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Criteria for the boundedness of a certain class of matrix operators from l_{pv} into l_{qu}

One of the main aims in the theory of matrices is to find necessary and sufficient conditions for the elements of any matrix so that the corresponding matrix operator maps continuously from one normed space into another one. Thus, it is very important to find the norm of the matrix operator, at least, to find upper and lower estimates of it. This problem in Lebesgue spaces of sequences in the general case is still open. This paper deals with the problem of boundedness of matrix operators from l_{pv} into l_{qu} for $1 < q < p < \infty$, and we obtain necessary and sufficient conditions of this problem when matrix operators belong to the classes $O_2^{\pm}$ satisfying weaker conditions than Oinarov's condition.

Keywords: matrix operator, conjugate operator, weight sequence, boundedness, weight inequalities, weight Lebesgue space, Oinarov's condition, Hardy operator, Hardy inequality, matrix.

Introduction

Let $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$. Let $u = \{u_i\}, v = \{v_i\}$ be sequences of positive numbers, which will be called the weight sequences. Let l_{pv} the space of all sequences $f = \{f_i\}_{i=1}^{\infty}$ of real numbers such that $\|f\|_{pv} = \left(\sum\limits_{i=1}^{\infty} |v_i f_i|^p \right)^{\frac{1}{p}}$, $1 \leq p < \infty$.

We consider the problem of boundedness for the following matrix operators

$$\left(A^+ f\right)_i = \sum_{j=1}^{i} a_{ij} f_j, \; i \geq 1, \tag{1}$$

$$\left(A^- f\right)_j = \sum_{i=j}^{\infty} a_{ij} f_i, \; j \geq 1 \tag{2}$$

from l_{pv} into l_{qu}, where $a_{ij} > 0, i \geq j \geq 1$, i.e. the vaidity of the inequality

$$\|A^{\pm} f\|_{qu} \leq C \|f\|_{pv}, \forall f \in l_{pv}. \tag{3}$$

The matrix operators (1), (2) were studied in many papers in different sequence spaces. The almost complete collection of these results is presented in the work by M. Stieglitz and H. Tietz [1]. There the mappings of matrix operators are considered in 11 sequence spaces except its mapping from l_{pv} into l_{qu}. The remaining case is still an open problem.

When $a_{ij} = 1, i \geq j \geq 1$ operators (1), (2) coincide with the discrete Hardy operators, which have been studied by many researchers, and main results were obtained in [2–7].

In the general case, the question is open on conditions on the entries of a matrix (a_{ij}) that giving boundedness of operators (1) and (2). For several classes of matrices, criteria for boundedness of the

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operators (1), (2) are known. One of the first studied classes was the class of operators matrices of which satisfy the following discrete Oinarov's condition: there exists $d \geq 1$ such that

$$\frac{1}{d}\left(a_{ik}+a_{kj}\right) \leq a_{ij} \leq d\left(a_{ik}+a_{kj}\right)$$

for all $i \geq k \geq j \geq 1$ (see [8], [9]).

In 2012 in paper [10] the wide classes O_n^+, O_n^-, $n \geq 0$ of matrices were presented, which defined by conditions on a matrix (a_{ij}) that are weaker than Oinarov's condition, and the necessary and sufficient conditions for boundedness of these operators for $1 < p \leq q < \infty$ were obtained, where their matrices belonged to these classes. However, the problem of boundedness of operators (1) and (2) with matrix from the classes O_n^+, O_n^-, $n > 1$ for the case $1 < q < p < \infty$ is still open. But the first results for this case - the criteria of boundedness for matrix operators from $O_1^{\pm}$ are found in [11], [12].

In the present paper, we find criteria of boundedness for operators (1), (2) from l_{pv} into l_{qu}, where their matrices belong to the class $O_2^{\pm}$ when $1 < q < p < \infty$.

Convention: The symbol $M << K$ means that $M \leq cK$, where c > 0 is a constant depending only on unessential parameters. If $M << K << M$, then we write $M \approx K$.

We assume $g_i = 0$ when $i < 1$ and $\Delta^- g_i = g_i - g_{i-1}, \Delta^+ g_i = g_i - g_{i+1}$.

1 Preliminaries

Let's give the definition of classes $O_1^{\pm}$, $O_2^{\pm}$.

Definition 1. Let (a_{ij}) be a matrix which is non-negative and non-decreasing in the first index for all $i \geq j \geq 1$. A matrix (a_{ij}) belongs to the class O_1^+, if there exist a non-negative matrix $(a_{ij}^{1,0})$, a number $r_1 \geq 1$ such that the estimates

$$\frac{1}{r_1}\left(a_{ik}^{1,0}+a_{kj}\right) \leq a_{ij} \leq r_1\left(a_{ik}^{1,0}+a_{kj}\right)$$

hold for all $i \geq k \geq j \geq 1$.

Definition 2. Let (a_{ij}) be a matrix which is non-negative and non-increasing in the second index for all $i \geq j \geq 1$. A matrix (a_{ij}) belongs to the class O_1^-, if there exist a non-negative matrix $(a_{ij}^{0,1})$, a number $\bar{r}_1 \geq 1$ such that the estimates

$$\frac{1}{\bar{r}_1}\left(a_{ik}+a_{kj}^{0,1}\right) \leq a_{ij} \leq \bar{r}_1\left(a_{ik}+a_{kj}^{0,1}\right)$$

hold for all $i \geq k \geq j \geq 1$.

Definition 3. Let (a_{ij}) be a matrix which is non-negative and non-decreasing in the first index for all $i \geq j \geq 1$. A matrix (a_{ij}) belongs to the class O_2^+, if there exist a non-negative matrices $(a_{ij}^{2,0})$, $(a_{ij}^{2,1})$, $(a_{ij}^{(1)})$, a number $r_2 \geq 1$ such that $(a_{ij}^{(1)}) \in O_1^+$,

$$\frac{1}{r_2}\left(a_{ik}^{2,0}+a_{ik}^{2,1}a_{kj}^{(1)}+a_{kj}\right) \leq a_{ij} \leq r_2\left(a_{ik}^{2,0}+a_{ik}^{2,1}a_{kj}^{(1)}+a_{kj}\right)$$

for all $i \geq k \geq j \geq 1$.

Definition 4. Let (a_{ij}) be a matrix which is non-negative and non-increasing in the second index for all $i \geq j \geq 1$. A matrix (a_{ij}) belongs to the class O_2^-, if there exist non-negative matrices $(a_{ij}^{0,2})$, $(a_{ij}^{1,2})$, $(a_{ij}^{(1)})$, a number $\bar{r}_1 \geq 1$ such that $(a_{ij}^{(1)}) \in O_1^-$,

$$\frac{1}{\bar{r}_2}\left(a_{ik}+a_{ik}^{(1)}a_{kj}^{1,2}+a_{kj}^{0,2}\right) \leq a_{ij} \leq \bar{r}_2\left(a_{ik}+a_{ik}^{(1)}a_{kj}^{1,2}+a_{kj}^{0,2}\right)$$

for all $i \geq k \geq j \geq 1$.

Let us consider some examples of matrices that belong to the classes $O_1^{\pm}$ and $O_2^{\pm}$.

Example 1. Let $\alpha > 0$. Let $\{a_i\}_{i=1}^{\infty}$ be a non-decreasing positive sequence and $\{b_i\}_{i=1}^{\infty}$ be an arbitrary positive sequence, such that $a_i \geq b_j$, $i \geq j \geq 1$. Then $a_{ij} = a_{ij}^{(1)} := \left(\ln \frac{a_i}{b_j}\right)^{\alpha} \in O_1^+$, when $i \geq j \geq 1$.

Indeed, for all $i \geq k \geq j \geq 1$

$$a_{ij}^{(1)} = \left(\ln \frac{a_i}{a_k} \cdot \frac{a_k}{b_j}\right)^{\alpha} \approx \left(\ln \frac{a_i}{a_k}\right)^{\alpha} + \left(\ln \frac{a_k}{b_j}\right)^{\alpha} = a_{ik}^{1,0} + a_{kj}^{(1)},$$

where $a_{ik}^{1,0} = \left(\ln \frac{a_i}{a_k}\right)^{\alpha}$.

Example 2. Let $\{a_i\}_{i=1}^{\infty}$ and $\{b_i\}_{i=1}^{\infty}$ satisfy the conditions from Example 1. Moreover, we assume that $\{\omega_i\}_{i=1}^{\infty}$ is a non-negative sequence. Then $a_{ij} = a_{ij}^{(2)} := \sum\limits_{s=j}^{i} \omega_s \left(\ln \frac{a_s}{b_j}\right)^{\alpha} \in O_2^+$, $i \geq j \geq 1$.

Indeed, for all $i \geq k \geq j \geq 1$ we have

$$a_{ij}^{(2)} = \sum_{s=j}^{i} \omega_s \left(\ln \frac{a_s}{b_j}\right)^{\alpha} \approx \sum_{s=j}^{k} \omega_s \left(\ln \frac{a_s}{b_j}\right)^{\alpha} + \sum_{s=k}^{i} \omega_s \left(\ln \frac{a_s}{b_j}\right)^{\alpha} \approx$$

$$\approx a_{kj}^{(2)} + \sum_{s=k}^{i} \omega_s \left(\ln \frac{a_s}{a_k}\right)^{\alpha} + \left(\ln \frac{a_k}{b_j}\right)^{\alpha} \sum_{s=k}^{i} \omega_s =$$

$$= a_{ik}^{2,0} + a_{ik}^{2,1} a_{kj}^{(1)} + a_{kj}^{(2)},$$

where $a_{kj}^{(1)} = \left(\ln \frac{a_k}{b_j}\right)^{\alpha} \in O_1^+$, $a_{ik}^{2,0} = \sum\limits_{s=k}^{i} \omega_s \left(\ln \frac{a_s}{a_k}\right)^{\alpha}$, $a_{ik}^{2,1} = \sum\limits_{s=k}^{i} \omega_s$, $i \geq k \geq j \geq 1$.

In the same way, one can show that $a_{ij}^{(1)} = \left(\ln \frac{a_i}{b_j}\right)^{\alpha} \in O_1^-$ and $a_{ij}^{(2)} := \sum\limits_{s=j}^{i} \omega_s \left(\ln \frac{a_s}{b_j}\right)^{\alpha} \in O_2^-$, $i \geq j \geq 1$, if $\{a_i\}_{i=1}^{\infty}$ is an arbitrary positive sequence and $\{b_i\}_{i=1}^{\infty}$ is a non-decreasing positive sequence, such that $a_i \geq b_j$, $i \geq j \geq 1$.

Remark 1. As it is shown in [10] the matrices $(a_{ij}^{2,0})$, $(a_{ij}^{2,1})$, $(a_{ij}^{(1)})$, $(a_{ij}^{0,2})$, $(a_{ij}^{1,2})$ can be considered non-decreasing in i and non-increasing in j.

Lemma A. [9] *Let $\gamma > 0$, $1 \leq n < N \leq \infty$ and let $\{h_k\}$ be a non-negative sequence. Then*

$$\left(\sum_{k=n}^{N} h_k\right)^{\gamma} \approx \sum_{k=n}^{N} \left(\sum_{i=n}^{k} h_i\right)^{\gamma-1} h_k, \tag{4}$$

$$\left(\sum_{k=n}^{N} h_k\right)^{\gamma} \approx \sum_{k=n}^{N} \left(\sum_{i=k}^{N} h_i\right)^{\gamma-1} h_k. \tag{5}$$

Let us state the necessary assertions from [5], [11] in a convenient form.

Theorem A. Let $1 < q < p < \infty$. The inequality

$$\left(\sum_{k=1}^{\infty} \left|\sum_{j=1}^{k} f_j\right|^q u_k^q\right)^{\frac{1}{q}} \leq C \left(\sum_{k=1}^{\infty} |f_k v_k|^p\right)^{\frac{1}{p}}, \forall f \in l_{pv} \tag{6}$$

holds if and only if

$$F = \left(\sum_{k=1}^{\infty} \left(\sum_{j=k}^{\infty} u_j^q \right)^{\frac{p}{p-q}} \left(\sum_{i=1}^{k} v_i^{-p'} \right)^{\frac{p(q-1)}{p-q}} v_k^{-p'} \right)^{\frac{p-q}{pq}} < \infty.$$

Moreover, $F \approx C$, where C is the best constant in (6).

Theorem B. Let $1 < q < p < \infty$ and the matrix (a_{ij}) belongs to the class O_1^+. Then inequality (3) for operator (1) holds if and only if $B = \max\{B_0, B_1\} < \infty$, where

$$B_0 = \left(\sum_{k=1}^{\infty} \left(\sum_{j=k}^{\infty} (a_{jk}^{1,0})^q u_j^q \right)^{\frac{p}{p-q}} \left(\sum_{i=1}^{k} v_i^{-p'} \right)^{\frac{p(q-1)}{p-q}} v_k^{-p'} \right)^{\frac{p-q}{pq}},$$

$$B_1 = \left(\sum_{k=1}^{\infty} \left(\sum_{j=k}^{\infty} u_j^q \right)^{\frac{q}{p-q}} \left(\sum_{i=1}^{k} a_{ki}^{p'} v_i^{-p'} \right)^{\frac{q(p-1)}{p-q}} u_k^q \right)^{\frac{p-q}{pq}}.$$

Moreover, $B \approx C$, where C is the best constant in (3).

2 *Main results*

Our main results read.

Theorem 1. Let $1 < q < p < \infty$ and $(a_{ij}) \in O_2^+$. Then operator (1) is bounded from l_{pv} into l_{qu} if and only if $M^+ = \max\{M_{2,0}^+, M_{2,1}^+, M_{2,2}^+\}$, where

$$M_{2,0}^+ = \left(\sum_{i=1}^{\infty} \left(\sum_{s=i}^{\infty} (a_{si}^{2,0})^q u_s^q \right)^{\frac{p}{p-q}} \left(\sum_{j=1}^{i} v_j^{-p'} \right)^{\frac{p(q-1)}{p-q}} v_i^{-p'} \right)^{\frac{p-q}{pq}},$$

$$M_{2,1}^+ = \left(\sum_{i=1}^{\infty} \left(\sum_{s=i}^{\infty} \left(a_{si}^{2,1} \right)^q u_s^q \right)^{\frac{p}{p-q}} \left(\sum_{j=1}^{i} \left(a_{ij}^{(1)} \right)^{p'} v_j^{-p'} \right)^{\frac{p(q-1)}{p-q}} \Delta^- \left(\sum_{j=1}^{i} \left(a_{ij}^{(1)} \right)^{p'} v_j^{-p'} \right) \right)^{\frac{p-q}{pq}},$$

$$M_{2,2}^+ = \left(\sum_{i=1}^{\infty} \left(\sum_{s=i}^{\infty} u_s^q \right)^{\frac{p}{p-q}} \left(\sum_{j=1}^{i} a_{ij}^{p'} v_j^{-p'} \right)^{\frac{p(q-1)}{p-q}} \Delta^- \left(\sum_{j=1}^{i} (a_{ij})^{p'} v_j^{-p'} \right) \right)^{\frac{p-q}{pq}}.$$

Moreover, $\|A^+\|_{pv \to qu} \approx M^+$, where $\|A^+\|_{pv \to qu}$ is the norm of operator A^+ from l_{pv} into l_{qu}.

Our corresponding result for operator (2) reads as follows.

Theorem 2. Let $1 < q < p < \infty$ and $(a_{ij}) \in O_2^-$. Then operator (2) is bounded from l_{pv} into l_{qu} if and only if $\mathcal{M}^- = \max\{\mathcal{M}_{0,2}^-, \mathcal{M}_{1,2}^-, \mathcal{M}_{2,2}^-\}$, where

$$\mathcal{M}_{0,2}^- = \left(\sum_{i=1}^{\infty} \left(\sum_{s=1}^{i} (a_{is}^{0,2})^q u_s^q \right)^{\frac{p}{p-q}} \left(\sum_{j=1}^{\infty} v_j^{-p'} \right)^{\frac{p(q-1)}{p-q}} v_i^{-p'} \right)^{\frac{p-q}{pq}},$$

$$\mathcal{M}_{1,2}^{-}=\left(\sum_{i=1}^{\infty}\left(\sum_{s=1}^{i}\left(a_{is}^{1,2}\right)^{q}u_s^q\right)^{\frac{p}{p-q}}\left(\sum_{j=i}^{\infty}\left(a_{ji}^{(1)}\right)^{p'}v_j^{-p'}\right)^{\frac{p(q-1)}{p-q}}\Delta^{+}\left(\sum_{j=i}^{\infty}\left(a_{ji}^{(1)}\right)^{p'}v_j^{-p'}\right)\right)^{\frac{p-q}{pq}},$$

$$\mathcal{M}_{2,2}^{-}=\left(\sum_{i=1}^{\infty}\left(\sum_{s=1}^{i}u_s^q\right)^{\frac{p}{p-q}}\left(\sum_{j=i}^{\infty}a_{ji}^{p'}v_j^{-p'}\right)^{\frac{p(q-1)}{p-q}}\Delta^{+}\left(\sum_{j=i}^{\infty}(a_{ji})^{p'}v_j^{-p'}\right)\right)^{\frac{p-q}{pq}}.$$

Moreover, $\|A^{-}\|_{pv\to qu}\approx\mathcal{M}^{-}$, where $\|A^{-}\|_{pv\to qu}$ is the norm of operator A^{-} from l_{pv} into l_{qu}.

Using the conjugacy of operators (1) and (2) from Theorem 1 and Theorem 2 we obtain the following results.

Theorem 3. Let $1<q<p<\infty$ and $(a_{ij})\in O_2^{+}$. Then operator (2) is bounded from l_{pv} into l_{qu} if and only if $M^{-}=\max\{M_{2,0}^{-},M_{2,1}^{-},M_{2,2}^{-}\}$, where

$$M_{2,0}^{-}=\left(\sum_{i=1}^{\infty}\left(\sum_{s=i}^{\infty}(a_{si}^{2,0})^{p'}v_s^{-p'}\right)^{\frac{q(p-1)}{p-q}}\left(\sum_{j=1}^{i}u_j^q\right)^{\frac{q}{p-q}}u_i^q\right)^{\frac{p-q}{pq}},$$

$$M_{2,1}^{-}=\left(\sum_{i=1}^{\infty}\left(\sum_{s=i}^{\infty}(a_{si}^{2,1})^{p'}v_s^{-p'}\right)^{\frac{q(p-1)}{p-q}}\left(\sum_{j=1}^{i}(a_{ij}^{(1)})^{q}u_j^q\right)^{\frac{q}{p-q}}\Delta^{-}\left(\sum_{j=1}^{i}(a_{ij}^{(1)})^{q}u_j^q\right)\right)^{\frac{p-q}{pq}},$$

$$M_{2,2}^{-}=\left(\sum_{i=1}^{\infty}\left(\sum_{s=i}^{\infty}v_s^{-p'}\right)^{\frac{q(p-1)}{p-q}}\left(\sum_{j=1}^{i}a_{ij}^{q}u_j^q\right)^{\frac{q}{p-q}}\Delta^{-}\left(\sum_{j=1}^{i}a_{ij}^{q}u_j^q\right)\right)^{\frac{p-q}{pq}}.$$

Moreover, $\|A^{-}\|_{pv\to qu}\approx M^{-}$, where $\|A^{-}\|_{pv\to qu}$ is the norm of operator A^{-} from l_{pv} into l_{qu}.

Theorem 4. Let $1<q<p<\infty$ and $(a_{ij})\in O_2^{-}$. Then operator (1) is bounded from l_{pv} into l_{qu} if and only if $\mathcal{M}^{+}=\max\{\mathcal{M}_{0,2}^{+},\mathcal{M}_{1,2}^{+},\mathcal{M}_{2,2}^{+}\}$, where

$$\mathcal{M}_{0,2}^{+}=\left(\sum_{i=1}^{\infty}\left(\sum_{s=1}^{i}(a_{is}^{2,0})^{p'}v_s^{-p'}\right)^{\frac{p}{p-q}}\left(\sum_{j=i}^{\infty}u_j^q\right)^{\frac{p(q-1)}{p-q}}u_i^q\right)^{\frac{p-q}{pq}},$$

$$\mathcal{M}_{1,2}^{+}=\left(\sum_{i=1}^{\infty}\left(\sum_{s=1}^{i}(a_{is}^{1,2})^{p'}v_s^{-p'}\right)^{\frac{p}{p-q}}\left(\sum_{j=i}^{\infty}(a_{ji}^{(1)})^{q}u_j^q\right)^{\frac{p(q-1)}{p-q}}\Delta^{+}\left(\sum_{j=i}^{\infty}(a_{ji}^{(1)})^{q}u_j^q\right)\right)^{\frac{p-q}{pq}},$$

$$\mathcal{M}_{2,2}^{+}=\left(\sum_{i=1}^{\infty}\left(\sum_{s=1}^{i}v_s^{-p'}\right)^{\frac{p}{p-q}}\left(\sum_{j=i}^{\infty}a_{ji}^{q}u_j^q\right)^{\frac{p(q-1)}{p-q}}\Delta^{+}\left(\sum_{j=i}^{\infty}a_{ji}^{q}u_j^q\right)\right)^{\frac{p-q}{pq}}.$$

Moreover, $\|A^{+}\|_{pv\to qu}\approx\mathcal{M}^{+}$, where $\|A^{+}\|_{pv\to qu}$ is the norm of operator A^{+} from l_{pv} into l_{qu}.

Since the proof of Theorem 2 is completely analogous to the proof of Theorem 1, we introduce the proof of Theorem 1.

Proof. Necessary. Let operator (1) be bounded from l_{pv} into l_{qu}, $\|A^+\|_{pv\to qu} < \infty$, i.e. the following inequality holds:

$$\left(\sum_{i=1}^{\infty}\left(\sum_{j=1}^{i} a_{ij} f_j\right)^q u_i^q\right)^{\frac{1}{q}} \leq \|A^+\|_{pv\to qu}\left(\sum_{i=1}^{\infty} f_i^p v_i^p\right)^{\frac{1}{p}}, \tag{7}$$

for all non-negative sequences $f \in l_{pv}$, in particular, for non-negative finite sequences $f \in l_{pv}$. By applying (4), a relation $a_{ik} >> a_{ij}^{2,0}, i \geq j \geq k \geq 1$ from Definition 3 and using the Abel transform, we obtain

$$\sum_{i=1}^{\infty}(a_{ij} f_j)^q u_i^q \approx \sum_{i=1}^{\infty}\sum_{j=1}^{i} a_{ij} f_j \left(\sum_{s=1}^{j} a_{is} f_s\right)^{q-1} u_i^q >>$$

$$>> \sum_{i=1}^{\infty}\sum_{j=1}^{i}(a_{ij}^{2,0})^q f_j \left(\sum_{s=1}^{j} f_s\right)^{q-1} u_i^q = \sum_{j=1}^{\infty} f_j \left(\sum_{s=1}^{j} f_s\right)^{q-1} \sum_{i=j}^{\infty}(a_{ij}^{2,0})^q u_i^q =$$

$$= \sum_{j=1}^{\infty}\Delta^-\left(\sum_{n=1}^{j} f_n \left(\sum_{s=1}^{n} f_s\right)^{q-1}\right)\sum_{i=j}^{\infty}(a_{ij}^{2,0})^q u_i^q =$$

$$= \sum_{j=1}^{\infty}\left(\sum_{n=1}^{j} f_n \left(\sum_{s=1}^{n} f_s\right)^{q-1}\right)\Delta^+\left(\sum_{i=j}^{\infty}(a_{ij}^{2,0})^q u_i^q\right) + \lim_{N\to\infty}\left(\sum_{n=1}^{N} f_n \left(\sum_{s=1}^{n} f_s\right)^{q-1}\right)\sum_{i=N+1}^{\infty}(a_{iN+1}^{2,0})^q u_i^q \approx$$

$$\approx \sum_{j=1}^{\infty}\left(\sum_{s=1}^{j} f_s\right)^q \Delta^+\left(\sum_{i=j}^{\infty}(a_{ij}^{2,0})^q u_i^q\right) + \lim_{N\to\infty}\left(\sum_{n=1}^{N} f_n \left(\sum_{s=1}^{n} f_s\right)^{q-1}\right)\sum_{i=N+1}^{\infty}(a_{iN+1}^{2,0})^q u_i^q.$$

Due to the finiteness of f and $a_{ij}^{2,0}$ is non-increasing in j, we have

$$\lim_{N\to\infty}\left(\sum_{n=1}^{N} f_n \left(\sum_{s=1}^{n} f_s\right)^{q-1}\right)\sum_{i=N+1}^{\infty}(a_{iN+1}^{2,0})^q u_i^q = 0.$$

Then

$$\sum_{i=1}^{\infty}\left(\sum_{j=1}^{i} a_{ij} f_j\right)^q u_i^q >> \sum_{j=1}^{\infty}\left(\sum_{s=1}^{j} f_s\right)^q \Delta^+\left(\sum_{i=j}^{\infty}(a_{ij}^{2,0})^q u_i^q\right).$$

Hence and from (7) it follows that

$$\left(\sum_{j=1}^{\infty}\left(\sum_{s=1}^{j} f_s\right)^q \Delta^+\left(\sum_{i=j}^{\infty}(a_{ij}^{2,0})^q u_i^q\right)\right)^{\frac{1}{q}} << \|A^+\|_{pv\to qu}\left(\sum_{i=1}^{\infty}(f_i v_i)^p\right)^{\frac{1}{p}}.$$

Then according to Theorem A, we get

$$\infty > \|A^+\|_{pv\to qu} >> \left(\sum_{k=1}^{\infty}\left(\sum_{j=k}^{\infty}\Delta^+\left(\sum_{i=j}^{\infty}(a_{ij}^{2,0})^q u_i^q\right)\right)^{\frac{p}{p-q}}\left(\sum_{s=1}^{k} v_s^{-p'}\right)^{\frac{p(q-1)}{p-q}} v_k^{-p'}\right)^{\frac{p-q}{pq}} =$$

$$= \left(\sum_{k=1}^{\infty}\left(\sum_{i=k}^{\infty}(a_{ik}^{2,0})^q u_i^q\right)^{\frac{p}{p-q}}\left(\sum_{s=1}^{k} v_s^{-p'}\right)^{\frac{p(q-1)}{p-q}} v_k^{-p'}\right)^{\frac{p-q}{pq}} = M_{2,0}^+. \tag{8}$$

Inequality (7) holds if and only if the following dual inequality

$$\left(\sum_{j=1}^{\infty}\left(\sum_{i=j}^{\infty}a_{ij}g_i\right)^{p'}v_j^{-p'}\right)^{\frac{1}{p'}} << \|A^+\|_{pv\to qu}\left(\sum_{i=1}^{\infty}(g_iu_i^{-1})^{q'}\right)^{\frac{1}{q'}} \tag{9}$$

holds for all non-negative sequences $g\in l_{q',u^{-1}}$, in particular, for non-negative finite sequences $g\in l_{q',u^{-1}}$. Using (5), a relation $a_{ij}<<a_{kj}, k\geq i$ from Definition 3 and applying the Abel transform, we obtain

$$\sum_{j=1}^{\infty}\left(\sum_{i=j}^{\infty}a_{ij}g_i\right)^{p'}v_j^{-p'}\approx\sum_{j=1}^{\infty}\sum_{i=j}^{\infty}a_{ij}g_i\left(\sum_{s=i}^{\infty}a_{sj}g_s\right)^{p'-1}v_j^{-p'}>>$$

$$>>\sum_{j=1}^{\infty}\sum_{i=j}^{\infty}a_{ij}^{p'}g_i\left(\sum_{s=i}^{\infty}g_s\right)^{p'-1}v_j^{-p'}=\sum_{i=1}^{\infty}g_i\left(\sum_{s=i}^{\infty}g_s\right)^{p'-1}\sum_{j=1}^{i}a_{ij}^{p'}v_j^{-p'}=$$

$$=\sum_{i=1}^{\infty}\Delta^+\left(\sum_{n=i}^{\infty}g_n\left(\sum_{s=n}^{\infty}g_s\right)^{p'-1}\right)\sum_{j=1}^{i}a_{ij}^{p'}v_j^{-p'}=$$

$$=\sum_{i=1}^{\infty}\left(\sum_{n=i}^{\infty}g_n\left(\sum_{s=n}^{\infty}g_s\right)^{p'-1}\right)\Delta^-\left(\sum_{j=1}^{i}a_{ij}^{p'}v_j^{-p'}\right)+\lim_{N\to\infty}\left(\sum_{n=N+1}^{\infty}g_n\left(\sum_{s=n}^{\infty}g_s\right)^{p'-1}\right)\sum_{j=1}^{N}a_{Nj}^{p'}v_j^{-p'}\approx$$

$$\approx\sum_{i=1}^{\infty}\left(\sum_{s=i}^{\infty}g_s\right)^{p'}\Delta^-\left(\sum_{j=1}^{i}a_{ij}^{p'}v_j^{-p'}\right)+\lim_{N\to\infty}\left(\sum_{n=N+1}^{\infty}g_n\left(\sum_{s=n}^{\infty}g_s\right)^{p'-1}\right)\sum_{j=1}^{N}a_{Nj}^{p'}v_j^{-p'}.$$

Due to the finiteness of g we have, that

$$\lim_{N\to\infty}\left(\sum_{n=N+1}^{\infty}g_n\left(\sum_{s=n}^{\infty}g_s\right)^{p'-1}\right)\sum_{j=1}^{N}a_{Nj}^{p'}v_j^{-p'}=0.$$

Since $\Delta^-(\sum_{j=1}^{i}a_{ij}^{p'}v_j^{-p'})\geq 0$, we assume $\omega_i=\left(\Delta^-(\sum_{j=1}^{i}a_{ij}^{p'}v_j^{-p'})\right)^{\frac{1}{p'}}$. Then

$$\sum_{j=1}^{\infty}\left(\sum_{i=j}^{\infty}a_{ij}g_i\right)^{p'}v_j^{-p'}>>\sum_{i=1}^{\infty}\left(\sum_{s=i}^{\infty}g_s\right)^{p'}\omega_i^{p'}.$$

Hence and from (9) it follows

$$\left(\sum_{i=1}^{\infty}\left(\sum_{s=i}^{\infty}g_s\right)^{p'}\omega_i^{p'}\right)^{\frac{1}{p'}}<<\|A^+\|_{pv\to qu}\left(\sum_{i=1}^{\infty}(g_iu_i^{-1})^{q'}\right)^{\frac{1}{q'}}. \tag{10}$$

We pass to dual inequality (10), i.e.

$$\left(\sum_{j=1}^{\infty}\left(\sum_{s=1}^{j}f_s\right)^{q}u_j^{q}\right)^{\frac{1}{q}}<<\|A^+\|_{pv\to qu}\left(\sum_{i=1}^{\infty}\left(f_i\omega_i^{-1}\right)^{p}\right)^{\frac{1}{p}},\quad 0\leq f\in l_{pv}.$$

Then by applying Theorem A, we obtain

$$\infty > \|A^{+}\|_{pv\to qu} >> \left(\sum_{k=1}^{\infty}\left(\sum_{j=k}^{\infty}u_j^q\right)^{\frac{p}{p-q}}\left(\sum_{s=1}^{k}\omega_s^{p'}\right)^{\frac{p(q-1)}{p-q}}\omega_k^{p'}\right)^{\frac{p-q}{pq}} =$$

$$= \left(\sum_{k=1}^{\infty}\left(\sum_{j=k}^{\infty}u_j^q\right)^{\frac{p}{p-q}}\left(\sum_{i=1}^{k}\Delta^{-}\left(\sum_{j=1}^{i}a_{ij}^{p'}v_j^{-p'}\right)\right)^{\frac{p(q-1)}{p-q}}\Delta^{-}\left(\sum_{j=1}^{k}a_{kj}^{p'}v_j^{-p'}\right)\right)^{\frac{p-q}{pq}} =$$

$$= \left(\sum_{k=1}^{\infty}\left(\sum_{j=k}^{\infty}u_j^q\right)^{\frac{p}{p-q}}\left(\sum_{j=1}^{k}a_{kj}^{p'}v_j^{-p'}\right)^{\frac{p(q-1)}{p-q}}\Delta^{-}\left(\sum_{j=1}^{k}a_{kj}^{p'}v_j^{-p'}\right)\right)^{\frac{p-q}{pq}} = M_{2,2}^{+}. \tag{11}$$

From Definition 3 it follows, that $a_{ij} >> a_{ik}^{2,1}a_{kj}^{(1)}, i \geq k \geq j \geq 1$. Then for $i \geq k \geq j \geq 1$

$$a_{ij} >> a_{ik}^{2,1}a_{kj}^{(1)} = a_{ik}^{2,1}a_{kj}^{(1)}\theta_k, \tag{12}$$

where

$$\theta_k = \begin{cases} 1, & j \leq k \leq i, \\ 0, & k > i, \quad k < j. \end{cases}$$

Let $\varphi = \{\varphi_i\}_{i=1}^{\infty}$ be a sequence of non-negative numbers such that $\sum_{i=1}^{\infty}\varphi_i = 1$. Multiplying both parts of (12) to φ and summing up by $k \in N$, we have

$$a_{ij} >> \sum_{k=j}^{i} a_{ik}^{2,1}a_{kj}^{(1)}\varphi_k. \tag{13}$$

Then using (13) and changing the order of summation twice, we have

$$\sum_{i=1}^{\infty}u_i^q\left(\sum_{j=1}^{i}a_{ij}f_j\right)^q = \sum_{i=1}^{\infty}u_i^q\sum_{j=1}^{i}a_{ij}f_j\left(\sum_{s=1}^{i}a_{is}f_s\right)^{q-1} \geq$$

$$\geq \sum_{i=1}^{\infty}u_i^q\sum_{j=1}^{i}\left(\sum_{k=j}^{i}a_{ik}^{2,1}a_{kj}^{(1)}\varphi_k\right)f_j\left(\sum_{s=1}^{i}\left(\sum_{\tau=s}^{i}a_{i\tau}^{2,1}a_{\tau s}^{(1)}\varphi_\tau\right)f_s\right)^{q-1} =$$

$$= \sum_{i=1}^{\infty}u_i^q\sum_{k=1}^{i}\varphi_k a_{ik}^{2,1}\sum_{j=1}^{k}a_{kj}^{(1)}f_j\left(\sum_{\tau=1}^{i}\varphi_\tau a_{i\tau}^{2,1}\sum_{s=1}^{\tau}a_{\tau s}^{(1)}f_s\right)^{q-1} \geq$$

$$\geq \sum_{k=1}^{\infty}\varphi_k\left(\sum_{j=1}^{k}a_{kj}^{(1)}f_j\right)\sum_{i=k}^{\infty}u_i^q a_{ik}^{2,1}\left(\sum_{\tau=k}^{i}\varphi_\tau a_{i\tau}^{2,1}\sum_{s=1}^{\tau}a_{\tau s}^{(1)}f_s\right)^{q-1} >>$$

$$>> \sum_{k=1}^{\infty}\left(\sum_{j=1}^{k}a_{kj}^{(1)}f_j\right)^q\sum_{i=k}^{\infty}u_i^q a_{ik}^{2,1}\left(\sum_{\tau=k}^{i}\varphi_\tau a_{i\tau}^{2,1}\right)^{q-1}\varphi_k =$$

$$= \sum_{k=1}^{\infty} \left(\sum_{j=1}^{k} a_{kj}^{(1)} f_j \right)^q h_k, \tag{14}$$

where $h_k = \sum\limits_{i=k}^{\infty} u_i^q a_{ik}^{2,1} \left(\sum\limits_{\tau=k}^{i} \varphi_\tau a_{i\tau}^{2,1} \right)^{q-1} \varphi_k$. From (7) and (14) it follows

$$\left(\sum_{k=1}^{\infty} \left(\sum_{j=1}^{k} a_{kj}^{(1)} f_j \right)^q h_k \right)^{\frac{1}{q}} << \|A^+\|_{pv\to qu} \left(\sum_{i=1}^{\infty} (f_i v_i)^p \right)^{\frac{1}{p}}, \ 0 \le f \in l_{pv}.$$

By applying Theorem B and taking into account (4), we get

$$\|A^+\|_{pv\to qu} >> B_1 := \left(\sum_{k=1}^{\infty} \left(\sum_{j=k}^{\infty} h_j \right)^{\frac{q}{p-q}} \left(\sum_{i=1}^{k} (a_{ki}^{(1)})^{p'} v_i^{-p'} \right)^{\frac{q(p-1)}{p-q}} h_k \right)^{\frac{p-q}{pq}}.$$

Using that $B_1 < \infty$ and $\sum\limits_{i=1}^{k} (a_{ki}^{(1)})^{p'} v_i^{-p'}$ is increasing in k, we have

$$0 = \lim_{N\to\infty} \sum_{k=N}^{\infty} h_k \left(\sum_{j=k}^{\infty} h_j \right)^{\frac{q}{p-q}} \left(\sum_{i=1}^{k} (a_{ki}^{(1)})^{p'} v_i^{-p'} \right)^{\frac{q(p-1)}{p-q}} \ge$$

$$\ge \lim_{N\to\infty} \sum_{k=N}^{\infty} h_k \left(\sum_{j=k}^{\infty} h_j \right)^{\frac{q}{p-q}} \left(\sum_{i=1}^{N} (a_{Ni}^{(1)})^{p'} v_i^{-p'} \right)^{\frac{q(p-1)}{p-q}}.$$

Further, using this relation to the Abel transform in B_1, (5) and the following elementary estimate

$$b^\gamma - a^\gamma \approx b^{\gamma-1}(b-a), \tag{15}$$

where $b > a > 0, \gamma > 0$, we obtain

$$\|A^+\|_{pv\to qu} >> B_1 \approx \left(\sum_{k=1}^{\infty} \Delta^+ \left(\sum_{j=k}^{\infty} h_j \left(\sum_{s=j}^{\infty} h_s \right)^{\frac{q}{p-q}} \right) \left(\sum_{i=1}^{k} (a_{ki}^{(1)})^{p'} v_i^{-p'} \right)^{\frac{q(p-1)}{p-q}} \right)^{\frac{p-q}{pq}} =$$

$$= \left(\sum_{k=1}^{\infty} \left(\sum_{j=k}^{\infty} h_j \left(\sum_{s=j}^{\infty} h_s \right)^{\frac{q}{p-q}} \right) \Delta^- \left(\sum_{i=1}^{k} (a_{ki}^{(1)})^{p'} v_i^{-p'} \right)^{\frac{q(p-1)}{p-q}} \right)^{\frac{p-q}{pq}} \approx$$

$$\approx \left(\sum_{k=1}^{\infty} \left(\sum_{j=k}^{\infty} h_j \right)^{\frac{p}{p-q}} \Delta^- \left(\sum_{i=1}^{k} (a_{ki}^{(1)})^{p'} v_i^{-p'} \right)^{\frac{q(p-1)}{p-q}} \right)^{\frac{p-q}{pq}} \ge$$

$$\ge \left(\sum_{k=1}^{\infty} \left(\sum_{j=k}^{\infty} h_j \right)^{\frac{p}{p-q}} \left(\sum_{i=1}^{k} (a_{ki}^{(1)})^{p'} v_i^{-p'} \right)^{\frac{p(q-1)}{p-q}} \Delta^- \left(\sum_{i=1}^{k} (a_{ki}^{(1)})^{p'} v_i^{-p'} \right) \right)^{\frac{p-q}{pq}},$$

where

$$\sum_{j=k}^{\infty} h_j = \sum_{j=k}^{\infty}\sum_{i=j}^{\infty} u_i^q a_{ij}^{(2,1)} \left(\sum_{\tau=j}^{i} \varphi_\tau a_{i\tau}^{(2,1)}\right)^{q-1} \varphi_j = \sum_{i=k}^{\infty} u_i^q \sum_{j=k}^{i} a_{ij}^{(2,1)} \varphi_j \left(\sum_{\tau=j}^{i} \varphi_\tau a_{i\tau}^{(2,1)}\right)^{q-1} \approx$$

$$\approx \sum_{i=k}^{\infty} u_i^q \left(\sum_{j=k}^{i} \varphi_j a_{ij}^{(2,1)}\right)^{q}.$$

Therefore, due to $\forall \varphi : \sum\limits_{k=1}^{\infty} \varphi_k = 1$, we have
$\|A^+\|_{pv\to qu} >>$

$$>> \sup_{\varphi} \left(\sum_{k=1}^{\infty} \left(\sum_{i=k}^{\infty} u_i^q \left(\sum_{j=k}^{i} \varphi_j a_{ij}^{2,1} \right)^q \right)^{\frac{p}{p-q}} \left(\sum_{s=1}^{k} (a_{ks}^{(1)})^{p'} v_s^{-p'} \right)^{\frac{p(q-1)}{p-q}} \Delta^- \left(\sum_{s=1}^{k} (a_{ks}^{(1)})^{p'} v_s^{-p'} \right) \right)^{\frac{p-q}{pq}}.$$

Assume, that $\varphi_j = \delta_j(m), m \geq 1$, where

$$\delta_j(m) = \begin{cases} 1, & j = m, \\ 0, & j \neq m. \end{cases}$$

Then taking into account that $a_{ij}^{2,1}$ is non-increasing in j
$\|A^+\|_{pv\to qu} >>$

$$>> \sup_{m\geq 1} \left(\sum_{k=1}^{\infty} \left(\sum_{i=k}^{\infty} u_i^q \left(\sum_{j=k}^{i} a_{ij}^{2,1} \delta_j(m) \right)^q \right)^{\frac{p}{p-q}} \left(\sum_{s=1}^{k} (a_{ks}^{(1)})^{p'} v_s^{-p'} \right)^{\frac{p(q-1)}{p-q}} \Delta^- \left(\sum_{s=1}^{k} (a_{ks}^{(1)})^{p'} v_s^{-p'} \right) \right)^{\frac{p-q}{pq}} =$$

$$= \left(\sum_{k=1}^{\infty} \left(\sum_{i=k}^{\infty} u_i^q (a_{ik}^{2,1})^q \right)^{\frac{p}{p-q}} \left(\sum_{s=1}^{k} (a_{ks}^{(1)})^{p'} v_s^{-p'} \right)^{\frac{p(q-1)}{p-q}} \Delta^- \left(\sum_{s=1}^{k} (a_{ks}^{(1)})^{p'} v_s^{-p'} \right) \right)^{\frac{p-q)}{pq}} = M_{2,1}^+. \tag{16}$$

Thus, from (8), (11) and (16) it follows

$$M^+ = \max\{M_{2,0}^+, M_{2,1}^+, M_{2,2}^+\} << \|A^+\|_{pv\to qu} < \infty. \tag{17}$$

Sufficiency. Let $M^+ < \infty$ and $0 \leq f \in l_{pv}$. $\mathbb{Z}$ is the set of integer numbers. Let's assume $\sum\limits_{i=k}^{n} = 0$ when $k > n$ and $a_{ij} = 0$ when $i < j$.

For all $i \geq 1$ we define the following set of integer numbers:

$$T_i = \{k \in \mathbb{Z} : (r_2+1)^k \leq (A^+ f)_i\},$$

where r_2 is the constant from Definition 3 and we assume that $k_i = \max T_i$. Then

$$(r_2+1)^{k_i} \leq (A^+ f)_i < (r_2+1)^{k_i+1}, \forall i \in \mathbb{N}. \tag{18}$$

Let $m_1 = 1$ and $M_1 = \{i \in \mathbb{N} : k_i = k_1 = k_{m_1}\}$. Suppose that m_2 is such that $\sup M_1 + 1 = m_2$. Obviously $m_2 > m_1$ and if the set M_1 is upper bounded, then $m_2 < \infty$ and $m_2 - 1 = \max M_1 = \sup M_1$.

Suppose that we have found numbers $1 = m_1 < m_2 < ... < m_s < \infty$, $s \geq 1$, then we define m_{s+1} by $m_{s+1} = \sup M_s + 1$, where $M_s = \{i \in \mathbb{N} : k_i = k_{m_s}\}$.

Let $N_0 = \{s \in \mathbb{N} : m_s < \infty\}$. Further, we assume that $k_{m_s} = n_s, s \in N_0$. From the definition of m_s and from (18) it follows that, for $s \in N_0$

$$(r_2+1)^{n_s} \leq (A^+f)_i < (r_2+1)^{n_s+1}, m_s \leq i \leq m_{s+1}-1 \tag{19}$$

and $\mathbb{N} = \bigcup_{s\in N_0}[m_s, m_{s+1}-1]$, where $[m_s, m_{s+1}) \cap [m_l, m_{l+1}) \neq 0$.

By using (19), Definition 3 and $n_{s-2}+1 \leq n_s - 1$, which follows from the inequality $n_{s-2} < n_{s-1} < n_s$, we can estimate the value $(r_2+1)^{n_s-1}$:

$$(r_2+1)^{n_s-1} = (r_2+1)^{n_s} - r_2(r_2+1)^{n_s-1} \leq (r_2+1)^{n_s} - r_2(r_2+1)^{n_{s-2}+1} \leq$$

$$\leq (A^+f)_{m_s} - r_2(A^+f)_{m_{s-1}-1} = \sum_{i=1}^{m_s} a_{m_s i} f_i - r_2 \sum_{i=1}^{m_{s-1}-1} a_{m_{s-1} i} f_i =$$

$$= \sum_{i=m_{s-1}}^{m_s} a_{m_s i} f_i + \sum_{i=1}^{m_{s-1}-1} [a_{m_s i} - r_2 a_{m_{s-1}-1 i}] f_i <<$$

$$<< \sum_{i=m_{s-1}}^{m_s} a_{m_s i} f_i + \sum_{i=1}^{m_{s-1}-1} [r_2 a^{2,0}_{m_s m_{s-1}-1} + r_2 a^{2,1}_{m_s m_{s-1}-1} a^{(1)}_{m_{s-1}-1 i}] f_i <<$$

$$<< \sum_{i=m_{s-1}}^{m_s} a_{m_s i} f_i + r_2 a^{2,0}_{m_s m_{s-1}-1} \sum_{i=1}^{m_{s-1}-1} f_i + r_2 a^{2,1}_{m_s m_{s-1}-1} \sum_{i=1}^{m_{s-1}-1} a^{(1)}_{m_{s-1}-1 i} f_i. \tag{20}$$

Then taking into account (20), we get

$$\|A^+f\|^q_{qu} = \sum_{s\in N_0} \sum_{i=m_s}^{m_{s+1}-1} u_i^q (A^+f)_i^q < \sum_{s\in N_0} (r_2+1)^{(n_s+1)q} \sum_{i=m_s}^{m_{s+1}-1} u_i^q \leq$$

$$\leq (r_2+1)^{2q} \sum_{s\in N_0} \left(\sum_{i=m_{s-1}}^{m_s} a_{m_s i} f_i + r_2 a^{2,0}_{m_s m_{s-1}-1} \sum_{i=1}^{m_{s-1}-1} f_i + r_2 a^{2,1}_{m_s m_{s-1}-1} \sum_{i=1}^{m_{s-1}-1} a^{(1)}_{m_{s-1}-1 i} f_i \right)^q \sum_{i=m_s}^{m_{s+1}-1} u_i^q <<$$

$$<< \sum_{s\in N_0} \left(\sum_{i=m_{s-1}}^{m_s} a_{m_s i} f_i \right)^q \sum_{i=m_s}^{m_{s+1}-1} u_i^q + \sum_{s\in N_0} \left(a^{2,0}_{m_s m_{s-1}-1} \right)^q \left(\sum_{i=1}^{m_{s-1}-1} f_i \right)^q \sum_{i=m_s}^{m_{s+1}-1} u_i^q +$$

$$+ \sum_{s\in N_0} \left(a^{2,1}_{m_s m_{s-1}-1} \right)^q \left(\sum_{i=1}^{m_{s-1}-1} a_{m_{s-1}-1 i} f_i \right)^q \sum_{i=m_s}^{m_{s+1}-1} u_i^q =$$

$$= S_{2,2} + S_{2,0} + S_{2,1}. \tag{21}$$

By applying Hölder's inequality twice and (5), we estimate $S_{2,2}$.

$$S_{2,2} = \sum_{s\in N_0} \left(\sum_{i=m_{s-1}}^{m_s} a_{m_s i} f_i v_i v_i^{-1} \right)^q \sum_{i=m_s}^{m_{s+1}-1} u_i^q \leq \sum_{s\in N_0} \left(\sum_{i=m_{s-1}}^{m_s} (f_i v_i)^p \right)^{\frac{q}{p}} \left(\sum_{i=m_{s-1}}^{m_s} a^{p'}_{m_s i} v_i^{-p'} \right)^{\frac{q}{p'}} \sum_{i=m_s}^{m_{s+1}-1} u_i^q \leq$$

$$\leq \left(\sum_{s\in N_0} \sum_{i=m_{s-1}}^{m_s} (f_i v_i)^p \right)^{\frac{q}{p}} \left(\sum_{s\in N_0} \left(\sum_{i=m_{s-1}}^{m_s} a_{m_s i}^{p'} v_i^{-p'} \right)^{\frac{q(p-1)}{p-q}} \left(\sum_{i=m_s}^{m_{s+1}-1} u_i^q \right)^{\frac{p}{p-q}} \right)^{\frac{p-q}{p}} \leq$$

$$\leq 2^{\frac{q}{p}} \|f\|_{pv}^q \left(\sum_{s\in N_0} \left(\sum_{i=m_{s-1}}^{m_s} a_{m_s i}^{p'} v_i^{-p'} \right)^{\frac{q(p-1)}{p-q}} \sum_{i=m_s}^{m_{s+1}-1} u_i^q \left(\sum_{j=i}^{m_{s+1}-1} u_j^q \right)^{\frac{q}{p-q}} \right)^{\frac{p-q}{p}} \leq$$

$$\leq \|f\|_{pv}^q \left(\sum_{i=1}^{\infty} u_i^q \left(\sum_{j=i}^{\infty} u_j^q \right)^{\frac{q}{p-q}} \left(\sum_{n=1}^{i} a_{in}^{p'} v_n^{-p'} \right)^{\frac{q(p-1)}{p-q}} \right)^{\frac{p-q}{p}} = \|f\|_{l_{pv}}^q \tilde{M}_{2,2}^q,$$

where

$$\tilde{M}_{2,2}^{\frac{pq}{p-q}} = \sum_{i=1}^{\infty} u_i^q \left(\sum_{j=i}^{\infty} u_j^q \right)^{\frac{q}{p-q}} \left(\sum_{n=1}^{i} a_{in}^{p'} v_n^{-p'} \right)^{\frac{q(p-1)}{p-q}}.$$

Using the Abel transform, (5) and (15), we have

$$\tilde{M}_{2,2}^{\frac{pq}{p-q}} = \sum_{i=1}^{\infty} \Delta^+ \left(\sum_{k=i}^{\infty} u_k^q \left(\sum_{j=k}^{\infty} u_j^q \right)^{\frac{q}{p-q}} \right) \left(\sum_{n=1}^{i} a_{in}^{p'} v_n^{-p'} \right)^{\frac{q(p-1)}{p-q}} =$$

$$= \sum_{i=1}^{\infty} \left(\sum_{k=i}^{\infty} u_k^q \left(\sum_{j=k}^{\infty} u_j^q \right)^{\frac{q}{p-q}} \right) \Delta^- \left(\sum_{n=1}^{i} a_{in}^{p'} v_n^{-p'} \right)^{\frac{q(p-1)}{p-q}} \approx$$

$$\approx \sum_{i=1}^{\infty} \left(\sum_{k=i}^{\infty} u_k^q \right)^{\frac{p}{p-q}} \left(\sum_{n=1}^{i} a_{in}^{p'} v_n^{-p'} \right)^{\frac{p(q-1)}{p-q}} \Delta^- \left(\sum_{n=1}^{i} a_{in}^{p'} v_n^{-p'} \right) = M_{2,2}^+ < \infty.$$

Therefore

$$S_{2,2} << (M_{2,2}^+)^q \|f\|_{pv}^q. \tag{22}$$

To estimate $S_{2,0}$, we assume

$$\eta_i(m_{s-1}-1) = \begin{cases} \sum\limits_{s\in N_0} \left(a_{m_s m_{s-1}-1}^{2,0} \right)^q \sum\limits_{i=m_s}^{m_{s+1}-1} u_i^q, & i = m_{s-1}-1, \\ 0, & i \neq m_{s-1}-1. \end{cases}$$

and we use Theorem A.

$$S_{2,0} = \sum_{s\in N_0} \left(a_{m_s m_{s-1}-1}^{2,0} \right)^q \left(\sum_{i=1}^{m_{s-1}-1} f_i \right)^q \sum_{i=m_s}^{m_{s+1}-1} u_i^q = \sum_{n=1}^{\infty} \left(\sum_{i=1}^{n} f_i \right)^q \eta_n <<$$

$$<< \left(\sum_{n=1}^{\infty} \left(\sum_{i=n}^{\infty} \eta_j \right)^{\frac{p}{p-q}} \left(\sum_{j=1}^{n} v_j^{-p'} \right)^{\frac{p(q-1)}{p-q}} v_n^{-p'} \right)^{\frac{p-q}{p}} \|f\|_{pv}^q. \tag{23}$$

Taking into account Remark 1, we estimate $\sum\limits_{i=n}^{\infty}\eta_i$:

$$\sum_{i=n}^{\infty}\eta_i=\sum_{s:m_{s-1}-1\geq n}\left(a^{2,0}_{m_s m_{s-1}-1}\right)^q\sum_{i=m_s}^{m_{s+1}-1}u_i^q=\sum_{s:m_{s-1}-1\geq n}\sum_{i=m_s}^{m_{s+1}-1}\left(a^{2,0}_{m_s m_{s-1}-1}\right)^q u_i^q<<\sum_{i=n}^{\infty}\left(a^{2,0}_{in}\right)^q u_i^q.$$

Hence and from (23), we have

$$S_{2,0}<<\left(\sum_{n=1}^{\infty}\left(\sum_{i=n}^{\infty}\left(a^{2,0}_{in}\right)^q u_i^q\right)^{\frac{p}{p-q}}\left(\sum_{j=1}^{n}v_j^{-p'}\right)^{\frac{p(q-1)}{p-q}}v_n^{-p'}\right)^{\frac{p-q}{pq}}\|f\|^q_{pv}=(M^+_{2,0})^q\|f\|^q_{pv}. \qquad (24)$$

Now, by using Theorem B we estimate $S_{2,1}$.

$$S_{2,1}=\sum_{s\in N_0}\left(a^{2,1}_{m_s m_{s-1}-1}\right)^q\left(\sum_{i=1}^{m_{s-1}-1}a^{(1)}_{m_{s-1}-1 i}f_i\right)^q\sum_{i=m_s}^{m_{s+1}-1}u_i^q<<$$

$$<<\sum_{s\in N_0}\left(\sum_{i=1}^{m_{s-1}-1}a^{(1)}_{m_{s-1}-1\,i}f_i\right)^q\sum_{i=m_s}^{m_{s+1}-1}\left(a^{2,1}_{i\,m_{s-1}-1}\right)^q u_i^q=$$

$$=\sum_{k=1}^{\infty}\left(\sum_{i=1}^{k}a^{(1)}_{ki}f_i\right)^q\theta_k\leq\left(\max\{\tilde{B}_0,\tilde{B}_1\}\right)^q\|f\|^q_{l_{pv}}, \qquad (25)$$

where $\theta_k=\sum\limits_{s\in N_0}\sum\limits_{n=m_s}^{m_{s+1}-1}\left(a^{2,1}_{n\,m_{s-1}-1}\right)^q u_n^q$ when $k=m_{s-1}-1$ and $\theta_k=0$ when $k\neq m_{s-1}-1$,

$$\tilde{B}_0=\left(\sum_{k=1}^{\infty}\left(\sum_{j=k}^{\infty}\left(a^{1,0}_{jk}\right)^q\theta_j\right)^{\frac{p}{p-q}}\left(\sum_{i=1}^{k}v_i^{-p'}\right)^{\frac{p(q-1)}{p-q}}v_k^{-p'}\right)^{\frac{p-q}{pq}},$$

$$\tilde{B}_1=\left(\sum_{k=1}^{\infty}\left(\sum_{j=k}^{\infty}\theta_j\right)^{\frac{q}{p-q}}\left(\sum_{i=1}^{k}\left(a^{(1)}_{ki}\right)^{p'}v_i^{-p'}\right)^{\frac{q(p-1)}{p-q}}\theta_k\right)^{\frac{p-q}{pq}}.$$

Let's evaluate the expression $\sum\limits_{j=k}^{\infty}\left(a^{1,0}_{jk}\right)^q\theta_j$ in $\tilde{B}_0$.

$$\sum_{j=k}^{\infty}\left(a^{1,0}_{jk}\right)^q\theta_j=\sum_{s:m_{s-1}-1\geq k}\left(a^{1,0}_{m_{s-1}-1\,k}\right)^q\sum_{n=m_s}^{m_{s+1}-1}\left(a^{2,1}_{n\,m_{s-1}-1}\right)^q u_n^q=$$

$$=\sum_{s:m_{s-1}-1\geq k}\sum_{n=m_s}^{m_{s+1}-1}\left(a^{2,1}_{n\,m_{s-1}-1}\right)^q\left(a^{1,0}_{m_{s-1}-1\,k}\right)^q u_n^q.$$

In [10] it is shown that $a^{2,1}_{n\,m_{s-1}-1}a^{1,0}_{m_{s-1}-1\,k}<<a^{2,0}_{n\,k}$ when $n\geq m_{s-1}-1\geq k\geq 1$. Then

$$\sum_{j=k}^{\infty}\left(a^{1,0}_{jk}\right)^q\theta_j<<\sum_{n=k}^{\infty}\left(a^{2,0}_{nk}\right)^q u_n^q.$$

Thus

$$\tilde{B}_0 << \left(\sum_{k=1}^{\infty}\left(\sum_{n=k}^{\infty}\left(a_{nk}^{2,0}\right)^q u_n^q\right)^{\frac{p}{p-q}}\left(\sum_{i=1}^{k} v_i^{-p'}\right)^{\frac{p(q-1)}{p-q}} v_k^{-p'}\right)^{\frac{p-q}{pq}} = M_{2,0}^{+} < \infty. \tag{26}$$

By using the Abel transform, (5) and (15) we estimate the value $\tilde{B}_1$.

$$\tilde{B}_1^{\frac{pq}{p-q}} = \sum_{k=1}^{\infty}\Delta^{+}\left(\sum_{i=k}^{\infty}\theta_i\left(\sum_{j=i}^{\infty}\theta_j\right)^{\frac{q}{p-q}}\right)\left(\sum_{i=1}^{k}\left(a_{ki}^{(1)}\right)^{p'} v_i^{-p'}\right)^{\frac{q(p-1)}{p-q}} =$$

$$= \sum_{k=1}^{\infty}\left(\sum_{i=k}^{\infty}\theta_i\left(\sum_{j=i}^{\infty}\theta_j\right)^{\frac{q}{p-q}}\right)\Delta^{-}\left(\sum_{i=1}^{k}\left(a_{ki}^{(1)}\right)^{p'} v_i^{-p'}\right)^{\frac{q(p-1)}{p-q}} \approx$$

$$\approx \sum_{k=1}^{\infty}\left(\sum_{i=k}^{\infty}\theta_i\right)^{\frac{p}{p-q}}\left(\sum_{i=1}^{k}\left(a_{ki}^{(1)}\right)^{p'} v_i^{-p'}\right)^{\frac{p(q-1)}{p-q}}\Delta^{-}\left(\sum_{i=1}^{k}\left(a_{ki}^{(1)}\right)^{p'} v_i^{-p'}\right). \tag{27}$$

Since

$$\sum_{i=k}^{\infty}\theta_i = \sum_{s:m_{s-1}-1\geq k}\ \sum_{n=m_s}^{m_{s+1}-1}\left(a_{n,m_{s-1}-1}^{2,1}\right)^q u_n^q \leq \sum_{n=k}^{\infty}\left(a_{n\,k}^{2,1}\right)^q u_n^q,$$

hence and from (27), it follows

$$\tilde{B}_1 << \left(\sum_{k=1}^{\infty}\left(\sum_{n=k}^{\infty}\left(a_{nk}^{2,1}\right)^q u_n^q\right)^{\frac{p}{p-q}}\left(\sum_{i=1}^{k}\left(a_{ki}^{(1)}\right)^{p'} v_i^{-p'}\right)^{\frac{p(q-1)}{p-q}}\Delta^{-}\left(\sum_{i=1}^{k}\left(a_{ki}^{(1)}\right)^{p'} v_i^{-p'}\right)\right)^{\frac{p-q}{pq}} = M_{2,1}^{+}. \tag{28}$$

Thus, from (25), (26) and (28), we obtain

$$S_{2,1} << \left(\max\{M_{2,0}^{+}, M_{2,1}^{+}\}\right)^q \|f\|_{l_{pv}}^q.$$

Hence and from (21), (22), (24) we have

$$\|A^{+}f\|_{qu} << \max\{M_{2,0}^{+}, M_{2,1}^{+}, M_{2,2}^{+}\}\|f\|_{l_{pv}} = M^{+}\|f\|_{l_{pv}},$$

i.e. the operator A^{+} is bounded from l_{pv} into l_{qu} and takes place for the norm $\|A^{+}\|_{pv\to qu} << M^{+}$, which with (17) gives us $\|A^{+}\|_{pv\to qu} \approx M^{+}$.

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Матрицалық операторлар бір класының l_{pv}-дан l_{qu}-ға шенелгендік критерийі

Матрицалар теориясының негізгі міндеттерінің бірі матрицалық оператордың бір нормалы кеңістіктен басқа нормалы кеңістікке үзіліссіз өту үшін матрицалар элементтеріне қажетті және жеткілікті шарттарын анықтау. Сонымен қатар матрицалық оператордың нормасын немесе оның дәл жоғарғы және төменгі бағалауын табу маңызды. Бұл есеп жалпы жағдайда Лебег тізбектер кеңістігінде ашық есеп. Берілген мақалада матрицалық операторының l_{pv}-дан l_{qu}-ға $1 < q < p < \infty$ болғанда шенелгендігі қарастырылған және бұл есептің қажетті және жеткілікті шарттары алынды, мұндағы матрица $O_2^{\pm}$ дискретті Ойнаров класына тиісті.

Кілт сөздер: матрицалық оператор, түйіндес оператор, салмақты тізбек, шенелгендік, салмақты теңсіздіктер, Лебег салмақты кеңістігі, Ойнаров шарты, Харди операторы, Харди теңсіздігі, матрица.

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Критерий ограниченности некоторого класса матричных операторов из l_{pv} в l_{qu}

Одной из основных задач теории матриц является нахождение необходимых и достаточных условий для элементов матрицы, при которых матричный оператор непрерывно действует из одного нормированного пространства в другое. При этом очень важно найти значение нормы матричного оператора, в крайнем случае, зафиксировать точные верхние и нижние оценки. Эта задача в лебеговых пространствах последовательностей в общем случае остается открытой. В статье рассмотрена проблема ограниченности матричных операторов из l_{pv} в l_{qu} при $1 < q < p < \infty$ и получены необходимые и достаточные условия этой задачи, когда матричные операторы принадлежат классам $O_2^{\pm}$, удовлетворяющим более слабым условиям, чем условие Ойнарова.

Ключевые слова: матричный оператор, сопряженный оператор, весовая последовательность, ограниченность, весовые неравенства, весовое пространство Лебега, условие Ойнарова, оператор Харди, неравенство Харди, матрица.

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