

## Spectral analysis of second order quantum difference operator over the sequence space $l_p$ ( $1 < p < \infty$ )

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In this article, we study the spectrum, fine spectrum and boundedness property of second order quantum difference operator  $\Delta_q^2$  ( $0 < q < 1$ ) over the class of sequence  $l_p$  ( $1 < p < \infty$ ), the  $p^{\text{th}}$  summable sequence space. The second order quantum difference operator  $\Delta_q^2$  is a lower triangular triple band matrix  $\Delta_q^2(1, -(1+q), q)$ . We also determine the approximate point spectrum, defect spectrum, compression spectrum, and Goldberg classification of the operator on the class of sequence. We obtained the results by solving an infinite system of linear equations and computing the inverse of a lower triangular infinite matrix. We also provide appropriate examples along with graphical representations where necessary.

*Keywords:* spectrum, difference operator, infinite matrices, triple-band matrix.

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### Introduction

Spectral theory of bounded linear operators on Banach or Hilbert spaces holds a significant place in different branches of Mathematics due to its many applications. The fundamental principle of the modern spectral theorem is that certain linear operators on infinite dimensional spaces can be represented in a “diagonal” matrix form. From this diagonal form, we can determine the spectrum of the operator. The spectrum of an operator can be classified into three parts: the point spectrum, the continuous spectrum, and the residual spectrum. These three disjoint parts together are referred to as the “fine spectrum”.

In operator theory, one of the most important linear operators is the difference operator. The spectrum of this operator and its different forms on various sequence spaces have been studied by many researchers. In recent times, researchers have started analyzing the spectrum of the quantum version of some well-known operators, one of which is the difference operator. In our study, we analyze the spectrum of a second order quantum difference operator. The  $q$ -analog of the second order difference operator is defined as  $(\Delta_q^2 u)_k = u_k - (1+q)u_{k-1} + qu_{k-2}$ , for all  $k \in \mathbb{N}$  and any term with negative indices are zero. The matrix representation of this operator is given below

$$\Delta_q^2 = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots \\ -(1+q) & 1 & 0 & 0 & \dots \\ q & -(1+q) & 1 & 0 & \dots \\ 0 & q & -(1+q) & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The  $n^{\text{th}}$  order  $q$ -difference operator is defined as  $(\Delta_q^n u)_k = \sum_{i=0}^n (-1)^i \binom{n}{i}_q q^{\binom{i}{2}} u_{n+k-i}$ , which was introduced by Bustoz and Gordillo [1].

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The initial study of the spectrum of the difference operator  $\Delta$ , ( $\Delta y_k = y_k - y_{k+1}$ ) was conducted by Altay and Basar [2], Kayaduman and Furkan [3], and Akhmedov and Basar [4, 5] in the spaces  $c_0$ ,  $c$ ;  $l_1$ ,  $bv$  and  $l_p$ ,  $bv_p$  respectively. The spectrum of the second-order difference operator  $\Delta^2$  was studied by Dutta and Baliarsingh [6] over the space  $c_0$ . After this the operator  $\Delta$  was generalized to  $B(r, s)$ ,  $B(r, s)(y_k) = (ry_k + sy_{k-1})$ . The spectrum of this operator was studied by Altay and Basar [7], Kayaduman et al. [8], Bilgic and Furkan [9], Dutta and Tripathy [10] over the spaces  $c_0$ ,  $c$ ;  $l_1$ ,  $bv$ ;  $l_p$ ,  $bv_p$  ( $1 \leq p < \infty$ ) and  $cs$  respectively.  $B(r, s)$  was further generalized to  $B(r, s, t)$ ,  $B(r, s, t)(y_k) = (ry_k + sy_{k-1} + ty_{k-2})$ . The spectrum of this operator was studied by Bilgic and Furkan [11]; Furkan et al. [12, 13] over the spaces  $l_1$ ,  $bv$ ,  $c_0$ ,  $c$  and  $l_p$ ,  $bv_p$  respectively. Srivastava and Kumar [14, 15]; Akhmedov and El-Shabrawy [16] studied the spectrum of the generalised difference operator  $\Delta_v$ , where  $\Delta_v(y_k) = (v_k y_k - v_{k-1} y_{k-1})$  over the sequence spaces  $c_0$ ,  $l_1$  and  $c$ ,  $l_p$  respectively. Akhmedov and El-Shabrawy [17, 18]; Dutta and Baliarsingh [19] obtained the spectrum of the operator  $\Delta_{ab}$ , where  $\Delta_{ab}(y_k) = (a_k y_k + b_{k-1} y_{k-1})$  over the sequence spaces  $c_0$ ,  $c$  and  $l_p$ ,  $bv_p$  respectively. Panigrahi and Srivastava [20, 21] also analyzed the spectrum of  $\Delta_{uv}^2$ , where  $\Delta_{uv}^2(y_k) = (u_k y_k - v_{k-1} y_{k-1} + u_{k-2} y_{k-2})$  and  $\Delta_{uvw}^2$ , where  $\Delta_{uvw}^2(y_k) = (u_k y_k + v_{k-1} y_{k-1} + w_{k-2} y_{k-2})$  over the sequence spaces  $c_0$  and  $l_1$  respectively. Then, Altundağ and Abay [22] studied on the fine spectrum of generalized upper triangular triple-band matrices  $(\Delta_{uvw}^2)^t$  where the transpose of matrix operator  $\Delta_{uvw}^2$  over the sequence space  $l_1$ . Patra and Srivastava [23] considered a new generalized difference operator  $A(p_1, p_2; q_1, q_2; r_1, r_2)$  and determined its spectrum over the sequence space  $l_p$  ( $1 \leq p < \infty$ ). The operators mentioned above can be expressed using a lower triangular band matrix. Spectral analysis of the quantum versions of some well-known operators has been conducted in recent years. The spectrum of  $q$ -Cesàro matrix was studied by Yildirim [24], Durna and Turkay [25] over the sequence space  $c_0$  and  $c$  respectively. Yaying et al. [26, 27] studied the spectrum of second order  $q$ -difference operator over the sequence space  $c_0$ ,  $l_1$  respectively. Spectrum of weighted  $q$ -difference operator was studied by Yaying et al. [28] over the sequence space  $c_0$ .

*q-analog*: A  $q$ -analog of a number, a theorem, an identity or an expression is a generalization that involves a new parameter  $q$  and it reduced to the original number, theorem, identity as the limit  $q \rightarrow 1^-$ . In the 19<sup>th</sup> century, the basic hypergeometric series became the first  $q$ -analog to be extensively studied. In recent research of many areas of Mathematics like combinatorics, approximation theory, difference and integral equations, etc.,  $q$ -calculus have been used extensively.

The  $q$ -analog  $[m]_q$  of  $m$  for  $q \in (0, 1)$  can be determined as

$$[m]_q = \begin{cases} \sum_{k=0}^{m-1} q^k, & m = 1, 2, 3, \dots, \\ 0, & m = 0. \end{cases}$$

One might observe that  $[m]_q = m$  whenever  $q \rightarrow 1^-$ . The  $q$ -analog  $\binom{m}{k}_q$  of binomial coefficient  $\binom{m}{k}$  can be determined as

$$\binom{m}{k}_q = \begin{cases} \frac{[m]_q!}{[m-k]_q! [k]_q!}, & m \geq k, \\ 0, & k > m, \end{cases}$$

where the  $q$ -analog of the factorial, i.e.,  $q$ -factorial, is defined as

$$[m]_q! = \begin{cases} \prod_{k=1}^m [k]_q, & m = 1, 2, 3, \dots, \\ 1, & m = 0. \end{cases}$$

The  $q$ -analog of some specific binomials such as  $\binom{0}{0} = \binom{m}{0} = \binom{m}{m} = 1$ , also  $\binom{m}{m-k}_q = \binom{m}{k}_q$ . For an in-depth study of quantum calculus, we refer to the book [29].

1 Some Definitions and Preliminaries

Consider  $M : U \rightarrow V$  be a bounded linear operator, in which  $U$  and  $V$  are Banach spaces, the following collections

$$R(M) = \{v \in V : v = Mu, u \in U\}$$

$$\text{and } B(U, V) = \{M : U \rightarrow V : M \text{ is continuous and linear}\}$$

are termed as the range of the operator  $M$  and the set of all bounded linear operators from  $U$  to  $V$  respectively. The adjoint operator  $M^*$  of  $M$  is defined from  $V^*$  to  $U^*$ , where  $V^*$  and  $U^*$  represent the dual space of  $V$  and  $U$  respectively. Again, it is defined as  $(M^*f)(u) = f(Mu)$ , for all  $f \in V^*$  and  $u \in U$ .

Let  $M : D(M) \rightarrow U$ , where  $D(M)$  denotes the domain of  $M$ . From  $M$  we can get an operator,

$$M_\mu = M - \mu I,$$

where  $\mu \in \mathbb{C}$  and  $I$  is the identity operator. A regular value  $\mu \in \mathbb{C}$  of  $M$  is such that  $M_\mu$  is invertible, and its inverse  $(M_\mu^{-1})$  is bounded and defined on a set  $A$  and call it the resolvent operator of  $M$ , where  $A$  is dense in  $U$ . The collections of such  $\mu$  is called the resolvent set and is denoted by  $\rho(M, U)$ . In the complex plane  $\mathbb{C}$ , the compliment of  $\rho(M, U)$  is denoted by  $\sigma(M, U)$ , and is called the spectrum of  $M$ .

Further,  $\sigma(M, U)$  is classified into three disjoint subsets, namely, the point spectrum  $\sigma_p(M, U)$ , the continuous spectrum  $\sigma_c(M, U)$ , and the residual spectrum  $\sigma_r(M, U)$ . In point spectrum,  $M_\mu^{-1}$  does not exist for any  $\mu \in \sigma_p(M, U)$ , while in continuous spectrum,  $M_\mu^{-1}$  exist but unbounded for every  $\mu \in \sigma_c(M, U)$ , and also defined on a set that is dense in  $U$ . On the other hand, in the residual spectrum,  $M_\mu^{-1}$  exists but may or may not be bounded for  $\mu \in \sigma_r(M, U)$  and is not dense in  $U$ .

There are more subdivisions of the spectrum of a bounded operator such as approximate point spectrum  $\sigma_{ap}(M, U)$ , defect spectrum  $\sigma_\delta(M, U)$  and compression spectrum  $\sigma_{co}(M, U)$ , which are defined as follows:

- $\sigma_{ap}(M, U) = \{\mu \in \mathbb{C} : (M - \mu I) \text{ is not bounded below}\};$
- $\sigma_\delta(M, U) = \{\mu \in \mathbb{C} : (M - \mu I) \text{ is not surjective}\};$
- $\sigma_{co}(M, U) = \{\mu \in \mathbb{C} : R(M - \mu I) \neq U\}.$

2 Goldberg's classification of spectrum

A detailed classification of the spectrum of an operator was given by Goldberg [30]. This classification is based on the nature of the set  $R(M_\mu)$  and the inverse  $M_\mu^{-1}$ .

If  $M \in B(U, U)$ , then there are three possibilities for  $R(M_\mu)$ :

- (P)  $R(M_\mu) = U,$
- (Q)  $\overline{R(M_\mu)} = U,$  but  $R(M_\mu) \neq U,$
- (R)  $\overline{R(M_\mu)} \neq U$

and three possibilities for  $M_\mu^{-1}$ :

- (1) Exist and continuous,
- (2) Exist but discontinuous,
- (3) Does not exist.

Combination of the possibilities  $P, Q, R$  and 1, 2, 3 leads to nine different states. They are identified as  $P_1, P_2, P_3, Q_1, Q_2, Q_3, R_1, R_2,$  and  $R_3$ .

If  $M_\mu \in P_1$  or  $M_\mu \in Q_1$ , then  $\mu \in \rho(M, X)$ . If  $M_\mu \in R_2$ , then  $M_\mu^{-1}$  exists and is unbounded, and  $\overline{R(M_\mu)} \neq X$  and we can write  $\mu \in R_2\sigma(M, X)$ . We can summarize this classification in the following Table 1.

Table 1

Goldberg's classification of spectrum of linear operator

		1	2	3
		$M_\mu^{-1}$ exist and bounded	$M_\mu^{-1}$ exist and unbounded	$M_\mu^{-1}$ does not exist
P	$R(M_\mu) = U$	$\mu \in \rho(M, U)$	-	$\mu \in \sigma_p(M, U)$ $\mu \in \sigma_{ap}(M, U)$
Q	$\overline{R(M_\mu)} = U$	$\mu \in \rho(M, U)$	$\mu \in \sigma_c(M, U)$ $\mu \in \sigma_\delta(M, U)$ $\mu \in \sigma_{ap}(M, U)$	$\mu \in \sigma_p(M, U)$ $\mu \in \sigma_\delta(M, U)$ $\mu \in \sigma_{ap}(M, U)$
R	$\overline{R(M_\mu)} \neq U$	$\mu \in \sigma_r(M, U)$ $\mu \in \sigma_\delta(M, U)$ $\mu \in \sigma_{co}(M, U)$	$\mu \in \sigma_r(M, U)$ $\mu \in \sigma_\delta(M, U)$ $\mu \in \sigma_{co}(M, U)$ $\mu \in \sigma_{ap}(M, U)$	$\mu \in \sigma_p(M, U)$ $\mu \in \sigma_\delta(M, U)$ $\mu \in \sigma_{co}(M, U)$ $\mu \in \sigma_{ap}(M, U)$

*Proposition 1.* ([31], p. 28) Spectral and sub-spectral relationships of an operator  $M$  and its adjoint operator  $M^*$  are provided below.

- (a)  $\sigma(M^*, Y^*) = \sigma(M, Y)$ ,
- (b)  $\sigma_{ap}(M^*, Y^*) = \sigma_\delta(M, Y)$ ,
- (c)  $\sigma_\delta(M^*, Y^*) = \sigma_{ap}(M, Y)$ ,
- (d)  $\sigma_p(M^*, Y^*) = \sigma_{co}(M, Y)$ ,
- (e)  $\sigma(M, Y) = \sigma_{ap}(M, Y) \cup \sigma_p(M^*, Y^*) = \sigma_p(M, Y) \cup \sigma_{ap}(M^*, Y^*)$ .

*Lemma 1.* ([30], p. 60) The adjoint operator  $M^*$  of  $M$  is onto if and only if  $M$  has a bounded inverse.

*Lemma 2.* ([30], p. 59) The bounded linear operator  $M : U \rightarrow V$  has dense range if and only if  $M^*$  is one to one.

Throughout this work, the aforementioned spaces  $c_0$ ,  $c$ ,  $l_1$ ,  $l_p$ ,  $bv$ ,  $bv_p$ ,  $cs$  and  $l_\infty$  represent the spaces of all null, convergent, absolutely summable,  $p$ -absolutely summable, bounded variation,  $p$ -bounded variation, convergent series, and bounded sequences, respectively.

Before going to the main results we state a remark, "if  $z$  is a complex number, then  $\sqrt{z}$  means the square root of  $z$  with non-negative real part. If  $Re\sqrt{z} = 0$ , then  $\sqrt{z}$  means the square root of  $z$  with  $Im(z) \geq 0$ ".

### 3 Spectrum of $\Delta_q^2(1, -(1+q), q)$ on $l_p$

*Theorem 1.*  $\Delta_q^2 \in B(l_p)$  with  $(1 + (1+q)^p + q^p)^{1/p} \leq \|\Delta_q^2\|_{l_p} \leq 2(q+1)$  for  $0 < q < 1$ , where  $1 < p < \infty$ .

*Proof.* The linearity of  $\Delta_q^2$  is straightforward to prove, so it is omitted. Now, consider  $e^{(0)} = (1, 0, 0, \dots)$  in  $l_p$ . Then,  $(\Delta_q^2)e^{(0)} = (1, -(1+q), q, 0, 0, \dots)$  and it is obtained that

$$\frac{\|(\Delta_q^2)e^{(0)}\|_{l_p}}{\|e^{(0)}\|_{l_p}} = (1 + (1+q)^p + q^p)^{1/p}.$$

From this we get,  $(1 + (1+q)^p + q^p)^{1/p} \leq \|(\Delta_q^2)e^{(0)}\|$ . Again for any  $u = (u_k) \in l_p$  and using the

Minkowski inequality, we get

$$\begin{aligned} \|\Delta_q^2 u\|_{l_p} &= \left( \sum_k |qu_{k-1} + (-(1+q))u_k + u_{k+1}|^p \right)^{1/p} \\ &\leq \left( \sum_k |qu_{k-1}|^p \right)^{1/p} + \left( \sum_k |(1+q)u_k|^p \right)^{1/p} + \left( \sum_k |u_{k+1}|^p \right)^{1/p} \\ &= \left( q^p \sum_k |u_{k-1}|^p \right)^{1/p} + \left( (1+q)^p \sum_k |u_k|^p \right)^{1/p} + \left( \sum_k |u_{k+1}|^p \right)^{1/p} \\ &= (q + (1+q) + 1) \|u\|_{l_p} \\ &= 2(1+q) \|u\|_{l_p}. \end{aligned}$$

As a result, we get  $(1 + (1 + q)^p + q^p)^{1/p} \leq \|\Delta_q^2\|_{l_p} \leq 2(q + 1)$ .

*Theorem 2.* The point spectrum  $\sigma_p(\Delta_q^2, l_p) = \phi$  (the empty set).

*Proof.* We prove this theorem by the method of contradiction. Consider  $\sigma_p(C_1(q), l_p) \neq \phi$ . Then for any  $0 \neq u \in l_p$  with  $(\Delta_q^2)u = \lambda u$ , we get the following equalities:

$$\begin{aligned} u_0 &= \lambda u_0, \\ -(1+q)u_0 + u_1 &= \lambda u_1, \\ qu_0 - (1+q)u_1 + u_2 &= \lambda u_2, \\ qu_1 - (1+q)u_2 + u_3 &= \lambda u_3, \\ &\vdots \\ qu_{m-2} - (1+q)u_{m-1} + u_m &= \lambda u_m, \\ &\vdots \end{aligned}$$

If  $u_m$  is the first non zero entry of the sequence  $u = (u_m)$ , then from the above equations we get  $\lambda = 1$ . Putting the value of  $\lambda = 1$  in the proceeding equation, we get  $u_m = 0$ , which contradicts our assumption. Thus,  $\sigma_p(\Delta_q^2, l_p) = \phi$ .

*Lemma 3.* ([32], p. 126) The matrix  $A = (a_{nk})$  defines a bounded linear operator  $T \in B(l_1)$ , mapping  $l_1$  to itself, if and only if the supremum of  $l_1$  norms of the columns of  $A$  is bounded.

*Lemma 4.* ([32], p. 126) The matrix  $A = (a_{nk})$  defines a bounded linear operator  $T \in B(l_\infty)$ , mapping  $l_\infty$  to itself, if and only if the supremum of  $l_1$  norms of the rows of  $A$  is bounded.

*Lemma 5.* ([33], p. 174, Theorem 9) If  $1 < p < \infty$  and  $A \in (l_1, l_1) \cap (l_\infty, l_\infty)$ . Then  $A \in (l_p, l_p)$ .

*Theorem 3.*  $\sigma(\Delta_q^2, l_p) = \left\{ \mu \in \mathbb{C} : |2(1 - \mu)| \leq \left| (1 + q) + \sqrt{(1 + q)^2 - 4(1 - \mu)q} \right| \right\}$ .

*Proof.* We consider

$$S = \left\{ \mu \in \mathbb{C} : |2(1 - \mu)| \leq \left| (1 + q) + \sqrt{(1 + q)^2 - 4(1 - \mu)q} \right| \right\}$$

and let  $\mu \notin S$ . So, we get  $\mu \neq 1$ , and this implies that  $(\Delta_q^2 - \mu I)$  has an inverse. Now,

$$(\Delta_q^2 - \mu I) = \begin{bmatrix} 1 - \mu & 0 & 0 & 0 & 0 & \dots \\ -(1 + q) & 1 - \mu & 0 & 0 & 0 & \dots \\ q & -(1 + q) & 1 - \mu & 0 & 0 & \dots \\ 0 & q & -(1 + q) & 1 - \mu & 0 & \dots \\ 0 & 0 & q & -(1 + q) & 1 - \mu & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Since  $(\Delta_q^2 - \mu I)$  is a lower triangular matrix, its inverse can be obtained easily and has been given below:

$$(\Delta_q^2 - \mu I)^{-1} = \begin{bmatrix} m_1 & 0 & 0 & 0 & \dots \\ m_2 & m_1 & 0 & 0 & \dots \\ m_3 & m_2 & m_1 & 0 & \dots \\ m_4 & m_3 & m_2 & m_1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

where

$$\begin{aligned} m_1 &= \frac{1}{1 - \mu}, \\ m_2 &= \frac{q + 1}{(1 - \mu)^2}, \\ m_3 &= \frac{(q + 1)^2 - q(1 - \mu)}{(1 - \mu)^3}, \\ &\vdots \end{aligned}$$

here the sequence  $(m_k)$  satisfies the following recurrence relation

$$m_k = \frac{(q + 1)m_{k-1} - qm_{k-2}}{1 - \mu}, \text{ for } k \geq 3.$$

From this recurrence relation, we get the characteristic equation as

$$(1 - \mu)w^2 - (1 + q)w + q = 0,$$

whose solutions are:

$$\begin{aligned} w_1 &= \frac{(1 + q) + \sqrt{(1 + q)^2 - 4(1 - \mu)q}}{2(1 - \mu)}, \\ w_2 &= \frac{(1 + q) - \sqrt{(1 + q)^2 - 4(1 - \mu)q}}{2(1 - \mu)}. \end{aligned}$$

From elementary calculations on recurrence sequence, we get

$$m_k = \frac{w_1^k - w_2^k}{\sqrt{(1 + q)^2 - 4(1 - \mu)q}}. \quad (1)$$

Now, we can proceed to the proof in two cases.

Case 1: If  $(1 + q)^2 = 4q(1 - \mu)$ , then we get

$$m_k = \left( \frac{2k}{1 + q} \right) \left[ \frac{(1 + q)}{2(1 - \mu)} \right]^k.$$

It can be easily proved that  $(m_k) \in l_p$  if  $\left| \frac{(1 + q)}{2(1 - \mu)} \right| < 1$ . So,  $\mu \notin S$  implies that  $(m_k) \in l_p$ .

Case 2: If  $(1 + q)^2 \neq 4q(1 - \mu)$ . Since  $\mu \notin S$ , we have  $|w_1| < 1$ . Again, using the inequality  $|1 - \sqrt{z}| \leq |1 + \sqrt{z}|$  for any  $z \in \mathbb{C}$ , we get

$$\begin{aligned} \left| \frac{(1 + q)}{2(1 - \mu)} - \frac{\sqrt{(1 + q)^2 - 4(1 - \mu)q}}{2(1 - \mu)} \right| &\leq \left| \frac{(1 + q)}{2(1 - \mu)} + \frac{\sqrt{(1 + q)^2 - 4(1 - \mu)q}}{2(1 - \mu)} \right| \\ \implies |w_2| &\leq |w_1| < 1. \end{aligned}$$

Using this in equation (1), it is obtained that  $(m_k) \rightarrow 0$  as  $k \rightarrow \infty$ . Now

$$\begin{aligned} \|(\Delta_q^2 - \mu I)^{-1}\|_{(l_1; l_1)} &= \sup_{k \in \mathbb{N}} \sum_{i=k}^{\infty} |m_i| = \sum_{i=1}^{\infty} |m_i| \\ &\leq \frac{1}{|\sqrt{(1 + q)^2 - 4(1 - \mu)q}|} \left( \sum_{i=1}^{\infty} |w_1|^i + \sum_{i=1}^{\infty} |w_2|^i \right) < \infty. \end{aligned}$$

Since  $|w_1| < 1$  and  $|w_2| < 1$ , it follows that  $(\Delta_q^2 - \mu I)^{-1} \in (l_1, l_1)$ . Again, since  $(m_k) \in l_1$ , the supremum of  $l_1$  norms of the rows of  $(\Delta_q^2 - \mu I)^{-1}$  is finite. This results in  $(\Delta_q^2 - \mu I)^{-1} \in (l_\infty, l_\infty)$ . Now, using Lemma 5, we get  $(\Delta_q^2 - \mu I)^{-1} \in (l_1, l_1) \cap (l_\infty, l_\infty) \implies (\Delta_q^2 - \mu I)^{-1} \in (l_p, l_p)$ . It proves that  $\sigma(\Delta_q^2, l_p) \subseteq S$ .

Now consider  $\mu \in S$ . If  $\mu = 1$ , then we get

$$(\Delta_q^2 - \mu I) = (\Delta_q^2 - I) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \dots \\ -(1 + q) & 0 & 0 & 0 & 0 & \dots \\ q & -(1 + q) & 0 & 0 & 0 & \dots \\ 0 & q & -(1 + q) & 0 & 0 & \dots \\ 0 & 0 & q & -(1 + q) & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Now  $(\Delta_q^2 - I)u = \theta \implies u = \theta$ . So, we get  $(\Delta_q^2 - I) : l_p \rightarrow l_p$  is one-one but not onto. This implies  $(\Delta_q^2 - I)$  is not invertible.

If we take  $\mu$  from  $S$  other than 1, then it is obtained from Case 1 that  $\left| \frac{(1 + q)}{2(1 - \mu)} \right| \geq 1$ . It implies  $(\Delta_q^2 - \mu I)^{-1} \notin B(l_p)$ . Again, from Case 2, it is obtained that  $|w_1| \geq 1$  and the inequality  $|w_1| > |w_2|$ , which directly implies  $(w_k) \not\rightarrow 0$ , and so  $\sum_{i=1}^{\infty} |w_i|^p$  diverges. Consider  $v = (1, 0, 0, \dots) \in l_p$ . Then  $(\Delta_q^2 - \mu I)^{-1}v = (m_1, m_2, m_3, \dots)$ , which doesn't belong to  $l_p$ . Therefore,  $(\Delta_q^2 - \mu I)^{-1} \notin B(l_p)$  and this proves that  $S \subseteq \sigma(\Delta_q^2, l_p)$ . Finally, it is obtained  $\sigma(\Delta_q^2, l_p) = S$ .

Lemma 6. ([34], p. 215) For any  $A \in B(l_p)$  ( $1 < p < \infty$ ), the adjoint operator  $A^* \in B(l_q)$ , where  $\frac{1}{p} + \frac{1}{q} = 1$  and can be represented by the transpose of  $A$  matrix.

Theorem 4.  $\sigma_p((\Delta_q^2)^*, l_p^* \cong l_q) = \left\{ \mu \in \mathbb{C} : |2(1 - \mu)| < \left| (1 + q) + \sqrt{(1 + q)^2 - 4(1 - \mu)q} \right| \right\}$ .

*Proof.* Consider the set  $S_1 = \left\{ \mu \in \mathbb{C} : |2(1 - \mu)| < \left| (1 + q) + \sqrt{(1 + q)^2 - 4(1 - \mu)q} \right| \right\}$ , and let  $\mu \in S_1$ . Now, solving  $(\Delta_q^2)^* u = \mu u$ , for  $\theta \neq u \in l_p^* \cong l_q$ , i.e.,

$$\begin{bmatrix} 1 & -(1+q) & q & 0 & 0 & 0 & \dots \\ 0 & 1 & -(1+q) & q & 0 & 0 & \dots \\ 0 & 0 & 1 & -(1+q) & q & 0 & \dots \\ 0 & 0 & 0 & 1 & -(1+q) & q & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ \vdots \end{bmatrix} = \mu \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ \vdots \end{bmatrix},$$

we get the following linear equations

$$\begin{aligned} u_0 - (1+q)u_1 + qu_2 &= \mu u_0, \\ u_1 - (1+q)u_2 + qu_3 &= \mu u_1, \\ u_2 - (1+q)u_3 + qu_4 &= \mu u_2, \\ &\vdots \\ u_k - (1+q)u_{k+1} + qu_{k+2} &= \mu u_k, \\ &\vdots \end{aligned}$$

Now, if we take  $\mu = 1$ , then we get an eigenvector  $(1, 0, 0, \dots)$  corresponding to  $\mu = 1$ . We consider  $\mu \in S_1$  other than 1. The above linear equations can also be expressed in terms of  $u_1$  and  $u_0$  as

$$\begin{aligned} u_2 &= \frac{(1+q)}{q} u_1 - \frac{1-\mu}{q} u_0, \\ u_3 &= \frac{(1+q)^2 - q(1-\mu)}{q^2} u_1 - \frac{(1-\mu)(1+q)}{q^2}, \\ &\vdots \\ u_k &= \frac{m_k(1-\mu)^k}{q^{k-1}} u_1 - \frac{m_{k-1}(1-\mu)^k}{q^{k-1}} u_0, \quad k \geq 2, \end{aligned} \tag{2}$$

in which  $(m_k)$  comes from equation (1). Now, we can find the eigenvector  $(u_k)$ , for  $\mu \neq 1$ . Here we can make a choice for  $u_0$  and  $u_1$ . Let  $u_0 = 1$  and  $u_1 = \frac{2(1-\mu)}{(1+q) + \sqrt{(1+q)^2 - 4q(1-\mu)}}$ .

Already, we have obtained that  $w_1$  and  $w_2$  are roots of the characteristic equation  $(1-\mu)w^2 - (1+q)w + q = 0$ . So, we get  $w_1 \cdot w_2 = \frac{q}{1-\mu}$ , and  $w_1 - w_2 = \frac{\sqrt{(1+q)^2 - 4q(1-\mu)}}{1-\mu}$ .

It can also be seen that  $u_1 = \frac{1}{w_1}$ . Using these facts in the relation of the sequence  $(u_k)$ , we get

$$\begin{aligned}
 u_k &= \frac{m_k(1-\mu)^k}{q^{k-1}}u_1 - \frac{m_{k-1}(1-\mu)^k}{q^{k-1}}u_0 \\
 &= \left(\frac{1-\mu}{q}\right)^{k-1} (1-\mu)(-m_{k-1}u_0 + m_k u_1) \\
 &= \frac{1}{(w_1 w_2)^{k-1}} (1-\mu)(-m_{k-1} + m_k \frac{1}{w_1}) \\
 &= \frac{1}{(w_1 w_2)^{k-1}} (1-\mu) \left( \frac{-w_1^{k-1} + w_2^{k-1}}{\sqrt{(1+q)^2 - 4q(1-\mu)}} + \frac{w_1^k - w_2^k}{\sqrt{(1+q)^2 - 4q(1-\mu)}} w_1^{-1} \right) \\
 &= \frac{1}{w_1^{k-1} w_2^{k-1}} \frac{1-\mu}{\sqrt{(1+q)^2 - 4q(1-\mu)}} (-w_1^{k-1} + w_2^{k-1} + w_1^{k-1} - w_2^k w_1^{-1}) \\
 &= \frac{1}{w_1^{k-1} w_2^{k-1}} \left( \frac{1}{w_1 - w_2} \right) w_2^{k-1} \left( 1 - \frac{w_2}{w_1} \right) \\
 &= \frac{1}{w_1^{k-1}} \frac{1}{w_1} \\
 &= \frac{1}{w_1^k} \\
 &= u_1^k.
 \end{aligned}$$

Finally, we obtained that the sequence  $(u_k) = u_1^k$ , for all  $k \geq 2$ . If we consider  $w_1 = w_2$ , i.e., for the case  $(1+q)^2 = 4q(1-\mu)$ , we get the same result. Thus, it is clearly seen that the sequence  $u = (u_k) \in l_p^*$ , since  $|u_1| < 1$ . As a result,  $S_1 \subseteq \sigma_p((\Delta_q^2)^*, l_q)$ .

Now, we consider  $\mu \notin S_1$ , i.e.,

$$\begin{aligned}
 |2(1-\mu)| &\geq \left| (1+q) + \sqrt{(1+q)^2 - 4(1-\mu)q} \right| \\
 \implies \left| \frac{1}{w_1} \right| &\geq 1 \\
 \implies |w_1| &\leq 1.
 \end{aligned}$$

Here, we have to show that  $\mu \notin \sigma_p((\Delta_q^2)^*, l_q)$ . From equation (2), it is obtained that

$$\frac{u_{k+1}}{u_k} = \left( \frac{1-\mu}{q} \right) \left( \frac{m_k}{m_{k-1}} \right) \begin{pmatrix} -u_0 + \frac{m_{k+1}}{m_k} u_1 \\ -u_0 + \frac{m_k}{m_{k-1}} u_1 \end{pmatrix}.$$

Based on the roots  $w_1$  and  $w_2$ , we will consider three cases.

*Case 1:*  $|w_2| < |w_1| \leq 1$ .

For this case, we get  $(1+q)^2 \neq 4q(1-\mu)$  and

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{m_k}{m_{k-1}} &= \lim_{k \rightarrow \infty} \frac{m_{k+1}}{m_k} = \lim_{k \rightarrow \infty} \frac{w_1^{k+1} - w_2^{k+1}}{w_1^k - w_2^k} \\ &= \lim_{k \rightarrow \infty} \frac{w_1^{k+1} \left[ 1 - \left( \frac{w_2}{w_1} \right)^{k+1} \right]}{w_1^k \left[ 1 - \left( \frac{w_2}{w_1} \right)^k \right]} \\ &= w_1. \end{aligned}$$

Again, from equation (2), we get

$$u_k = \left( \frac{1-\mu}{q} \right)^{k-1} (1-\mu)(-m_{k-1}u_0 + m_k u_1). \quad (3)$$

If  $-u_0 + w_1 u_1 = 0$ , then from equation (3), we find  $(u_k) = \left( \frac{u_0}{w_1^k} \right)$ , which doesn't belong to  $l_q$  as  $|w_1| \leq 1$ . Otherwise,

$$\lim_{k \rightarrow \infty} \left| \frac{u_{k+1}}{u_k} \right|^q = \frac{1}{|w_1|^q |w_2|^q} |w_1|^q = \frac{1}{|w_2|^q} > 1.$$

*Case 2:*  $|w_2| = |w_1| < 1$ .

For this case, we get  $(1+q)^2 = 4q(1-\mu)$  and, using the formula

$$m_k = \left( \frac{2k}{1+q} \right) \left[ \frac{1+q}{2(1-\mu)} \right]^k, \text{ for all } k \geq 1,$$

we get that

$$\lim_{k \rightarrow \infty} \left| \frac{m_k}{m_{k-1}} \right|^q = \left| \frac{1+q}{2(1-\mu)} \right|^q = |w_1|^q$$

which leads to

$$\lim_{k \rightarrow \infty} \left| \frac{u_{k+1}}{u_k} \right|^q = \frac{1}{|w_1|^q |w_2|^q} |w_1|^q = \frac{1}{|w_2|^q} > 1.$$

This implies that  $(u_k)$  doesn't belong to  $l_q$ .

*Case 3:*  $|w_1| = |w_2| = 1$ .

For this case, we get  $(1+q)^2 = 4q(1-\mu)$  and  $\left| \frac{1+q}{2q} \right| = 1$ . Assume that  $\mu \in \sigma_p((\Delta_q^2)^*, l_q)$ , then

there exists  $\theta \neq u \in l_q$ . Now, rewriting equation (2), we get

$$\begin{aligned} u_k &= \frac{m_k(1-\mu)^k}{q^{k-1}}u_1 - \frac{m_{k-1}(1-\mu)^k}{q^{k-1}}u_0 \\ &= \frac{\binom{2k}{1+q} \left(\frac{1+q}{2(1-\mu)}\right)^k (1-\mu)^k}{q^{k-1}}u_1 - \frac{\binom{2(k-1)}{1+q} \left(\frac{1+q}{2(1-\mu)}\right)^{k-1} (1-\mu)^k}{q^{k-1}}u_0 \\ &= \frac{k(1+q)^{k-1}}{(2q)^{k-1}}u_1 - \frac{(k-1) \left(\frac{1+q}{2}\right)^{k-2} (1-\mu)}{q^{k-1}}u_0 \\ &= \frac{k(1+q)^{k-1}}{(2q)^{k-1}}u_1 - \frac{(k-1) \left(\frac{1+q}{2}\right)^{k-2} \frac{(1+q)^2}{4q}}{q^{k-1}}u_0 \\ &= \frac{k(1+q)^{k-1}}{(2q)^{k-1}}u_1 - \frac{(k-1)(1+q)^k}{(2q)^k}u_0 \\ &= \left(\frac{1+q}{2q}\right)^{k-1} \left[ku_1 - (k-1)\frac{1+q}{2q}u_0\right]. \end{aligned}$$

Since  $\lim_{k \rightarrow \infty} u_k = 0 \implies \lim_{k \rightarrow \infty} \left[ku_1 - (k-1)\frac{1+q}{2q}u_0\right] = 0$ , and we must have  $u_0 = u_1 = 0$ . Consequently, it implies  $u = \theta$ , a contradiction. So, we get  $\mu \notin \sigma_p((\Delta_q^2)^*, l_q)$ . Thus,  $\sigma_p((\Delta_q^2)^*, l_q) \subseteq S_1$ , and hence  $\sigma_p((\Delta_q^2)^*, l_q) = S_1$ .

*Theorem 5.* The residual spectrum:

$$\sigma_r(\Delta_q^2, l_p) = \left\{ \mu \in \mathbb{C} : |2(1-\mu)| < \left| (1+q) + \sqrt{(1+q)^2 - 4(1-\mu)q} \right| \right\}.$$

*Proof.* From Lemma 2, we get  $\sigma_r(\Delta_q^2, l_p) = \sigma_p((\Delta_q^2)^*, l_q) \setminus \sigma_p(\Delta_q^2, l_p)$ . Now, applying the Theorems 2 and 4, we get the required result.

*Theorem 6.* The continuous spectrum:

$$\sigma_c(\Delta_q^2, l_p) = \left\{ \mu \in \mathbb{C} : |2(1-\mu)| = \left| (1+q) + \sqrt{(1+q)^2 - 4(1-\mu)q} \right| \right\}.$$

*Proof.* We have  $\sigma(\Delta_q^2, l_p) = \sigma_p(\Delta_q^2, l_p) \cup \sigma_r(\Delta_q^2, l_p) \cup \sigma_c(\Delta_q^2, l_p)$  and the corresponding sets are pairwise disjoint. Now, applying Theorems 3, 4 and 5, we get the required result.

*Theorem 7.*  $P_3\sigma(\Delta_q^2, l_p) = Q_3\sigma(\Delta_q^2, l_p) = R_3\sigma(\Delta_q^2, l_p) = \phi$ .

*Proof.* From Table 1, we get  $\sigma_p(\Delta_q^2, l_p) = P_3\sigma(\Delta_q^2, l_p) \cup Q_3\sigma(\Delta_q^2, l_p) \cup R_3\sigma(\Delta_q^2, l_p)$ . Again, from Theorem 2, we get  $\sigma_p(\Delta_q^2, l_p) = \phi$ , and consequently we get the required result.

*Theorem 8.* The operator  $\Delta_q^2$  satisfies the following relations:

- (a)  $Q_2\sigma(\Delta_q^2, l_p) = \left\{ \mu \in \mathbb{C} : |2(1-\mu)| = \left| (1+q) + \sqrt{(1+q)^2 - 4(1-\mu)q} \right| \right\}$ ,
- (b)  $R_2\sigma(\Delta_q^2, l_p) \supseteq \left\{ \mu \in \mathbb{C} : |2(1-\mu)| < \left| (1+q) + \sqrt{(1+q)^2 - 4(1-\mu)q} \right| \right\} \setminus \{1\}$ ,
- (c)  $R_1\sigma(\Delta_q^2, l_p) \subseteq \{1\}$ .

*Proof.* We have from Table 1 that  $\sigma_c(\Delta_q^2, l_p) = Q_2\sigma(\Delta_q^2, l_p)$ . Now, using Theorem 6, we get the result of (a).

Again, from Theorem 3, if for any  $\mu \in \sigma_r(\Delta_q^2, l_p) \setminus \{1\}$  then the operator  $(\Delta_q^2 - \mu I)^{-1} \notin \Delta_q^2(l_p)$ . From this, we get  $\sigma_r(\Delta_q^2, l_p) \setminus \{1\} \subseteq R_2\sigma(\Delta_q^2, l_p)$  and  $R_1\sigma(\Delta_q^2, l_p) \subseteq \{1\}$ .

*Theorem 9.* For the operator  $\Delta_q^2$  the following results hold.

- (a)  $\sigma_{ap}(\Delta_q^2, l_p) \supseteq \left\{ \mu \in \mathbb{C} : |2(1 - \mu)| \leq \left| (1 + q) + \sqrt{(1 + q)^2 - 4(1 - \mu)q} \right| \right\} \setminus \{1\}$ ,
- (b)  $\sigma_{ap}((\Delta_q^2)^*, l_p^*) = \left\{ \mu \in \mathbb{C} : |2(1 - \mu)| \leq \left| (1 + q) + \sqrt{(1 + q)^2 - 4(1 - \mu)q} \right| \right\}$ ,
- (c)  $\sigma_\delta(\Delta_q^2, l_p) = \left\{ \mu \in \mathbb{C} : |2(1 - \mu)| \leq \left| (1 + q) + \sqrt{(1 + q)^2 - 4(1 - \mu)q} \right| \right\}$ ,
- (d)  $\delta_{co}(\Delta_q^2, l_p) = \left\{ \mu \in \mathbb{C} : |2(1 - \mu)| < \left| (1 + q) + \sqrt{(1 + q)^2 - 4(1 - \mu)q} \right| \right\}$ .

*Proof.* (a) From Table 1, we get

$$\sigma_{ap}(\Delta_q^2, l_p) = \sigma(\Delta_q^2, l_p) \setminus R_1 \sigma(\Delta_q^2, l_p).$$

Now, applying the Theorems 3 and 8, we get the required result.

- (b) The result in (b) is obtained from the relation (e) in Proposition 1.
- (c) The result in (c) is obtained from the relation (b) in Proposition 1.
- (d) The result in (d) is obtained from the relation (d) in Proposition 1.

#### 4 Example

Taking particular values for  $q \in (0, 1)$ , we construct some examples of spectrum of  $\Delta_q^2$  that are given below.

- (i) If  $q = \frac{1}{2}$ , then the spectrum of  $\Delta_{1/2}^2$  is given by

$$\sigma(\Delta_{1/2}^2, l_p) = \left\{ \mu \in \mathbb{C} : |2(1 - \mu)| \leq \left| \frac{3}{2} + \sqrt{\frac{9}{4} - 2(1 - \mu)} \right| \right\}.$$

- (ii) If  $q = \frac{1}{4}$ , then the spectrum of  $\Delta_{1/4}^2$  is given by

$$\sigma(\Delta_{1/4}^2, l_p) = \left\{ \mu \in \mathbb{C} : |2(1 - \mu)| \leq \left| \frac{5}{4} + \sqrt{\frac{25}{16} - (1 - \mu)} \right| \right\}.$$

The graphical representation of the spectra for these examples is presented in Figures 1 and 2.

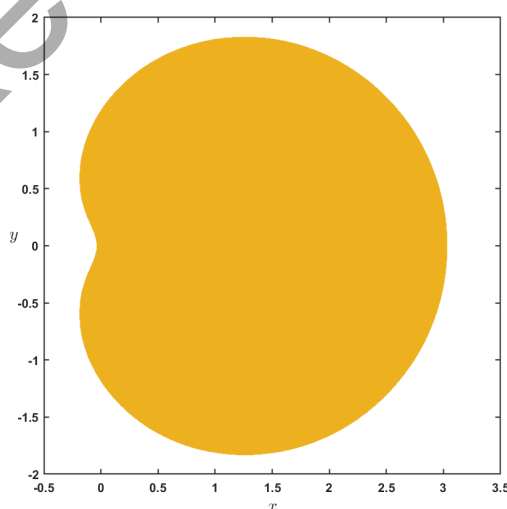
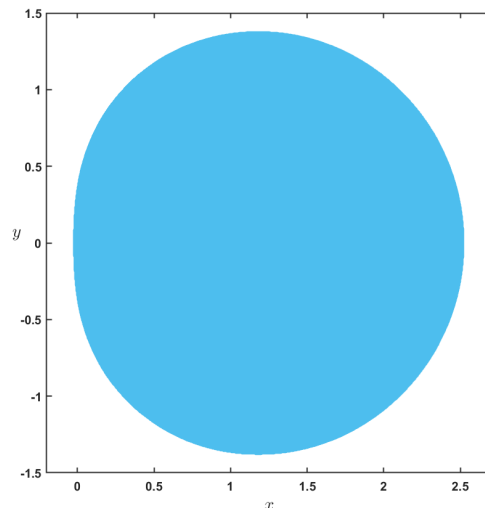


Figure 1. Spectrum of  $\Delta_{1/2}^2$

Figure 2. Spectrum of  $\Delta_{1/4}^2$ 

### Conclusion

In our study, we have determined the spectrum and fine spectrum of  $q$ -analog of second order difference operator. This operator reduces to second order difference operator when  $q \rightarrow 1$ . Like a generalized difference operator, we have a generalized quantum difference operator. Spectral analysis of generalized quantum difference operator can also be done in different sequence spaces.

### Author Contributions

All authors contributed equally to this work.

### Conflict of Interest

The authors declare no conflict of interest.

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