Neumann problem

Let us consider time-fractional diffusion equation

$$D_{0+,t}^\alpha u(t, x) - t^\beta u_{xx}(t, x) = 0, \quad (t, x) \in (0, \infty) \times (0, 1), \tag{17}$$

with the Neumann boundary condition

$$u_x(t, 0) = u_x(t, 1) = 0, \quad t > 0, \quad x \in [0, 1], \tag{18}$$

and the Cauchy initial condition

$$I_{0+,t}^{1-\alpha} u(0, x) = \phi(x). \quad (19)$$

Definition 2. The solution to problem (17)–(19) is $t^{1-\alpha} u \in C((0, \infty); L^2(0, 1))$, which satisfies $t^{1-\alpha-\beta} D_{0+,t}^\alpha u, t^{1-\alpha} u_x, t^{1-\alpha} u_{xx} \in C((0, \infty); L^2(0, 1))$.

Theorem 2. Let $\phi(x) \in H^2(0, 1)$. The unique solution to problem (17)–(19) is the function u , which has form

$$u(t, x) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} \sum_{k=0}^{\infty} \phi_k E_{\alpha, 1+\frac{\beta}{\alpha}, 1+\frac{\beta-1}{\alpha}}(-\pi^2 k^2 t^{\beta+\alpha}) \cos \pi k x, \quad (t, x) \in (0, \infty) \times (0, 1),$$

where $\phi_0 = \int_0^1 \phi(x) dx$ and $\phi_k = 2 \int_0^1 \phi(x) \cos \pi k x dx$, $k \in N$ and $E_{\alpha, m, l}(z)$ is a Kilbas–Saigo function, which is defined by the formula (4) and (5).

It can be easily proven by the idea of Theorem 1.

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Бөлшек ретті туындылы өзгешеленген диффузия теңдеулері үшін бастапқы шеттік есептің қисындылығы

Мақалада t^β , $\beta > 1 - \alpha$ диффузиялық коэффициенттері бар бір өлшемді сызықтық бөлшек ретті туындылы өзгешеленген диффузия теңдеулері үшін бастапқы шеттік есептерді шешу қарастырылған. $\in (0, 1)$ үшін бөлшек ретті Риман-Лиувилль интегралы $I_{0+,t}^{1-\alpha}$ және $\alpha \in (0, 1)$ үшін бөлшек ретті Риман-Лиувилль туындысы $D_{0+,t}^\alpha$ бар бір өлшемді уақыт бойынша бөлшек ретті туындылы өзгешеленген диффузия теңдеулері үшін бастапқы шеттік есептердің шешімдері көрсетілген. Бөлшек ретті диффузиялық теңдеулердің шешімдері $E_{\alpha,m,t}(z)$ Килбас-Сайго функциясы арқылы берілген. Есептердің шешімі айнымалыларды ажырату әдісі арқылы, бір айнымалысы бар екі есепті табу арқылы анықталады. Демек, Дирихле немесе Нейман шекаралық шарттарымен t параметріне тәуелді бөлшек ретті есебінің шешімін және x параметріне тәуелді Штурм-Лиувилль есебіне қойылған бастапқы шарты бөлшек ретті Риман-Лиувилль интегралы арқылы берілген есептің шешімін табу арқылы. Есептің шешімінің бар болуы мен жалғыздығы дәлелденген. Шешімнің жинақтылығы Kilbas-Saigo $E_{\alpha,m,t}(z)$ функциясының бағалауы көмегімен және Парсевал теңдігін қолдану арқылы дәлелденді.

Кілт сөздер: бөлшек ретті туындылы өзгешеленген диффузия теңдеуі, айнымалыларын ажырату әдісі, Килбас-Сайго функциясы.

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Корректность начально-краевых задач для дробных вырожденных диффузионных уравнений

Статья посвящена решению начально-краевых задач для одномерных дробных вырожденных линейных диффузионных уравнений коэффициентами диффузии t^β при $\beta > 1 - \alpha$, начально-краевых задач для одномерных уравнений вырождающейся диффузии с дробным временем с дробным интегралом Римана-Лиувилля $I_{0+,t}^{1-\alpha}$ порядка $\alpha \in (0, 1)$ и с дробной производной Римана-Лиувилля $D_{0+,t}^\alpha$ порядка $\alpha \in (0, 1)$ по переменной. Решения этих дробных диффузионных уравнений представлены с помощью функции Килбаса-Сайго $E_{\alpha,m,t}(z)$, их получили методом разделения переменных, путем нахождения двух задач с одной переменной. Вернее, путем нахождения решения дробной задачи, зависящей от

параметра t , с граничными условиями Дирихле или Неймана, и решение задачи Штурма–Лиувилля, зависящей от переменной x с начальным дробно-интегральным условием Римана–Лиувилля. Доказаны существование и единственность решения задачи. Сходимость в решения подтверждена с помощью оценки функции Килбаса–Сайго $E_{\alpha,m,l}(z)$ и тождества Парсевалья.

Ключевые слова: дробно-вырожденное диффузионное уравнение, метод разделения переменных, функция Килбаса–Сайго.

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