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F.N. Dekhkonov[*]

Namangan State University, Namangan, Uzbekistan;
Department of Mathematics, New Uzbekistan University, Tashkent, Uzbekistan
(E-mail: f.n.dehqonov@mail.ru)

On the time-optimal control problem for a heat equation

In previous works, we have considered some control problems for parabolic type equations, namely, control problems for parabolic type equations were studied as boundary value problems of the first type, and the weight function was expanded into a Fourier series by sines. In this paper, we consider boundary control problem for a heat equation on the interval. In the part of the bound of the given domain it is given value of a solution and it is required to find a control to get the average value of the solution. By the mathematical-physics methods it is proved that like this control exists and the estimate of a minimal time for achieving the given average temperature over some domain is found.

Keywords: heat equation, minimal time, admissible control, integral equation, initial-boundary value problem.

Introduction

Consider the heat equation

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}, \quad (x,t) \in \Omega = \{(x,t) : 0 < x < l, \quad t > 0\}, \tag{1}$$

with boundary value conditions

$$u_x(0,t) \ = \ -\mu(t), \quad u_x(l,t) \ = \ 0, \quad t > 0, \tag{2}$$

and an initial condition

$$u(x,0) \ = \ 0, \quad 0 \le x \le l. \tag{3}$$

Definition 1. A function $\mu(t)$ is an *admissible control* if this function is piecewise smooth on $t \ge 0$ and satisfies the conditions

$$\mu(0) = 0, \quad |\mu(t)| \ \le \ M, \quad \text{where} \quad M = \text{const} > 0.$$

Consider the function $\rho(x) \in W_2^2[0,l]$ satisfying the conditions

$$\rho'(x) \le 0, \quad \rho''(x) \ge 0, \quad \frac{1}{l}\int_0^l \rho(x)\,dx = 1. \tag{4}$$

Let

$$\rho(x) = \sum_{k=1}^{\infty} \rho_k \cos\frac{k\pi x}{l}, \quad x \in (0,l),$$

where

$$\rho_k = \frac{2}{l}\int_0^l \rho(x)\cos\frac{k\pi x}{l}\,dx, \quad k = 1, 2, ... \tag{5}$$

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[*] Corresponding author.
E-mail: f.n.dehqonov@mail.ru

Problem H. *Let $\theta > 0$ be a given constant. Problem H consists in looking for the minimal value of $T > 0$ so that for $t > 0$ the solution $u(x,t)$ of problem (1)–(3) with a control function $\mu(t)$ exists and for some $T_1 > T$ satisfies the equation*

$$\int_0^l \rho(x)\, u(x,t)\, dx \;=\; \theta, \quad T \le t \le T_1. \tag{6}$$

We recall that the time-optimal control for parabolic type equations was first investigated in [1] and [2]. Recent results concerned with this problem were established in [3–12]. Some boundary control problems for hyperbolic type equations are studied in [13]. The same result as in this article was seen in detail in [5] case. Detailed information on the problems of optimal control for distributed parameter systems is given in [14] and in the monographs [15, 16] and [17]. Close to this work, boundary control problems for the pseudo-parabolic equation were studied in works [18, 19].

Overall numerical optimization and optimal control have been studied in a great number of publications such as [20]. The practical approaches to the optimal control of the heat equation are described in publications such as [21].

Theorem 1. Let

$$0 < \theta < \frac{\rho_1\, l^2\, M}{\pi^2}.$$

Set

$$T_0 = -\frac{l^2}{\pi^2} \ln\left(1 - \frac{\theta\, \pi^2}{\rho_1\, l^2\, M}\right).$$

Then a solution T_{min} of the Problem H exists and the estimate $T_{min} \le T_0$ is valid.

1 *Main integral equation*

Let $T > 0$ and B be a Banach space. Set by $C([0,T] \to B)$ the Banach space of all continuous mappings $u : [0,T] \to B$ with the norm

$$\|u\| = \max_{0 \le t \le T} \|u(t)\|.$$

Now by symbol $\widetilde{W}_2^1(\Omega)$ we denote the subspace of the Sobolev space $W_2^1(\Omega)$ formed by functions trace of which is equal to $\partial\Omega$ zero. Note that since $\widetilde{W}_2^1(\Omega)$ is closed and the sum of a series of functions from $\widetilde{W}_2^1(\Omega)$ converging in metric $W_2^1(\Omega)$ also in $\widetilde{W}_2^1(\Omega)$ (see, [10]).

Definition 2. By the solution of the problem (1) - (3) we mean function $u(x,t)$, expressed the form

$$u(x,t) = \mu(t)\, \frac{(l-x)^2}{2l} - v(x,t),$$

where the function $v(x,t)$ is a generalized solution from $C([0,T] \to \widetilde{W}_2^1(\Omega))$ of the problem

$$v_t(x,t) - v_{xx}(x,t) \;=\; \mu'(t)\, \frac{(l-x)^2}{2l} - \frac{1}{l}\, \mu(t),$$

with initial and boundary conditions

$$v_x(0,t) = v_x(l,t) = 0, \quad v(x,0) = 0, \quad 0 \le x \le l.$$

Consequently, we get (see, [22, 23])

$$v(x,t) \;=\; \frac{l}{6}\,\mu(t)-\frac{1}{l}\int\limits_0^t \mu(s)\,ds+\frac{2\,l}{\pi^2}\sum_{k=1}^{\infty}\frac{\cos\frac{k\pi x}{l}}{k^2}\int\limits_0^t e^{-(k\pi/l)^2(t-s)}\,\mu'(s)\,ds.$$

Note that the class $C([0,T]\to \widetilde{W}_2^1(\Omega))$ is a subset of the class $W_2^1(\Omega)$ considered in the monograph [24] in order to define a problem with homogeneous boundary conditions. So, the generalized solution given above is also a generalized solution in the sense of monograph [24].

Proposition 1. Let $\mu\in W_2^1(\mathbb{R}_+)$ and $\mu(0)=0$. Then the function

$$u(x,t)=\frac{1}{l}\int\limits_0^t\left(1+2\sum_{k=1}^{\infty}e^{-(k\pi/l)^2(t-s)}\,\cos\frac{k\pi x}{l}\right)\mu(s)\,ds \tag{7}$$

is a solution of problem (1)–(3).

Proof. We write the function $u(x,t)$ again in the form

$$u(x,t)\;=\;\mu(t)\,\frac{(l-x)^2}{2l}-$$

$$-\frac{l}{6}\,\mu(t)+\frac{1}{l}\int\limits_0^t\mu(s)\,ds-\frac{2\,l}{\pi^2}\sum_{k=1}^{\infty}\frac{\cos\frac{k\pi x}{l}}{k^2}\int\limits_0^t e^{-(k\pi/l)^2(t-s)}\,\mu'(s)\,ds.$$

Now we show that function $v(x,t)$ belongs to the class $C([0,T]\to\widetilde{W}_2^1(\Omega))$. For this, it is enough to prove that the gradient of this function, taken in $x\in\Omega$, continuously depends on $t\in[0,T]$ in the norm of the space $L_2(\Omega)$. According to Parseval's equality, the norm of this gradient is

$$\|v_x(\cdot,t)\|^2_{L_2(\Omega)}\;=\;\frac{2\,l}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\left(\int\limits_0^t e^{-(k\pi/l)^2(t-s)}\,\mu'(s)\,ds\right)^2\le$$

$$\le C\,\|\mu'\|^2\,\sum_{k=1}^{\infty}\frac{1}{k^4}\le C_1\,\|\mu'\|^2.$$

Proposition 1 is proved.

From (7) and condition (6), we can write

$$\theta(t)=\int\limits_0^l\rho(x)\,u(x,t)\,dx=$$

$$=\int\limits_0^t\left(\frac{1}{l}\int\limits_0^l\rho(x)\,dx+\frac{2}{l}\sum_{k=1}^{\infty}e^{-(k\pi/l)^2(t-s)}\int\limits_0^l\rho(x)\cos\frac{k\pi x}{l}dx\right)\mu(s)\,ds.$$

Then according to (4) and (5), we have

$$\theta(t)=\int\limits_0^t\left(1+\sum_{k=1}^{\infty}\rho_k\,e^{-(k\pi/l)^2(t-s)}\right)\mu(s)\,ds.$$

Set

$$B(t) = 1 + \sum_{k=1}^{\infty} \rho_k \, e^{-(k\pi/l)^2 t}, \quad t > 0. \tag{8}$$

Then we get the main integral equation

$$\int\limits_0^t B(t-s)\mu(s)ds = \theta(t), \quad t > 0.$$

Lemma 1. [6] Let $g(y) \geq 0$ and $g'(y) \leq 0$. Then the inequality holds

$$\int\limits_0^{n\pi} g(y)\sin y dy \geq 0, \quad y \in [0,\infty), \quad n = 1,2,....$$

Proposition 2. For the coefficients $\{\rho_k\}_{k\in N}$ defined by (5) the estimate

$$0 \ \leq \ \rho_k \ \leq \ \frac{C}{k^2}, \quad k = 1,2,...$$

is valid.

Proof. From (5), we write

$$\rho_k \ = \ \frac{2}{l}\int\limits_0^l \rho(x) \, \cos\frac{k\pi x}{l} \, dx \ = \ \frac{2}{k\pi} \, \rho(x) \, \sin\frac{k\pi x}{l}\bigg|_{x=0}^{x=l} -$$

$$- \ \frac{2}{k\pi}\int\limits_0^l \rho'(x)\sin\frac{k\pi x}{l} \, dx = -\frac{2}{k\pi}\int\limits_0^l \rho'(x)\sin\frac{k\pi x}{l} \, dx. \tag{9}$$

By conditions (4) and Lemma 1 we obtain $\rho_k \geq 0$. Then, from (9) we can write

$$\rho_k \ = \ -\frac{2}{k\pi}\int\limits_0^l \rho'(x)\sin\frac{k\pi x}{l} \, dx = \frac{2\,l}{k^2\pi^2} \ \rho'(x) \ \cos\frac{k\pi x}{l}\bigg|_{x=0}^{x=l} -$$

$$-\frac{2\,l}{k^2\pi^2}\int\limits_0^l \rho''(x)\cos\frac{k\pi x}{l} \, dx = \frac{2\,l}{k^2\,\pi^2}\left[\rho'(l)\,(-1)^k - \rho'(0)\right] + \frac{o(1)}{k^2},$$

where $\rho'(l)\,(-1)^k - \rho'(0) \geq 0$.

Then we obtain

$$0 \ \leq \ \rho_k \ \leq \ \frac{C}{k^2}.$$

Proposition 2 is proved.

Proposition 3. A function $B(t)$ defined by (8) is continuous on the half-line $t \geq 0$.

Proof. Indeed, from (8) and Proposition 2 we obtain

$$1 \ \leq \ B(t) \ \leq \ 1 + \text{const}\sum_{k=1}^{\infty}\frac{1}{k^2}\, e^{-(k\pi/l)^2 t}.$$

Proposition 3 is proved.

2 Estimate for the Minimal Time

Consider the Volterra integral equation

$$\int_0^t B(t-s)\,\mu(s)\,ds \;=\; \theta, \quad t \ge T,$$

where

$$B(t) \;=\; 1 + \sum_{k=1}^{\infty} \rho_k \; e^{-(k\pi/l)^2\, t}. \tag{10}$$

Proposition 4. For the function defined by Eq. (10) the following estimate

$$B(t) \ge \rho_1 e^{-(\pi/l)^2\, t}$$

is valid.

Proof. Proof of the proposition comes from functional series defined by (10) is non-negative.

Proposition 4 is proved.

We introduce a function as follows

$$Q(t) = \int_0^t B(t-s)ds = \int_0^t B(s)\,ds.$$

It is clear that physical meaning of this function $Q(t)$ equals the average temperature of Ω in case where the heater is acting unit load (see, [3,10]). We know that $Q(0)=0$ and $Q'(t)=B(t)>0$. Set

$$Q^* = \lim_{t\to\infty} Q(t) = \int_0^\infty B(s)ds.$$

Proposition 5. Let $0<\theta<MQ^*$. In that case there is $T>0$ and a real measurable function $\mu(t)$ and the equality

$$\int_0^T B(T-s)\mu(s)ds = \theta \tag{11}$$

is valid.

Proof. Obviously, if we set $\mu(t)=M$ then we obtain

$$\int_0^t B(t-s)\mu(s)ds = M\int_0^t B(t-s)ds = MQ(t),$$

and since from (11) there exists $T>0$ so that $MQ(T)=\theta$.

Proposition 5 is proved.

Remark 1. We know that the value T found in Proposition 5 gives a solution to the problem. Clearly, T is a root of the following equation

$$Q(T) = \frac{\theta}{M}. \tag{12}$$

Proposition 6. Let

$$0 < \theta < \frac{\rho_1 \, l^2 \, M}{\pi^2}. \tag{13}$$

Then there exists $T > 0$ and

$$T < -\frac{l^2}{\pi^2} \ln\left(1 - \frac{\theta \, \pi^2}{\rho_1 \, l^2 \, M}\right),$$

and the Eq. (12) is fulfilled.

Proof. Now we use Proposition 4. As result, we can write

$$Q(t) = \int\limits_0^t B(s) \, ds \geq \rho_1 \int\limits_0^t e^{-(\pi/l)^2 \, s} \, ds = \rho_1 \, l^2 \, \frac{1 - e^{-(\pi/l)^2 \, t}}{\pi^2}. \tag{14}$$

Consider the equation for the defining of T_0:

$$\rho_1 \, l^2 \, \frac{1 - e^{-(\pi/l)^2 \, T_0}}{\pi^2} = \frac{\theta}{M}. \tag{15}$$

Then we have

$$T_0 = -\frac{l^2}{\pi^2} \ln\left(1 - \frac{\theta \, \pi^2}{\rho_1 \, l^2 \, M}\right).$$

From (14) and (15), we can write

$$0 < \frac{\theta}{M} \leq Q(T_0).$$

Obviously, there exists T, $0 < T < T_0$, which is a solution of Eq. (12).

Proposition 6 is proved.

Proposition 7. Let $T > 0$ satisfies Eq. (12) and condition (13). Then there exist $T_1 > T$ and the measurable function $\mu(t)$ so that $|\mu(t)| \leq M$ and the equality

$$\int\limits_0^l \rho(x) u(x,t) \, dx = \theta, \quad T \leq t \leq T_1$$

is valid.

Proof. According to the following

$$\int\limits_0^t B(t-s)\mu(s) ds = \theta,$$

it is enough to prove that there exists a solution of the equation

$$\int\limits_0^t B(t-s)\mu(s) ds = f(t), \quad 0 \leq t \leq T_1, \tag{16}$$

where

$$f(t) = \begin{cases} MQ(t), & \text{if } 0 \leq t \leq T, \\ \theta, & \text{if } T < t \leq T_1. \end{cases} \tag{17}$$

Solution (17) is piecewise smooth and, according to Eq. (12), is continuous.

Set

$$\mu(t) = \begin{cases} M, & \text{if } 0 \le t \le T, \\ \mu_1(t), & \text{if } T < t \le T_1, \end{cases} \tag{18}$$

where $\mu_1(t)$ is a solution of the following integral equation

$$\int\limits_0^T B(t-s)M ds + \int\limits_T^t B(t-s)\mu_1(s)ds = \theta, \quad T \le t \le T_1. \tag{19}$$

Then differentiating this equation we obtain

$$B(0)\mu_1(t) + \int\limits_T^t B'(t-s)\mu_1(s)\, ds = M\,[B(t-T) - B(t)]. \tag{20}$$

According to Proposition 2,

$$B(0) = 1 + \sum_{k=1}^{\infty} \rho_k < \infty.$$

We know that the function $B(t)$ is convergence function on given interval. Therefore, equation (20) has a unique solution $\mu_1(t)$ for $t \ge T$, which is continuous function on $t \ge T$. Besides,

$$\mu_1(T) = M\,\big(1 - \frac{B(T)}{B(0)}\big) < M,$$

and there exists $T_1 > T$ so that

$$|\mu_1(t)| \le M, \quad T \le t \le T_1.$$

We know that this function is the unique solution of equation (19). Hence, function (18) is piecewise continuous and satisfies equation (16). Consequently, this function $\mu(t)$, which has a jump at the point $t = T$, is the required solution.

Proposition 7 is proved.

Proof of Theorem 1 follows from Propositions 6 and 7.

Conclusions

Note that in case where the temperature θ is small enough, the value of T_0 can be replaced by the following one:

$$T_0 \; = \; \frac{\theta}{\rho_1 M}.$$

Hence, in this case the estimate of optimal time given by Theorem 1 is proportional to required temperature θ and inversely proportional to size of the rod l and to the maximum output of heat source M.

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Ф.Н. Дехконов

Наманган мемлекеттік университеті, Наманган, Өзбекстан;
Жаңа Өзбекстан университеті, Ташкент, Өзбекстан

Жылутеңдеуі үшін оңтайлы уақыт мәселесі туралы

Алдыңғы жұмыстарда параболалық типті теңдеулер үшін кейбір басқару есептері қарастырылған. Яғни параболалық типті теңдеулердің басқару есептері бірінші типті шекаралық есептер ретінде зерттеліп, салмақ функциясы синустар бойынша Фурье қатарына кеңейтілді. Мақалада интервалдағы жылу теңдеуі үшін шекті бақылау мәселесі зерттелген. Өріс шекарасының бұл бөлігінде бақылаудың мәні берілген және температураның орташа мәнін алу үшін басқару элементін табу қажет. Математикалық-физикалық әдістерді қолдана отырып, мұндай бақылаудың бар екендігі дәлелденді және белгілі бір аумақта берілген орташа температураға жету үшін ең аз уақыттың бағасы табылды.

Кілт сөздер: жылу теңдеуі, ең аз уақыт, рұқсат етілген бақылау, интегралдық теңдеу, бастапқы-шекаралық есеп.

Ф.Н. Дехконов

Наманганский государственный университет, Наманган, Узбекистан;
Университет Новый Узбекистан, Ташкент, Узбекистан

О задаче быстродействия для уравнения теплопроводности

В предыдущих работах мы рассмотрели некоторые задачи управления для уравнений параболического типа, а именно: задачи управления для уравнений параболического типа изучались как краевые задачи первого типа, а весовая функция разлагалась в ряд Фурье по синусам. В настоящей работе рассмотрена задача граничного управления для уравнения теплопроводности на отрезке. В части границы данной области задано значение решения и требуется найти управление, чтобы получить среднее значение решения. Методами математической физики доказано, что подобное управление существует, и находится оценка минимального времени достижения заданной средней температуры по некоторой области.

Ключевые слова: уравнение теплопроводности, минимальное время, допустимое управление, интегральные уравнения, начально-краевая задача.

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