UDC 517.9

D.R. Beisenova[1,2], K.N. Ospanov[2]

[1] *Ye.A Buketov Karaganda State University, Kazakhstan;*
[2] *L.N. Gumilyov Eurasian National University, Astana, Kazakhstan*
(E-mail: dana_68_11@mail.ru)

Maximal regularity and compactness conditions for a high order system of difference equations

In this paper we study an infinite linear system of difference equations of high even order with the right-hand side from the Hilbert space of numerical sequences. Sequences formed from the coefficients of the equations of the system for the same orders of difference can be unlimited, and their growth may not be subject to the growth of the potential. The previously developed methods, which essentially use the dominant potential growth in the difference systems of Sturm-Liouville type equations, do not pass here, since In the case under consideration, the potential may turn out to be zero, or not having a definite sign by a sequence. We give conditions for the correct solvability of the system, as well as optimal estimates of the norms of the solution and its differences up to the highest order. Conditions for the compactness of the resolvent of the corresponding system of a degenerate operator are obtained. We prove some difference weight inequalities of Hardy type having independent scientific interest. They are used in the proof of the main results of the paper. It is shown that, in comparison with degenerate differential equations, in the case of a difference system, it is possible to remove the condition for oscillations of the coefficients of the system.

Keywords: difference system, intermediate coefficient, correctness of solution, maximum regularity, compactness of resolution.

1 Introduction and main results

The present paper is devoted to the study of the correct solvability and differential properties of the solution of the following high order infinite system of difference equations:

$$L_0 y = \Delta^{(2n)} y + r\Delta^{(2n-1)} + s\overline{\Delta^{(2n-1)} y} + \sum_{j=1}^{2n-1} \left(Q^{(j)} \Delta^{(2n-j-1)} y + P^{(j)} \overline{\Delta^{(2n-j-1)}} \right) = f, \qquad (1.1)$$

where

$$y = \{y_k\}_{k=-\infty}^{+\infty}, \quad \Delta_+ y_k = y_{(k+1)} - y_k, \quad \Delta^{(2)} y_k = \Delta_- \Delta_+ y_k = y_{k+1} - 2y_k + y_{k-1}\,(k \in Z)\,,$$

$$\Delta^{(2s)} y = \Delta^{(2)} \Delta^{(2s-2)} y, \quad \Delta^{(2s-1)} y = \Delta_+ \underbrace{\Delta^{(2)} \Delta^{(2)} \ldots \Delta^{(2)}}_{(s-1)}\,(s \in N)\,,$$

and

$$r = \{\mathrm{diag}, r_{jj}\}_{j=-\infty}^{+\infty}, \quad s = \{\mathrm{diag}, s_{jj}\}_{j=-\infty}^{+\infty},$$

$$Q^{(\theta)} = \{\mathrm{diag}, q_{jj}^{(\theta)}\}_{j=-\infty}^{+\infty}, \quad P^{(\theta)} = \{\mathrm{diag}, p_{jj}^{(\theta)}\}_{j=-\infty}^{+\infty}, \quad \theta = \overline{1, 2n-1}$$

are given diagonal matrices, $f \in l_2$.

Many dynamic problems in practice, according to the nature of the formulation, are given either to differential equations, or to infinite difference systems, or to differential-difference equations. Functional analysis accelerated the development of the theory of infinite systems of difference equations. Recently, much attention is paid to the study of differential equations and second-order systems with unlimited intermediate coefficients, in view of their important applications. In addition, questions of modeling the propagation of vibrations in viscoelastic and compressible media [1, 2], as well as some problems in the theory of stochastic processes and stochastic differential equations [3–6] lead to them. Known representatives of such equations are, for example, the Fokker-Planck equation and the Oinstein-Uhlenbeck equation used to describe the Brownian motion. Along with this, a number of issues of the solution of second-order systems depend on infinite higher-order difference equations.

We present the main results of this paper. Let $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$ and $u = \{\mathrm{diag}, u_n\}_{n=-\infty}^{+\infty}$, $v = \{\mathrm{diag}, v_n\}_{n=-\infty}^{+\infty}$, sequence of real numbers. We introduce the following notations.

$$T_{m,u,v} = \sup_{n=0,1,2,\ldots} \left(\sum_{j=0}^{n} |u_j|^p \right)^{\frac{1}{p}} \left(\sum_{j=n}^{+\infty} \left(j^{(m-1)p'} \right)^{p'} |v_j|^{-p'} \right)^{\frac{1}{p'}};$$

$$T''_{m,u,v} = \sup_{\tau<0} \left(\sum_{j=\tau}^{0} |u_j|^p \right)^{\frac{1}{p}} \left(\sum_{j=-\infty}^{\tau} |j|^{(m-1)p'} |v_j|^{-p'} \right)^{\frac{1}{p'}};$$

$$\gamma_{m,u,v} = \sqrt[p]{\max\left[(T_{m,u,v})^p, (T''_{m,u,v})^p \right]}, \ (m = 2, 3, \ldots).$$

We denote by $\widetilde{l}$ the set of all finite sequences of real numbers.

Definition 1.1. $y = \{y_j\}_{j=-\infty}^{+\infty} \in l_2$ is called a solution of the system (1.1), if there exists a sequence $\{z^{(k)}\}_{k=1}^{+\infty} \subset \widetilde{l}$ such that $\|z^{(k)} - y\|_2 \to 0$, $\|L_0 z^{(k)} - f\|_2 \to 0$ $(k \to +\infty)$.

Theorem 1.1. Let the sequences $\widetilde{r} = \{r_{jj}\}_{j=-\infty}^{+\infty}$, $\widetilde{s} = \{s_{jj}\}_{j=-\infty}^{+\infty}$, $\widetilde{Q}^{(\theta)} = \{q_{jj}^{\theta}\}_{j=-\infty}^{+\infty}$, $\widetilde{P}^{(\theta)} = \{p_{jj}^{\theta}\}_{j=-\infty}^{+\infty}$; $(\theta = \overline{1, 2n-1})$ satisfy the following conditions:

$$\max\left(\gamma_{2n-1,e,\widetilde{r}}, \gamma_{\theta,\widetilde{Q}^{(\theta)},\widetilde{r}}, \gamma_{\theta,\widetilde{P}^{(\theta)},\widetilde{r}} \right) < \infty \quad (\theta = \overline{1, 2n-1}); \tag{1.2}$$

$$|s_{jj}| \le \alpha_1 r_{jj} \quad (j \in Z), \quad 0 < \alpha_1 < \frac{1}{5\sqrt{2}}, \tag{1.3}$$

where $e = \{e_n\}_{n=-\infty}^{+\infty}$, $e_n = 1$ $\forall n \in Z$. Then there exists a unique solution of the system (1.1). Furthermore, the solution satisfies the following estimate:

$$\left\| \Delta^{(2n)} y \right\|_2 + \left\| r \Delta^{(2n-1)} y \right\|_2 + \left\| s \overline{\Delta^{(2n-1)} y} \right\|_2 +$$
$$+ \sum_{j=1}^{2n-1} \left(\left\| Q^{(j)} \Delta^{(2n-j-1)} y \right\|_2 + \left\| P^{(j)} \Delta^{(2n-j-1)} y \right\|_2 \right) \le C_1 \|L_0 y\|_2. \tag{1.4}$$

We denote by L the closure in l_2 of the following difference expression

$$L_0 y = \Delta^{2n} y + r \Delta^{(2n-1)} y + s \overline{\Delta^{(2n-1)} y} + \sum_{j=1}^{2n-1} \left(Q^{(j)} \Delta^{(2n-j-1)} y + P^{(j)} \overline{\Delta^{(2n-j-1)} y} \right),$$

originally defined on a set $\widetilde{l}$ of all finite sequences. If the conditions of Theorem 1.1 hold, then there exists an inverse operator L^{-1} to L and it is continuous. The following assertion is important in questions of the approximate solution of the system (1.1).

Theorem 1.2. Let all of the conditions of Theorem 1.1 be satisfied and

$$\lim_{n\to\infty} \left(n \cdot \sum_{j=n}^{+\infty} r_{jj}^{-2} \right) = 0; \tag{1.5}$$

$$\lim_{k\to\infty} \left(k \cdot \sum_{j=-\infty}^{k} r_{jj}^{-2} \right) = 0. \tag{1.6}$$

Then the operator L_{-1} is compact in space l_2.

First part of this paper is devoted to study of the new difference weighted Hardy inequalities. In the second part, we apply this results together with the operator methods and theorems on small perturbations, to the proof of Theorems 1.1 and 1.2. A review of previous results obtained in these directions is contained in [7]. When $n = 1$ and $h = 1$, Theorem 1.1 coincides with the results of [7]. In the case of higher-order elliptic differential equations, the question of maximal regularity (coercivity) was considered in [8, 9]. One of the achievements of Theorem 1.1 obtained for the difference analogue of these equations, is that it removes the restriction on the oscillations of the coefficients. Qualitative research on systems of infinite difference equations can be found in [7, 10] and references therein, and the weight inequalities associated with these systems and questions of compactness in Sobolev difference spaces can be found in [11, 12].

2 Some weighted difference inequalities

Let $\widetilde{l}_+ = \{\{w_n\}_{n=-\infty}^{+\infty} \in \widetilde{l} : w_k = 0, \forall k < 0\}$

Lemma 2.1. Let $y = \{y_n\}_{n=-\infty}^{+\infty} \in \widetilde{l}_+$, and the numbers $P_{s,k}$ $(s \in N, k = 0, 1, 2, \dots)$ are defined as follows:

$$P_{1,k} = 1, P_{2,k} = k, P_{3,k} = \frac{k(k+1)}{2}, P_{m,k} = \sum_{j=0}^{k} P_{m-1,j} \quad (m = 4, 5, \dots). \tag{2.1}$$

Then holds the following equality:

$$y_n = \sum_{k-n}^{+\infty} P_{m,k-n} (-\Delta)^{(m)} y_k \quad (n = 0, 1, 2, \dots), \tag{2.2}$$

where m — is a fixed natural number.

Proof. Let $\{a_k\} \in \widetilde{l}_+$. If

$$y_n = \sum_{k=n}^{+\infty} a_k,$$

then $a_n = -\Delta y_n$. Therefore,

$$y_n = \sum_{k=n}^{+\infty} (-\Delta)\, y_k \quad (n = 0, 1, 2, \dots). \tag{2.3}$$

If we put $-\Delta y_k = z_k$, then by (2.3)

$$z_k = \sum_{s=k}^{+\infty} (-\Delta)\, z_s.$$

From here

$$y_n = \sum_{k=n}^{+\infty} \sum_{s=k}^{+\infty} (-\Delta)^{(2)} y_s = \sum_{s=n}^{+\infty} \sum_{k=n}^{s} (-\Delta)^{(2)} y_s =$$

$$= \sum_{s=n}^{+\infty} (s-n)\, (-\Delta)^{(2)} y_s, \quad n \geq 1.$$

Continuing this process, we obtain equalities (2.1) and (2.2). The lemma is proved.

We take the sequence $v = \{v_j\}_{j=0}^{+\infty}$, $v_j \neq 0$ $(j = 0, 1, \dots)$. We denote by $\widetilde{H}_{p,v}^{(k)}$ $(1 < p < \infty, k \in N)$ the space with a norm

$$\|a\|_{\widetilde{H}_{p,v}^{(k)}} = \left(\sum_{s=0}^{+\infty} \left| v_s \Delta^{(k)} a_s \right|^p \right)^{\frac{1}{p}} \left(a = \{a_s\}_{s=0}^{+\infty} \right).$$

Lemma 2.2. If $y = \{y_n\}_{n=-\infty}^{+\infty} \in \widetilde{l}_+$ and $(m \in N)$, then

$$\sup_{\|y\|_{\widetilde{H}_{p,v}^{(m)}} = 1} |y_n| = \left(\sum_{s=n}^{+\infty} |P_{m,s-n}|^{p'} \right)^{\frac{1}{p'}} (n = 0, 1, 2, \dots,). \tag{2.4}$$

Proof. From (2.2) by Holder inequality:

$$|y_n| \leq \left(\sum_{k=n}^{+\infty} P_{m,k-n}^{p'} |v_k|^{-p'} \right)^{\frac{1}{p'}} \left(\sum_{k=n}^{+\infty} |v_k|^p \left| (-\Delta)^{(m)} y_k \right|^p \right)^{\frac{1}{p}} \leq \left(\sum_{k=n}^{+\infty} P_{m,k-n}^{p'} |v_k|^{-p'} \right)^{\frac{1}{p'}} \|y\|_{\widetilde{H}_{p,v}^{(m)}} ,$$

so

$$\sup_{\|y\|_{\widetilde{H}_{p,v}^{(m)}} = 1} |y_n| \leq \left(\sum_{k=n}^{+\infty} P_{m,k-n}^{p'} |v_k|^{-p'} \right)^{\frac{1}{p'}} . \tag{2.5}$$

We choose the sequence $\widetilde{y} = \{\widetilde{y_{n,j}}\}_{j=0}^{+\infty}$ by the equalities:

$$(-\Delta)^{(m)}\,\widetilde{y}_{n,j} = \begin{cases} P_{m,j-n}^{p'-1}\,|v_j|^{-p'}, \text{if } j \in [n,N],\ N \geq n+1; \\ 0, \text{if } j \notin [n,N]. \end{cases} \tag{2.6}$$

Then by (2.2) $\widetilde{y}_{n,j} = \sum_{s=j}^{N} P_{m,s-j}\,(P_{m,s-j})^{p'-1}\,|v_s|^{-p'}$, and when $j = n$

$$\widetilde{y}_{n,n} = \sum_{s=n}^{N} P_{m,s-n}^{p'}\,|v_s|^{-p'}. \tag{2.7}$$

Further

$$\|y\|_{\widetilde{H}_{p,v}^{(m)}}^{p} = \sum_{s=n}^{N}\left[|v_s|\left(P_{m,s-j}^{p'-1}\,|v_s|^{-p'}\right)\right]^{p} = \sum_{s=n}^{N} P_{m,s-n}^{(p'-1)p}\,|v_s|^{(1-p')p} = \sum_{s=n}^{N} P_{m,s-n}^{p'}\,|v_s|^{-p'}.$$

By this equality and (2.7),

$$\sup_{\|y\|_{\widetilde{H}_{p,v}^{(m)}}=1} |y_n| \geq \frac{|\widetilde{y}_{n,n}|}{\|\widetilde{y}_{n,n}\|_{\widetilde{H}_{p,v}^{(m)}}} = \frac{\sum_{s=n}^{N} P_{m,s-n}^{p'}\,|v_s|^{-p'}}{\left[\sum_{s=n}^{N} P_{m,s-n}^{p'}\,|v_s|^{-p'}\right]^{\frac{1}{p}}} = \left(\sum_{s=n}^{N} P_{m,s-n}^{p'}\,|v_s|^{-p'}\right)^{\frac{1}{p'}}. \tag{2.8}$$

From (2.8) and (2.5), since N is any number not less than $n+1$, we obtain (2.4). The lemma is proved. Consider the sum $S_m(n) = 1^m + 2^m + \cdots + (n-1)^m + n^m$ $(m,n = 1,2,\ldots)$.

Lemma 2.3. If $m,n = 1,2,\ldots, n \geq 2m+1$, then the following inequalities hold:

$$\frac{1}{10(m+1}\,(n+1)^{m+1} \leq S_m(n) \leq \frac{1}{m+1}\,(n+1)^{m+1}. \tag{2.9}$$

Proof. Equalities $S_1(n) = \frac{n(n+1}{2}$, $S_2(n) = \frac{n(n+1)(n+1/2)}{3}$ and

$$C_{m+1}^{1} S_m(n) = (n+1)^{m+1} - C_{m+1}^{2} S_{m-1}(n) - C_{m+1}^{3} S_{m-2}(n) - \cdots - C_{m+1}^{m-2} S_3(n) -$$

$$- C_{m+1}^{m-1} S_2(n) - C_{m+1}^{m} S_1(n) - n - 1; \tag{2.10}$$

$C_k^r = \frac{k!}{r!(k-r)!}$ $(k, r \in N, k \geq r)$ are well known. By (2.10):

$$S_m(n) \leq \frac{1}{m+1}(n+1)^{m+1}. \tag{2.11}$$

The lower estimate for $S_m(n)$ follows easily from (2.10) and (2.11). The lemma is proved.

Lemma 2.4. If $s \geq n \geq 2m+1$, $m \geq 2$ $(m,n,s \in N)$, then

$$\frac{1}{10^{m-2}(m-1)!}(s-n+m-2)^{m-1} \leq P_{m,s-n} \leq \frac{1}{(m-1)!}(s-n+m-2)^{m-1}. \tag{2.12}$$

Proof. If $m = 2$, then $P_{2,s-n} = s-n$ and is satisfied (2.12). Suppose that (2.12) holds for $m = k$:

$$\frac{1}{10^{k-2}(k-1)!}\sum_{j=n}^{s}(j-n+k-2)^{k-1} \leq P_{k,s-n} \leq \frac{1}{(k-1)!}\sum_{j=n}^{s}(j-n+k-2)^{k-1}.$$

Then by Lemma 2.1,

$$\frac{1}{10^{k-2}(k-1)!}S_{k-1}(s-n+k-2) = \frac{1}{10^{k-2}(k-1)!}(s-n+k-2)^{k-1} \leq$$

$$\leq P_{k+1,s-n} \leq \frac{1}{(k-1)!}(s-n+k-2)^{k-1} = \frac{1}{(k-1)!}S_{k-1}(s-n+k-2).$$

By inequality (2.9), we have

$$\frac{1}{10^{k-1}k!}(s-n+k-1)^k \leq P_{k+1,s-n} \leq \frac{1}{k!}(s-n+k-1)^k.$$

Thus, inequalities (2.12) also hold, when $m = k+1$. The principle of mathematical induction proves the lemma.

From (2.12), since m is a fixed number, we obtain the following assertion.

Corollary 2.1. If $m \geq 2$, then there exists a positive number j_0 , that for all $j \geq j_0$ the following inequalities hold

$$A_+ j^{m-1} \leq P_{m,j} \leq B_+ j^{m-1}, \tag{2.13}$$

where A_+ and B_+ are positive constants.

Theorem 2.1. Let $1 < p < \infty$, $1/p + 1/p' = 1$, $m \geq 2$ and

$$T_{m,u,v} = \sup_{n=0,1,2,\dots} \left(\sum_{j=0}^{n} |u_j|^p\right)^{\frac{1}{p}} \left(\sum_{j=n}^{+\infty} \left(j^{(m-1)p'}\right)^{p'} |v_j|^{-p'}\right)^{\frac{1}{p'}} < \infty.$$

Then

$$\left(\sum_{n=0}^{+\infty} |u_n y_n|^p\right)^{\frac{1}{p}} \leq C_{m,u,v} \left(\sum_{n=0} +\infty \left|v_n\left(-\Delta\right)^{(m)} y_n\right|^p\right)^{\frac{1}{p}}, \forall y = \{y_n\}_{n=0}^{\infty} \in \widetilde{l}_+. \tag{2.14}$$

In addition, if $C_{m,u,v}$ is the smallest constant satisfying (2.14), then

$$A_+ T_{0,m,u,v} \leq C_{m,u,v} \leq B_+ p^{\frac{1}{p}} \left(p'\right)^{\frac{1}{p'}} T_{m,u,v}, \tag{2.15}$$

where

$$T_{0,m,u,v} = \sup_{n=0,1,2,\dots} \left(\sum_{j=0}^{n} |u_j|^p\right)^{\frac{1}{p}} \left(\sum_{j=n}^{+\infty} \left((j-n)^{(m-1)p'}\right)^{p'} |v_j|^{-p'}\right)^{\frac{1}{p'}}$$

and A_+ и B_+ are constants in (2.13).

Proof. It suffices to verify that inequalities (2.15) hold. If we use (2.2), (2.13) and the well-known weighted difference Hardy type theorem [11]

$$\left(\sum_{n=0}^{+\infty} |u_n y_n|^p\right)^{\frac{1}{p}} = \left(\sum_{n=0}^{+\infty}\left|u_n \sum_{k=n}^{+\infty} P_{m,k} v_k^{-1}\left(v_k(-\Delta)^{(m)} y_k\right)\right|^p\right)^{\frac{1}{p}} \leq$$

$$\leq p^{\frac{1}{p}} (p')^{\frac{1}{p'}} \sup_{n=0,1,2,\dots} \left[\left(\sum_{j=0}^{n} |u_j|^p\right)^{\frac{1}{p}} \left(\sum_{j=n}^{+\infty} \left|P_{m,j} v_j^{-1}\right|^{p'}\right)^{\frac{1}{p'}}\right] \left(\sum_{j=0}^{+\infty}\left|v_j(-\Delta)^{(m)} y_j\right|^p\right)^{\frac{1}{p}} \leq$$

$$\leq B_+ p^{\frac{1}{p}} p'^{\frac{1}{p'}} T_{m,u,v} \left(\sum_{j=0}^{+\infty} |v_j(-\Delta)^m y_j|^p\right)^{\frac{1}{p}}.$$

This implies the right-hand inequality in (2.15). Now we prove the left-hand inequality in (2.15). For the numbers $\widetilde{y}_{n,n}$ $(n = 0, 1, \dots)$ chosen above, by virtue of (2.7), we have

$$\sum_{n=0}^{+\infty} |u_n \widetilde{y}_{n,n}|^p = \sum_{n=0}^{+\infty} |u_n|^p \left|\sum_{s=n}^{N} P_{m,s-n}^{p'} |v_s|^{-p'}\right|^p \geq \sum_{j=0}^{n} |u_j|^p \left(\sum_{s=n+1}^{N} P_{m,s-n}^{p'} |v_s|^{-p'}\right)^p.$$

And by (2.2) and (2.6),

$$\|\widetilde{y}\|^p_{\widetilde{H}^{(m)}_{p,v}} = \sum_{s=n+1}^{N} \left[|v_s| \left(-\Delta\right)^{(m)} \widetilde{y}_s\right]^p = \sum_{s=n+1}^{N} |v_s|^p \left(P_{m,s-n}\right)^{(p'-1)p} |v_s|^{-p'p} = \sum_{s=n+1}^{N} \left(P_{m,s-n}\right)^{p'} |v_s|^{-p'}.$$

Consequently

$$\sum_{n=0}^{+\infty} |u_n \widetilde{y}_{n,n}|^p \geq \sum_{j=0}^{n} |u_j|^p \left(\sum_{s=n+1}^{N} (P_{m,s-n})^{-p'} \right)^{p-1} \cdot \|\widetilde{y}\|^p_{\widetilde{H}^{(m)}_{p,v}}.$$

According to our choice, N is any positive integer. Therefore, by (2.14), we obtain the estimate $A_+ T_{0,m,u,v} \leq$ $\leq C_{m,u,v}$. The theorem is proved.

Let $\widetilde{l}_+ = \{\{w_n\}_{n=-\infty}^{+\infty} \in \widetilde{l} : w_k = 0 \quad \forall k \geq 0\}$. The following assertion is proved similarly to Lemma 2.1 and Lemma 2.4.

Lemma 2.5. If $m \geq 2$, $n \leq -m-1$, $j \leq n$ $(m,n,j \in Z)$, then for each $y = \{y_n\}_{n=-\infty}^{0} \in \widetilde{l}_-$ the following equality holds:

$$y_n = (-1)^m \sum_{j=-\infty}^{n} P_{l,-,n-j} \left(-\Delta\right)^{(m)} y_j (n \in Z),$$

where $P_{m,-,n-j}$ $(m = 1, 2, \dots)$ are defined by

$$P_{1,-,n-j} = 1, P_{k,-,n-j} = \sum_{s=j}^{n} P_{k-1,-,s-j} (k = 2, 3, \dots)$$

and they satisfy the following estimates:

$$\frac{1}{10^{m-1} \cdot (m-1)!} (n-j+m-2)^{m-1} \leq P_{m,-,n-j} \leq \frac{1}{(m-1)!} (n-j+m-2)^{m-1}.$$

Corollary 2.2. If $m \geq 2$, then there exists a number $j_0 < 0$, such that for all $j \leq j_0$ hold the following inequalities:

$$A_- |j|^{m-1} \leq P_{m,j} \leq B_- |j|^{m-1}, \tag{2.16}$$

where and are positive constants.

We denote by $\widehat{H}^{(k)}_{p,v}$ the space with the norm

$$\|a\|_{\widehat{H}^{(k)}_{p,v}} = \left(\sum_{s=-\infty}^{0} \left| v_s \Delta^{(k)} a_s \right|^p \right)^{\frac{1}{p}} (a = \{a_s\}_{s=-\infty}^{0}).$$

Using Lemma 2.5 we prove the following assertion. *Lemma 2.6.* Let $y = \{y_n\}_{n=-\infty}^{0} \in \widetilde{l}_-$. Then

$$\sup_{\|y\|_{\widehat{H}^{(m)}_{p,v}}=1} |y_n| = \left(\sum_{s=-\infty}^{n} |P_{m,-,n-s}|^{p'} v_s^{-p'} \right)^{\frac{1}{p'}} (n = 0, -1, -2, \dots).$$

Theorem 2.2 Let $1 < p < \infty$, $1/p + 1/p' = 1$, $m \geq 2$ and

$$T''_{m,u,v} = \sup_{\tau<0} \left(\sum_{j=\tau}^{0} |u_j|^p \right)^{\frac{1}{p}} \left(\sum_{j=-\infty}^{\tau} |j|^{(m-1)p'} |v_j|^{-p'} \right)^{\frac{1}{p'}} < \infty.$$

Then for $y = \{y_n\}_{n=-\infty}^{0} \in \widetilde{l}_-$ the following inequality holds:

$$\left(\sum_{n=-\infty}^{0} |u_n y_n|^p \right)^{\frac{1}{p}} \leq C_1 \left(\sum_{n=-\infty} 0 \left| v_n \left(-\Delta\right)^{(m)} y_n \right|^p \right)^{\frac{1}{p}}. \tag{2.17}$$

In addition, if C_1 is the smallest constant satisfying (2.17), then

$$A_- T''_{0,m,u,v} \leq C_1 \leq B_- p^{\frac{1}{p}} (p')^{\frac{1}{p'}} T''_{m,u,v}.$$

Here A_- and B_- are constants in (2.16), and where

$$T''_{0,m,u,v} = \sup_{n=0,-1,-2,\dots} \left(\sum_{j=\tau}^{0} |u_j|^p \right)^{\frac{1}{p}} \left(\sum_{j=-\infty}^{\tau} |(\tau - j|^{(m-1)p'} |v_j|^{-p'} \right)^{\frac{1}{p'}} .$$

This theorem is proved on the basis of Theorem 2.1, by performing some simple substitutions. We introduce the notations

$$\gamma_{m,u,v,B} = \sqrt[p]{\max \left[(B_+ T_{m,u,v})^p , \left(B_- T''_{m,u,v} \right)^p \right]}$$

and

$$\gamma'_{m,u,v,B} = \sqrt[p]{\min \left[(A_+ T_{0,m,u,v})^p , \left(A_- T''_{0,m,u,v} \right)^p \right]},$$

where A_+ , B_+ ,A_-, B_- are constants in (2.13) and (2.16).

Theorem 2.3 Let the sequences u, v satisfy the condition $\gamma_{m,u,v,B} < \infty$. Then for $y_j \in \widetilde{l}$ the following inequalities hold:

$$\sum_{j=-\infty}^{+\infty} |u_j y_j|^p \, leq C^p_{2,m,u,v} \sum_{j=-\infty}^{+\infty} \left| v_j \Delta^{(m)} y_j \right|^p . \tag{2.18}$$

In addition, if $C_{2,m,u,v}$ is a smallest constant for which (2.18) is true, then

$$\gamma'_{m,u,v,A} \le C_{2,m,u,v} \le p^{1/p} \left(p' \right)^{1/p'} \gamma_{m,u,v,B}. \tag{2.19}$$

Proof. By Theorem 2.1 and Theorem 2.2, we obtain estimates (2.14) and (2.17)

$$\sum_{n=0}^{+\infty} |u_n y_n|^p \, leq C^p_{m,u,v} \sum_{n=0}^{+\infty} \left| v_n \left(-\Delta \right)^{(m)} y_n \right|^p$$

and

$$\sum_{j=-\infty}^{-1} |u_n y_n|^p \, leq C^p_{1,m,u,v} \sum_{n=-\infty}^{-1} \left| v_n \Delta^{(m)} y_n \right|^p .$$

Summing them, we have (2.18). Now we estimate $C_{2,m,u,v}$. According to (2.14) and (2.17),

$$\sum_{j=-\infty}^{+\infty} |u_n y_n|^p \le \left[B_+ p^{\frac{1}{p}} \left(p' \right)^{\frac{1}{p'}} T_{m,u,v} \right]^p \sum_{n=0}^{\infty} \left| v_n \left(-\Delta \right)^{(m)} y_n \right|^p + \left[B_- p^{\frac{1}{p}} \left(p' \right)^{\frac{1}{p'}} T''_{m,u,v} \right]^p \sum_{n=-\infty}^{0} \left| v_n \Delta^{(m)} y_n \right|^p \le$$

$$\le p^{\frac{1}{p}} \left(p' \right)^{p-1} \gamma^p_{m,u,v,B} \sum_{n=-\infty}^{+\infty} \left| v_n \Delta^{(m)} y_n \right|^p .$$

In this way, $C_{2,m,u,v} \le p^{\frac{1}{p}} \left(p' \right)^{p-1} \gamma_{m,u,v,B}$.

By Theorem 2.1 and Theorem 2.2, respectively, $C_{m,u,v} \ge T_{0,m,u,v}$ and $C_{1,m,u,v} \ge T''_{0,m,u,v}$. In other words, there are sequences $z = \{z_n\}_{n=-\infty}^{1} \in \widetilde{l}_-$ and $\theta = \{\theta_k\}_{k=0}^{+\infty} \in \widetilde{l}_+$ such that

$$\sum_{n=-\infty}^{-1} |u_n z_n|^p \ge \left(A_- T''_{0,m,u,v} \right)^p \sum_{n=-\infty}^{-1} \left| v_n \Delta^{(m)} z_n \right|^p ;$$

$$\sum_{n=0}^{+\infty} |u_n \theta_n|^p \ge \left(A_+ T_{0,m,u,v} \right)^p \sum_{n=0}^{+\infty} \left| v_n \Delta^{(m)} \theta_n \right|^p .$$

Then, denoting $z_k = \theta_k$ $(k = 0, 1, \dots)$, for $\widetilde{z} = \{z_n\}_{n=-\infty}^{+\infty} \in \widetilde{l}$, we obtain

$$\sum_{n=-\infty}^{+\infty} |u_n z_n|^p \ge \left(A_+ T_{0,m,u,v} \right)^p \sum_{n=0}^{+\infty} \left| v_n \Delta^{(m)} z_n \right|^p + \left(A_- T''_{0,m,u,v} \right)^p \sum_{n=-\infty}^{-1} \left| v_n \Delta^{(m)} z_n \right|^p \ge$$

$$\ge \left(\gamma'_{m,u,v,A} \right)^p \sum_{n=-\infty}^{+\infty} \left| v_n \Delta^{(m)} z_n \right|^p .$$

This implies the left-hand inequality in (2.19). The theorem is proved.

3 Coercive estimates for a degenerate system of difference equations

Consider the following system

$$l_0 y_j = \Delta^{2n} y_j + r_{jj}\Delta^{2n-1} y_j = F_j, j \in Z. \tag{3.1}$$

Let $y = \{y_j\}_{j=-\infty}^{+\infty} \in \widetilde{l}$. From (3.1) we obtain

$$\sum_{j\in Z} \Delta^{2n} y_j \cdot \Delta^{2n-1} y_j + \sum_{j\in Z} r_{jj} \left[\Delta^{2n-1} y_j\right]^2 = \sum_{j\in Z} l_0 y_j \cdot r_j \Delta^{2n-1} y_j. \tag{3.2}$$

Putting $\Delta^{2n-1} y_j = z_j$, we rewrite this equality in the following form:

$$\sum_{j\in Z} \Delta_- z_j \cdot z_j + \sum_{j\in Z} r_j z_j^2 = \sum_{j\in Z} l_0 y_j \cdot z_j,$$

It is easy to see that the expression $A = \sum_{j\in Z} \Delta_- z_j \cdot z_j$ satisfies the equality $A = \sum_{j\in Z} [\Delta_- z_j]^2 - A$, so

$$A = \frac{1}{2}\sum_{j\in Z} [\Delta_- z_j]^2 = \frac{1}{2}\sum_{j\in Z} \left[\Delta^{2n} y_j\right]^2.$$

Then (3.2) implies the following estimate:

$$\frac{1}{2}\sum_{j\in Z} \left[\Delta^{2n} y_j\right]^2 + \sum_{j\in Z} r_{jj} \left[\Delta^{2n-1} y_j\right]^2 \le \left(\sum_{j\in Z} \left[\frac{l_0 y_j}{\sqrt{r_{jj}}}\right]^2\right)^{\frac{1}{2}} \left(\sum_{j\in Z} \left[\sqrt{r_{jj}}\Delta^{(2n-1)} y_j\right]^2\right)^{\frac{1}{2}}, \tag{3.3}$$

in particular,

$$\left(\sum_{j\in Z} \left[\sqrt{r_{jj}}\Delta^{(2n-1)} y_j\right]^2\right)^{\frac{1}{2}} \le \left(\sum_{j\in Z} \left[l_0 y_j \frac{1}{\sqrt{r_{jj}}}\right]^2\right)^{\frac{1}{2}}.$$

Using this inequality and condition $r_{jj} \ge 1$, from (3.3) we have

$$\frac{1}{2}\sum_{j\in Z} \left[\Delta^{2n} y_j\right]^2 + \sum_{j\in Z} r_{jj} \left[\Delta^{2n-1} y_j\right]^2 \le \sum_{j\in Z} \left[l_0 y_j\right]^2. \tag{3.4}$$

Then, in view of (3.1),

$$\sum_{j\in Z} r_{jj}^2 \left[\Delta^{2n-1} y_j\right]^2 \le 3\sum_{j\in Z} (l_0 y_j)^2.$$

From this and (3.4)

$$\left\|\Delta^{(2n)} y\right\|_2^2 + \left\|r\Delta^{(2n-1)} y\right\|_2^2 \le 5\left\|l_0 y\right\|_2^2,$$

then

$$\left\|\Delta^{(2n)} y\right\|_2 + \left\|r\Delta^{(2n-1)} y\right\|_2 \le 5\sqrt{2}\left\|l_0 y\right\|_2, \quad y \in \widetilde{l}. \tag{3.5}$$

We put $\widetilde{r} = \{r_{jj}\}_{j=-\infty}^{+\infty}$. If $\gamma_{2n-1,e,\widetilde{r}} < \infty$, then by Theorem 2.3,

$$\|y\|_2 \le 2\gamma_{2n-1,e,\widetilde{r}} \left\|r\Delta^{(2n-1)} y\right\|_2, y \in \widetilde{l}.$$

Taking this into account, from (3.5) we obtain

$$\left\|\Delta^{(2n)} y\right\|_2 + \left\|r\Delta^{(2n-1)} y\right\|_2 + \|y\|_2 \le \left(2\gamma_{2n-1,e,\widetilde{r}} + \sqrt{10}\right)\left\|l_0 y\right\|_2, y \in \widetilde{l}. \tag{3.6}$$

The following assertion is proved by a standard method.

Lemma 3.1. Let $r_{jj} \geq 1$ $(j \in Z)$ and $\gamma_{2n-1,e,\widetilde{r}} < \infty$. Then the operator l_0 $(D(l_0) = \widetilde{l})$ corresponding to the system (3.1) is closable in the norm of l_2 . We denote by l the closure of l_0 in l_2 .

Lemma 3.2. Let $r_{jj} \geq 1$ $(j \in Z)$ and $\gamma_{2n-1,e,\widetilde{r}} < \infty$. Then the inequality (3.6) holds for each $y \in D(l)$.

Proof If $y \in D(l)$, then there exists a sequence $\{y^{(k)}\}_{k=1}^{\infty}$ such that

$$\left\|y^{(k)} - y\right\|_2 \to 0, \left\|l_0 y^{(k)} - ly\right\|_2 \to 0$$

as $k \to \infty$. According to (3.6)

$$\left\|\Delta^{(2n)} y^{(k)}\right\|_2 + \left\|r\Delta^{(2n-1)} y^{(k)}\right\|_2 + \left\|y^{(k)}\right\|_2 \leq \left(2\gamma_{2n-1,e,\widetilde{r}} + \sqrt{10}\right) \left\|ly^{(k)} - ly^{(m)}\right\|_2 . \tag{3.7}$$

We denote by $\dot{w}_2^{(2n)}$ the completion of $\widetilde{l}$ in the norm $\|v\|_w = \left\|\Delta^{(2n)} v\right\|_2 + \left\|r\Delta^{(2n-1)} v\right\|_2 + \|v\|_2$. $\dot{w}_2^{(2n)}$ is a difference Sobolev space with weight. By (3.7), for $\forall k, m \in N$ we obtain

$$\left\|y^{(k)} - y^{(m)}\right\|_w \leq \left(2\gamma_{2n-1,e,\widetilde{r}} + \sqrt{10}\right) \left\|ly^{(k)} - ly^{(m)}\right\|_2 .$$

Consequently, the sequence $\{y^{(k)}\}_{k=1}^{\infty} \in \widetilde{l}$ is fundamental in a Banach space $w_2^{(2n)}$, so, there exists an element $v \in w_2^{(2n)}$ such that $\left\|y^{(k)} - v\right\|_w \to 0$ $(k \to \infty)$. According to our choice $\left\|ly^{(k)} - ly\right\|_2 \to 0$ $(k \to \infty)$. Then $v \in D(l)$ and $ly = lv$. By (3.7),

$$\|v\|_w \leq \left(2\gamma_{2n-1,e,\widetilde{r}} + \sqrt{10}\right) \|lv\|_2 = \left(2\gamma_{2n-1,e,\widetilde{r}} + \sqrt{10}\right) \|ly\|_2 . \tag{3.8}$$

Therefore $D(l) \subseteq w_2^{(2n)}$. In this way, $y \in w_2^{(2n)}$ and $v = y$. Then by (3.8), we obtain the inequality (3.6). The lemma is proved.

Definition 3.1. The element $y = \{y_j\}_{j=-\infty}^{+\infty}$ is called a solution of the system (3.1), if there is a sequence $\{z^{(k)}\}_{k=1}^{+\infty}$ such that the following relations are satisfied:

$$\left\|z^{(k)-y}\right\|_2 \to 0, \left\|l_0 z^{(k)-F}\right\|_2 \to 0, (k \to +\infty).$$

It's clear that $y = \{y_j h\}_{j=-\infty}^{+\infty} \in l_2$ is a solution of (3.1) if and only if $y \in D(l)$ and $ly = f$.

Theorem 3.1. Let $n \geq 2$, and the sequence $\widetilde{r} = \{r_{jj}\}_{j=-\infty}^{+\infty}$ satisfies the conditions $r_{jj} \geq 1$ $(j \in Z)$ and $\gamma_{2n-1,e,\widetilde{r}} < \infty$. Then the system (3.1) has a unique solution $y \in l_2$. In addition, for the solution y holds (3.6).

Proof. By lemma 3.2 and Definition 3.1, the inequality (3.6) holds for a solution of (3.1). The solution of the system (3.1) is unique. In fact, if $z_1, z_2 \in l_2$ are two solutions of (3.1), then for $w = z_1 - z_2$ we have $lw = 0$. By (3.6), $\|w\|_2 = 0$, so $z_1 = z_2$. Now we will prove that the solution of the system (3.1) does exist. It suffices to show that $R(l) = l_2$. Assume the contrary, let $R(l) \neq l_2$. Then there exists an element $v \in l_2 \;\; R(l)$, $v \neq 0$, such that $(l_0 y, v) = 0 \forall y \in D(l_0)$. Since the set $D(l_0) = \widetilde{l}$ is dense in l_2 , we have

$$\Delta^{(2n)} v_j - \Delta^{(2n-1)} \left(r_{jj} v_j\right) = 0, \forall j \in Z. \tag{3.9}$$

By choice, $v \in l_2$, therefore, $\lim_{|j|\to\infty} |v_j|^2 = 0$. Consequently $\forall \varepsilon > 0$, $\exists j_0 : \forall j \geq j_0$ $\left|\Delta^{(2n)} v_j\right| < \varepsilon$. Then by (3.9)

$$\Delta^{(2n-1)} \left(r_{jj} v_j\right) = 0.$$

According to the conditions imposed on the sequence $r = \{r_{jj}\}_{(j=-\infty}^{+\infty}$, the solution of this equation belonging to l_2 , is only $v = 0$. We obtain the contradiction. The theorem is proved.

4 Proofs of the main theorems

Proof of Theorem 1.1. By condition (1.2), hold (1.3). Then by Theorem 3.1, the minimal closed operator $\dot{l}$ in l_2 defined by equality $\dot{l}y = \Delta^{(2n)} y + r\Delta^{(2n-1)} y$, is continuously invertible and (3.5) holds for each $y \in D(\dot{l})$. In view of (3.5) and (1.3)

$$\left\|s\overline{\Delta^{(2n-1)} y}\right\|_2 < \alpha_1 \left\|r\Delta^{(2n-1)} y\right\|_2 \leq 5\sqrt{2}\alpha_1 \left\|\dot{l}y\right\|_2 . \tag{4.1}$$

Then, by (1.3), (4.1) and the well-known theorem on small perturbations, $\widehat{l}y = y + s\overline{\dot{\Delta}^{(2n-1)}y}$ is an closed and continuously invertible operator. And for $y \in D(\widehat{l})$ holds the inequality

$$\left\|\Delta^{(2n)}y\right\|_2 + \left\|r\Delta^{(2n-1)}y\right\|_2 + \left\|s\overline{\Delta^{(2n-1)}y}\right\|_2 \leq 5\sqrt{2}(1+\alpha_1)\left\|\dot{l}y\right\|_2. \tag{4.2}$$

On the other hand,

$$\left\|\dot{l}y\right\|_2 \leq \left\|\widehat{l}y\right\|_2 + \left\|s\overline{\Delta^{(2n-1)}y}\right\|_2 < \left\|\widehat{l}y\right\|_2 + 5\sqrt{2}\alpha_1\left\|\dot{l}y\right\|_2,$$

so

$$\left\|\dot{l}y\right\|_2 \leq \frac{1}{1-5\sqrt{2}\alpha_1}\left\|\widehat{l}y\right\|_2 \forall y \in D(\widehat{l}).$$

Then from (4.2)

$$\left\|\Delta^{(2n)}y\right\|_2 + \left\|r\Delta^{(2n-1)}y\right\|_2 + \left\|s\overline{\Delta^{(2n-1)}y}\right\|_2 \leq \frac{5\sqrt{2}(1+\alpha_1)}{1-5\sqrt{2}\alpha_1}\left\|\widehat{1}y\right\|_2 \forall y \in D(\widehat{l}). \tag{4.3}$$

Let, now, k be a positive constant, and $\tau = \frac{1}{k}$. If the values of $\widetilde{y}_{j\tau}$ $(j \in Z)$ are chosen such that $\widetilde{y}_{j\tau} = y_{j\tau}$ $(j\tau \in Z)$, then $\Delta_+ y_j = y_{j+1} - y_j = k\left(\widetilde{y}_{(j+1)\tau} - \widetilde{y}_{j\tau}\right) = k\Delta_+\widetilde{y}_{j\tau}$, $\Delta^{(2)}y_j = \Delta_+(\Delta_- y_j) = k^2\Delta^{(2)}\widetilde{y}_{j\tau}$ and $\Delta_-\left(\Delta^{(2)}y_j\right) = k^3\Delta^{(3)}\widetilde{y}_{j\tau}$. Also $\Delta^{(m)}y_j = k^m\Delta^{(m)}\widetilde{y}_{j\tau}$, $m \in N$. Therefore, if we introduce quantities $\widehat{r}_{jj\tau}$, $\widehat{s}_{jj\tau}$, $\widehat{q}_{jj\tau}$, $\widehat{p}_{jj\tau}$, $\widehat{f}_{jj\tau}$ $(j \in Z)$ according to $\widehat{r}_{jj\tau} = r_{jj\tau}$, $\widehat{s}_{jj\tau} = s_{jj\tau}$, $\widehat{q}_{jj\tau} = q_{jj\tau}$, $\widehat{p}_{jj\tau} = p_{jj\tau}$, $\widehat{f}_{jj\tau} = f_{jj\tau}$ $(j\tau = \frac{j}{k} \in Z)$, then equation (1.1) reduces to the form:

$$\widehat{L}_0 y = \Delta^{(2n)}\widetilde{y} + \frac{1}{k}\widehat{r}\Delta^{(2n-1)}\widetilde{y} + \frac{1}{k}\overline{\widehat{s}\Delta^{(2n-1)}\widetilde{y}} +$$

$$+\sum_{j=1}^{2n-1}\frac{1}{k^{j+1}}\left(\widehat{Q}^{(j)}\Delta^{2n-j-1}\widetilde{y} + \widehat{P}^{(j)}\overline{\Delta^{2n-j-1}\widetilde{y}}\right) = k^{-2n}\widehat{f}, \widehat{f} \in l_{2,\tau}, \tag{4.4}$$

where $\widetilde{y} = \{\widetilde{y}_{j\tau}\}_{j=-\infty}^{+\infty}$; $\widehat{r} = \{\mathrm{diag}, \widehat{r}_{jj\tau}\}_{j=-\infty}^{+\infty}$; $\widehat{s} = \{\mathrm{diag}, \widehat{s}_{jj\tau}\}_{j=-\infty}^{+\infty}$; $\widehat{Q}^{(\theta)} = \{\mathrm{diag}, \widehat{q}_{jj\tau}^{(\theta)}\}_{j=-\infty}^{+\infty}$; $\widehat{P}^{(\theta)} = \{\mathrm{diag}, \widehat{p}_{jj\tau}^{(\theta)}\}_{j=-\infty}^{+\infty}$; $(\theta = \overline{1, 2n-1})$, $\widetilde{f} = \{\widetilde{f}_{j\tau}\}_{j=-\infty}^{+\infty}$. We rewrite (4.4) in the following form:

$$\widehat{l}\widetilde{y} + \sum_{j=1}^{2n-1}\frac{1}{k^{j+1}}\left(\widehat{Q}^{(j)}\Delta^{2n-j-1}\widetilde{y} + \widehat{P}^{(j)}\overline{\Delta^{2n-j-1}\widetilde{y}}\right) = \frac{1}{2n}\widehat{f}, \widehat{f} \in l_{2,\tau}.$$

By condition (1.2) and Theorem 2.3,

$$\sum_{j=1}^{2n-1}\left\|\frac{1}{k^{j+1}}\widehat{Q}^{(j)}\Delta^{2n-j-1}\widetilde{y}\right\|_{2,\tau} \leq$$

$$\leq (2n-1)max_{j=\overline{1,2n-1}}\left[\frac{1}{k^{j+1}}\gamma_{(2n-1),\widehat{Q}^{(j)},[min(A_-,A_+)]^{-1}\widehat{r}}\right]\left\|\widehat{r}\Delta^{(2n-1)}\widetilde{y}\right\|_{2,\tau} \tag{4.5}$$

and

$$\sum_{j=1}^{2n-1}\left\|\frac{1}{k^{j+1}}\widehat{P}^{(j)}\Delta^{2n-j-1}\widetilde{y}\right\|_{2,\tau} \leq$$

$$\leq (2n-1)max_{j=\overline{1,2n-1}}\left[\frac{1}{k^{j+1}}\gamma_{(2n-1),\widehat{P}^{(j)},[min(A_-,A_+)]^{-1}\widehat{r}}\right]\left\|\widehat{r}\Delta^{(2n-1)}\widetilde{y}\right\|_{2,\tau}. \tag{4.6}$$

We denote

$$G(\Phi, \widetilde{r}, n, \tau) = (2n-1), max_{j=\overline{1,2n-1}}\left[\frac{1}{k^{j+1}}\gamma_{(2n-1),\Phi,[min(A_-,A_+)]^{-1}\widehat{r}}\right],$$

where Φ is the matrix equal to either $\widehat{Q}^{(\theta)}$, or $\widehat{P}^{(\theta)}$ $(\theta = \overline{1, 2n-1})$. If we choose k such that

$$k^{2n-1} > \frac{5\sqrt{2}(1+\alpha_1)}{\alpha(1-5\sqrt{2}\alpha_1)}\left\{4(2n-1)max\left[max_{\theta=\overline{1,2n-1}}G(\widehat{P}^{(\theta)}, \widehat{r}, n, \theta)\right]\right\}\frac{5\sqrt{2}(1+\alpha_1)}{(1-5\sqrt{2}\alpha_1)k^{2n-1}}\left\|\widehat{1}\widetilde{y}\right\|_2$$

for some $\alpha \in (0,1)$, then according to (4.5) and (4.6),

$$\left\| \sum_{j=1}^{2n-1} \frac{1}{k^{j+1}} \left(\widehat{Q}^{(j)} \Delta^{2n-j-1} \widetilde{y} + \widehat{P}^{(j)} \overline{\Delta^{2n-j-1} \widetilde{y}} \right) \right\|_{2,\tau} \le$$

$$\le \left\{ 4(2n-1) max \left[max_{\theta=\overline{1,2n-1}} G(\widehat{Q}^{(\theta)}, \widehat{r}, n, \theta), max_{\theta=\overline{1,2n-1}} G(\widehat{P}^{(\theta)}, \widehat{r}, n, \theta) \right] \right\} \frac{5\sqrt{2}(1+\alpha_1)}{(1-5\sqrt{2}\alpha_1)k^{2n-1}} \left\| \widehat{l}\widetilde{y} \right\|_{2,\tau} <$$

$$< \alpha \left\| \widehat{l}\widetilde{y} \right\|_{2,\tau}. \tag{4.7}$$

Since the operator $\widehat{l}$ is closed, from the inequality (4.7), first, by the theorem on small perturbations it follows that the operator $\widehat{L}_0$ is closed. We denote its closure by $\widehat{L}$. Second, it follows that the operator $\widehat{L}$ is continuously invertible. Then, since $\widehat{L}\widetilde{y} = Ly$, the operator L is also closed and continuously invertible. Thus, by Definition 3.1, the solution of the equation (1.1) exists and is unique.

Let $\widetilde{y} \in D(\widehat{L}_0$. Then from inequalities (4.3) and (4.7) we obtain:

$$\left\| \Delta^{(2n)} \widetilde{y} \right\|_{2,\tau} + \left\| k^{-1} \widehat{r} \Delta^{(2n-1)} \widetilde{y} \right\|_{2,\tau} + \left\| k^{-1} \widehat{s} \overline{\Delta^{(2n-1)} \widetilde{y}} \right\|_{2,\tau} + \sum_{j=1}^{2n-1} \left\| \frac{1}{k^{j+1}} \widehat{Q}^{(j)} \Delta^{2n-j-1} \widetilde{y} \right\|_{2,\tau} +$$

$$+ \sum_{j=1}^{2n-1} \left\| \frac{1}{k^{j+1}} \widehat{P}^{(j)} \Delta^{2n-j-1} \widetilde{y} \right\|_{2,\tau} \le \left(\frac{5\sqrt{2}(1+\alpha_1)}{1-5\sqrt{2}\alpha_1} + \alpha \right) \left\| \widehat{l}\widetilde{y} \right\|_{2,\tau}. \tag{4.8}$$

Further

$$\left\| \widehat{l}\widetilde{y} \right\|_{2,\tau} \le \left\| \widehat{l}\widetilde{y} + \sum_{j=1}^{2n-2} \frac{1}{k^{j+1}} \left(\widehat{Q}^{(j)} \Delta^{2n-j-1} \widetilde{y} + \widehat{P}^{(j)} \overline{\Delta^{2n-j-1} \widetilde{y}} \right) \right\|_{2,\tau} +$$

$$+ \left\| \sum_{j=1}^{2n-2} \frac{1}{k^{j+1}} \left(\widehat{Q}^{(j)} \Delta^{2n-j-1} \widetilde{y} + \widehat{P}^{(j)} \overline{\Delta^{2n-j-1} \widetilde{y}} \right) \right\|_{2,\tau} \le \left\| \widehat{L}\widetilde{y} \right\|_{2,\tau} + \alpha \left\| \widehat{l}\widetilde{y} \right\|_{2,\tau};$$

$$\left\| \widehat{l}\widetilde{y} \right\|_{2,\tau} \le \frac{1}{1-\alpha} \left\| \widehat{L}\widetilde{y} \right\|_{2,\tau}. \tag{4.9}$$

By virtue of (4.8) and (4.9)

$$\left\| \Delta^{(2n)} \widetilde{y} \right\|_{2,\tau} + \left\| \frac{1}{k} \widehat{r} \Delta^{(2n-1)} \widetilde{y} \right\|_{2,\tau} + \left\| \frac{1}{k} \widehat{s} \overline{\Delta^{(2n-1)} \widetilde{y}} \right\|_{2,\tau} + \sum_{j=1}^{2n-1} \frac{1}{k^{j+1}} \left(\left\| \widehat{Q}^{(j)} \Delta^{2n-j-1} \widetilde{y} \right\|_{2,\tau} + \left\| \widehat{P}^{(j)} \overline{\Delta^{2n-j-1} \widetilde{y}} \right\|_{2,\tau} \right) \le$$

$$\le \frac{(1-5\sqrt{2}\alpha_1)\alpha + 5\sqrt{2}(1+\alpha_1)}{(1-\alpha)(1-5\sqrt{2}\alpha_1)} \left\| \widehat{L}\widetilde{y} \right\|_{2,\tau}. \tag{4.10}$$

From (4.10), making the substitutions, we obtain the inequality (1.4) for each $\widetilde{y} \in D(L)$, and, in addition, for the solution of the system (1.1). The theorem is proved. *Proof of Theorem 1.2.* By virtue of the inequality (1.4), the operator L^{-1} displays the entire space l_2 on the difference Sobolev space $H_{2,\widetilde{r}}^{(2n)}$ with the norm $\left\| \Delta^{(2n)} y \right\|_2 + \left\| r \Delta^{(2n-1)} y \right\|_2$. According to the results of [12, 13], when conditions (1.5) and (1.6) are satisfied, an embedding operator of a weighted space $H_{2,\widetilde{r}}^{(2n)}$ in l_2 is compact. Therefore, $L^{-1} : l_2 \to l_2$ is a compact operator. The theorem is proved. *Remark 4.1* If $h > 0$, then the assertions of Theorem 1.1 and Theorem 1.2 are also satisfied for an infinite difference system

$$L_0 y = h^{-2n} \Delta^{(2n)} y + h^{-2n+1} r \Delta^{(2n-1)} y + h^{-2n+1} s \overline{\Delta^{(2n-1)} y} +$$

$$+ \sum_{j=1}^{2n-1} \left(Q^{(j)} h^{-(2n-j-1)} \Delta^{(2n-j-1)} y + P^{(j)} h^{-(2n-j-1)} \overline{\Delta^{(2n-j-1)y}} \right) = f,$$

where $y = \{y_{jh}\}_{j=-\infty}^{+\infty}$; $\Delta_+ y_{kh} = y_{(k+1)h} - y_{kh}$; $\Delta^{(2)} y_{kh} = \Delta_- \Delta_+ y_{kh} = y_{(k+1)h} - 2y_{kh} + y_{(k-1)h}$ and $r = (r_{jh,jh})_{j \in Z} \, (r_{jh,jh} \ge 1)$; $s = (s_{jh,jh})_{j \in Z}$; $Q^\theta = \left(q_{jh,jh}^{(\theta)} \right)_{j \in Z}$; $P^\theta = \left(p_{jh,jh}^{(\theta)} \right)_{j \in Z}$ $\left(\theta = \overline{1, 2n-1} \right)$ are diagonal matrices, and $f \in l_2(h)$. Here $l_2(h)$ is the space of the real numerical sequences $f = \{f_{jh}\}_{j=-\infty}^{+\infty}$ with the norm $\|f\|_{2,h} = \left(\sum_{j=-\infty}^{+\infty} f_{jh}^2 h\right)^{1/2}$.

This work is partially supported by project AP05131649 of the Ministry of Education and Science of the Republic of Kazakhstan and by L.N. Gumilyov Eurasian National University Research Fund.

References

1 Тихонов А.Н. Уравнения математической физики / А.Н. Тихонов, А.А. Самарский. — М.: Наука, 1973. — 136 с.

2 Voit S.S. The propagation of the initial condensation in a viscous gas / S.S. Voit // Uchenye zapiski Moskovskogo gosudarstvennogo universiteta. Mechanics. — 1954. — Vol. 5. — P. 125–142.

3 Ornstein S. On the theory of Brownian motion / S. Ornstein, G.E. Uhlenbeck // Physical Review. — 1930. — Vol. 36. — P. 823–841.

4 Wang M.Ch. On the theory of the Brownian motion / M.Ch. Wang, G.E. Uhlenbek // Review of Modern Physics. — 1945. — Vol. 17. — No. 2-3. — P. 323–342.

5 Pruss J. The domain of elliptic operators on Lp(Rd) with unbounded drift coefficients / J. Pruss // Houston journal of mathematics. — 2006. — Vol. 32. — No. 2. — P. 563–576.

6 Bogachev V.I. Fokker–Planck–Kolmogorov Equations / V.I. Bogachev, N.V. Krylov, M. Reckner, S.V. Shaposhnikov // Alpha Magnetic Spectrometer. — 2015. — Vol. 207.

7 Оспанов Қ.Н. Комплекс коэффициентті шексіз айырымдық теңдеулер жүйесінің коэрцитивті шешілу шарттары / Қ.Н. Оспанов, Т.Н. Бекжан, Д.Р. Бейсенова // Қарағанды ун-нің хабаршысы. Математика сер. — 2017. — № 3(87). — С. 59–69.

8 Отелбаев М. Коэрцитивные оценки и теоремы разделимости для эллиптических уравнений в R^n / М. Отелбаев // Труды Математического ин-та им. В.А. Стеклова АН СССР. — 1983. — Т. 161. — С. 195–217.

9 Бойматов К.Х. Теоремы разделимости, весовые пространства и их приложения к краевым задачам / К.Х.Бойматов // Труды Математического ин-та им. В.А. Стеклова АН СССР. — 1984. — Т. 170. — С. 37–76.

10 Ospanov K.N. Coercive solvability of degenerate system of second order difference equations / K.N. Ospanov, A. Zulkhazhav // American Institute of Physics Conference Proceedings. — 2016. — Vol. 1759. — P. 1–5.

11 Мухамедиев Г.Х. Плотность финитных функций и нормы вложения для одного класса весовых пространств: дис. ... канд. физ.-мат. наук: 01.01.01 – «Математический анализ» / Г.Х. Мухамедиев. — Баку, 1986.

12 Мустафина Л.М. О некоторых разностных весовых теоремах вложения: автореф. дис. ... канд. физ.-мат. наук: 01.01.01 – «Математический анализ» /Л.М. Мустафина. — Алма-Ата, 1989. — 16 с.

13 Апышев О.Д. О спектре одного класса дифференциальных операторов и некоторые теоремы вложения / О.Д. Апышев, М. Отелбаев // Известия АН СССР. — 1979. — Т. 43. — № 4. — С. 739–764.

Д.Р. Бейсенова, Қ.Н. Оспанов

Жоғарғы ретті айырымдық теңдеулер жүйесі үшін максималды регулярлық және шағын шарттары

Мақалада жұп ретті шексіз айырымдық теңдеулер жүйесі қарастырылған, жүйенің оң жағы гильберт кеңістігінің сандық тізбектерінен тұрады. Жұмыста қарастырылған жүйе теңдеулерінің коэффициенттерінен құралған тізбектер бірдей ретті айырымдарда шексіз болуы мүмкін, сонымен бірге олардың өсуі потенциалдың өсуіне бағынбауы мүмкін. Біздің қарастырып отырған жағдайда потенциал нольдік тізбек, немесе таңбасы белгілі болмауы мүмкін, сондықтан Штурм-Лиувилль типті айырымдық теңдеулер жүйесінің потенциалы өсуін қолданатын бұрын-соңды жасалған әдістер бұл жерде қолданыла алмайды. Мақала авторлары жүйенің коррективті шешілуінің шарттарын келтіріп, шешімнің және оның айырмаларының нормаларының тиімді бағаларын берді. Нұқсанды оператор жүйесіне сәйкес резольвентаның компактылығы шарттары алынды. Өзіндік ғылыми қызығушылық тудыратын кейбір салмақты айырымдық Харди типтес теңсіздіктер дәлелденген. Олар жұмыстың негізгі нәтижелерін дәлелдеу барысында пайдаланылған. Нұқсанды дифференциалдық теңдеулермен салыстырғанда айырымдық жүйе жағдайында жүйе коэффициенттері тербелісіне қойылған шартты алып тастауға болатыны көрсетілген.

Кілт сөздер: айырымдық жүйе, аралық коэффициент, шешімнің коррективтілігі, максималды регулярлық, резольвентаның компактылығы.

Д.Р. Бейсенова, К.Н. Оспанов

Условия максимальной регулярности и компактности для системы разностных уравнений высокого порядка

В статье исследована бесконечная линейная система разностных уравнений высокого четного порядка с правой частью из гильбертова пространства числовых последовательностей. Последовательности, образованные из коэффициентов уравнений системы при одинаковых порядках разностей, могут быть неограниченными, а также их рост может не подчиняться росту потенциала. Ранее разработанные методы, существенно использующие доминирующий рост потенциала в разностных системах уравнений типа Штурма-Лиувилля, здесь не подходят, так как в рассматриваемом нами случае потенциал может оказаться нулевым, или не имеющим определенного знака последовательности. Авторами статьи приведены условия корректной разрешимости системы, а также оптимальные оценки норм решения и его разностей вплоть до самого старшего порядка. Получены условия компактности резольвенты, соответствующей системе вырожденного оператора. Доказаны некоторые разностные весовые неравенства типа Харди, имеющие самостоятельный научный интерес. Они использованы в доказательстве основных результатов работы. Показано, что, по сравнению с вырожденными дифференциальными уравнениями, в случае разностной системы удается снять условие на колебания коэффициентов системы.

Ключевые слова: разностная система, промежуточный коэффициент, корректность решения, максимальная регулярность, компактность разрешения.

References

1 Tikhonov, A.N., & Samarski, A.A. (1973). *Uravneniia matematicheskoi fiziki [Equations of Mathematical physics]*. Moscow: Nauka [in Russian].

2 Voit, S.S. (1954). The propagation of the initial condensation in a viscous gas. *Uchenye zapiski Moskovskoho hosudarstvennoho universiteta. Mechanics, Vol. 5,* 125–142.

3 Ornstein, S., & Uhlenbeck, G.E. (1930). On the theory of Brownian motion. *Physical Review, Vol. 36,* 823–841.

4 Wang, M.Ch., & Uhlenbek, G.E. (1945). On the theory of the Brownian motion. *Review of Modern Physics, Vol. 17, 2-3,* 323–342.

5 Pruss, J., Rhandi, A., & Schnaubelt, R. (2006). The domain of elliptic operators on Lp(Rd) with unbounded drift coefficients. *Houston journal of mathematics, Vol. 32, 2,* 563–576.

6 Bogachev, V.I., Krylov, N.V., Reckner, M., & Shaposhnikov, S.V. (2015). Fokker–Planck–Kolmogorov Equations. *Alpha Magnetic Spectrometer, Vol. 207.*

7 Ospanov, K.N., Bekjan, T.N., & Beissenova, D.R. (2017). Kompleks koeffitsientti sheksiz aiyrymdyk tendeuler zhuiesinin koertsitivti sheshilu sharttary [Coercive solvability conditions of an infinite system of difference equations with complex coefficients]. *Karagandy universitetinin khabarshysy. Matematika seriiasy – Bulletin of Karaganda University. Mathematics series, 3(87),* 59–69 [in Kazakh].

8 Otelbaev, M. (1984). Koertsitivnye otsenki i teoremy razdelimosti dlia ellipticheskikh uravnenii v R^n [Coercive estimates and separation theorems for elliptic equations in R^n]. *Trudy Matematicheskoho instituta imeni V.A. Steklova AN SSSR – Proceedings of the Mathematical Institute to them V.A. Steklov AS USSR, 3,* 195–217 [in Russian].

9 Boimatov, K.Kh. (1984). Teoremy razdelimosti, vesovye prostranstva i ikh prilozheniia k kraevym zadacham [Separation theorems, weighted spaces and its applications to the boundary value problems]. *Trudy Matematicheskoho instituta imeni V.A. Steklova AN SSSR – Proceedings of the Mathematical Institute to them V.A. Steklov AS USSR, Vol. 170,* 37–76 [in Russian].

10 Ospanov, K.N., & Zulkhazhav, A. (2016). Coercive solvability of degenerate system of second order difference equations. *American Institute of Physics Conference Proceedings, Vol. 1759,* 1–5.

11 Mukhamediev, G.Kh. (1986). Plotnost finitnykh funktsii i normy vlozheniia dlia odnoho klassa vesovykh prostranstv [The density of compactly supported functions and the norms of an imbedding for one class of weighted spaces]. *Extended abstract of candidate's thesis.* Baku [in Russian].

12 Mustaphina, L.M. (1989). O nekotorykh raznostnykh vesovykh teoremakh vlozheniia [On some difference weighted imbedding theorems]. *Extended abstract of candidate's thesis.* Alma-Ata [in Russian].

13 Apyshev, O.D., & Otelbaev, M. (1979). O spektre odnoho klassa differentsialnykh operatorov i nekotorye teoremy vlozheniia [On the spectrum of a class of differential operators and some imbedding theorems]. *Izvestiia AN SSSR – Proceedings of the USSR AS, Vol. 43, 4,* 739–764 [in Russian].