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On the behavior of solutions of a doubly nonlinear degenerate parabolic system with nonlinear sources and absorptions with variable densities

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In this paper, the problem of a doubly nonlinear degenerate parabolic system with nonlinear sources and absorption terms not located in a homogeneous medium was considered. It obeys zero Dirichlet boundary conditions in a smooth bounded domain. The comparison principle and self-similar approach was used to study the problem. In this paper, the nonlinear splitting method was used to prove the existence of global and blow-up in finite time solutions. It is shown that the role of the nonlinear source and nonlinear absorption is important for the existence and non-existence of the solution. The results contribute to a broader understanding of nonlinear parabolic systems.

Keywords: doubly nonlinear degenerate parabolic system, nonlinear source, nonlinear absorption, global existence, blow-up in finite time, comparison principle, variable density.

2020 Mathematics Subject Classification: 35A01, 35B44, 35K57, 35K65.

Introduction

In this paper, we consider the following doubly nonlinear degenerate parabolic system with both nonlinear sources and absorptions with variable densities:

$$\begin{aligned} &\frac{\partial u}{\partial t} = \nabla\left(|x|^{n_1} u^{m_1-1}\left|\nabla u^{k_1}\right|^{p_1-2}\nabla u\right) + v^{q_1} - \alpha_1 u^{r_1}, \quad x\in\Omega, t>0,\\ &\frac{\partial v}{\partial t} = \nabla\left(|x|^{n_2} v^{m_2-1}\left|\nabla v^{k_2}\right|^{p_2-2}\nabla v\right) + u^{q_2} - \alpha_2 v^{r_2}, \quad x\in\Omega, t>0,\\ &u(x,t) = v(x,t) = 0, \quad x\in\partial\Omega, t>0,\\ &u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x\in\Omega, \end{aligned} \tag{1}$$

where $p_i \geq 2$, $k_i, m_i \geq 1$, $n_i, q_i, r_i, \alpha_i \geq 0$ $(i=\overline{1,2})$ and Ω is a bounded domain of R^N, $N\geq 1$ with a smooth boundary $\partial\Omega$. The initial data $u_0(x), v_0(x) \in C^{2+\nu}\left(\overline{\Omega}\right)$, with $0<\nu<1$, $u_0(x), v_0(x)\geq 0$ and $u_0(x), v_0(x) \not\equiv 0$.

In recent years, the problem of reaction-diffusion processes with nonlinear interactions has attracted considerable attention because it arises in such fields as biology, chemistry and physics. Population dynamics, chemical reactions, heat transfer and other phenomena can be predicted if the conditions for the existence of global and blow-up in finite time solutions are known. For example, in a biological context, the presence of a variable density term means changing population density or resources that are not distributed uniformly. For more detailed information on physical models describing with the above and similar equations, we refer to literature [1–7] and references therein.

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However, there are few studies on doubly nonlinear degenerate parabolic systems including both reaction and absorption terms in an inhomogeneous medium.

Doubly nonlinear degenerate parabolic equations and systems have been studied by many scientists (see [5–12] and references therein). Especially, when $n_i = 0$, $p_i = 2$ and $m_i = k_i = 1$ $(i = \overline{1,2})$ the system (1) reduces to following semilinear form:

$$\begin{aligned} &\frac{\partial u}{\partial t} = \Delta u + v^{q_1} - \alpha_1 u^{r_1}, \quad x \in \Omega, t > 0, \\ &\frac{\partial v}{\partial t} = \Delta v + u^{q_2} - \alpha_2 v^{r_2}, \quad x \in \Omega, t > 0, \\ &u(x,t) = v(x,t) = 0, \quad x \in \partial\Omega, t > 0, \\ &u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x \in \Omega. \end{aligned} \tag{2}$$

In particular, Bedjaoui and Souplet [3] used the comparison principle to show the global solvability of problem (2). Authors of [4–8, 13–15] analyzed the blow-up properties of solutions to system (2) without absorption terms, when $\alpha_1 = \alpha_2 = 0$. The critical Fujita exponent for solutions with blow-up was found by the authors of [13–15]. Aripov and Bobokandov [8] obtained estimates of solutions and fronts (free boundaries) for equations with a single absorption term in an inhomogeneous medium.

Recently, many authors have addressed the problems with variable densities [8, 11, 12, 15–21]. Zhou et al. [9, 10] determined the global existence and blow-up of solutions to a degenerate singular parabolic system. The authors obtained the blow-up set and uniform blow-up conditions using the comparison principle and asymptotic analysis methods. Kong et al. [5] established uniform blow-up profiles for the weakly absorbed case of a semilinear parabolic system.

Anh et al. [20] and Niu et al. [21] investigated the long-time behavior of solutions to the following degenerate parabolic equation

$$\begin{aligned} &\frac{\partial u}{\partial t} = \nabla\left(\sigma(x)\left|\nabla u\right|^{p-2}\nabla u\right) + g - f(x,u), \quad x \in \Omega \times R^+, \\ &u(x,t) = 0, \quad x \in \partial\Omega \times R^+, \\ &u(x,0) = u_0(x), \quad x \in \Omega. \end{aligned} \tag{3}$$

The existence of a global attractor in L^q was shown using some estimates of the solution.

Other related works includes [22–29], where the authors studied doubly degenerate parabolic equations with nonlinear sources and absorption terms when $k_i = m_i$ $(i = \overline{1,2})$

$$\begin{aligned} &\frac{\partial u}{\partial t} = \nabla\left(\left|\nabla u^{m_1}\right|^{p_1-2}\nabla u^{m_1}\right) + v^{q_1} - \alpha_1 u^{r_1}, \quad x \in \Omega, t > 0, \\ &\frac{\partial v}{\partial t} = \nabla\left(\left|\nabla v^{m_2}\right|^{p_2-2}\nabla v^{m_2}\right) + u^{q_2} - \alpha_2 v^{r_2}, \quad x \in \Omega, t > 0, \\ &u(x,t) = v(x,t) = 0, \quad x \in \partial\Omega, t > 0, \\ &u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x \in \Omega. \end{aligned}$$

This is also known as the p-Laplacian system when $k_i = m_i = 1$ $(i = \overline{1,2})$. Xiulan Wu [30] provided criteria for the global existence or the finite-time blow-up of solutions to (3).

This study extends the results to the more general case of a doubly nonlinear degenerate parabolic system including variable density terms. This provides a more precise understanding of the behavior of the physical phenomena described by the system (3).

The rest of the paper is organized as follows. Section 1 gives preliminary notations and main results. Section 2 is devoted to the existence of global and finite-time exploding solutions. Finally, conclusions and observations are discussed.

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1 Preliminaries and Main results

Degenerate equations may not have classical solutions, so we define weak upper and weak lower solutions. In this paper we denote $Q_T = \Omega \times (0, T)$.

Definition 1. We call a non-negative function $(u, v) \in \left[C^{2,1}(Q_T) \cap C(Q_T)\right]^2$ a weak upper solution (a weak lower solution) of problem (1) in Q_T if the following fulfills:

$$\begin{aligned}
&\frac{\partial u}{\partial t} \geq (\leq) \nabla\left(|x|^{n_1} u^{m_1-1}\left|\nabla u^{k_1}\right|^{p_1-2} \nabla u\right) + v^{q_1} - \alpha_1 u^{r_1}, \quad x \in \Omega, t > 0,\\
&\frac{\partial v}{\partial t} \geq (\leq) \nabla\left(|x|^{n_2} v^{m_2-1}\left|\nabla v^{k_2}\right|^{p_2-2} \nabla v\right) + u^{q_2} - \alpha_2 v^{r_2}, \quad x \in \Omega, t > 0,\\
&u(x,t) \geq v(x,t) \geq 0, \quad x \in \partial\Omega, t > 0,\\
&u(x,0) \geq u_0(x), \quad v(x,0) \geq v_0(x), \quad x \in \Omega.
\end{aligned}$$

We also say that (u, v) is a weak solution of problem (1) in Q_T if (u, v) is both weak upper and weak lower solution of (1) in Q_T. Moreover, (u, v) is a global solution of problem (1) if it is a solution of (1) in Q_T for any $T > 0$, and any solution (u, v) blows up in the sense of the L^∞ norm if $T < \infty$:

$$\lim_{t \to T}\left(\|u(\cdot,t)\|_\infty + \|v(\cdot,t)\|_\infty\right) = \infty.$$

In order to state our results, we introduce some useful symbols. Let $\varphi(x)$ and $\psi(x)$ satisfy the following elliptic problem respectively:

$$-\nabla\left(|x|^{n_1} \varphi^{m_1-1}\left|\nabla \varphi^{k_1}\right|^{p_1-2} \nabla \varphi\right) = 1, \quad x \in \Omega, \quad \varphi(x) = 1, \quad x \in \partial\Omega, \tag{4}$$

$$-\nabla\left(|x|^{n_2} \psi^{m_1-1}\left|\nabla \psi^{k_1}\right|^{p_1-2} \nabla \psi\right) = 1, \quad x \in \Omega, \quad \psi(x) = 1, \quad x \in \partial\Omega. \tag{5}$$

It is known [19] that (4) and (5) have unique solutions with the following properties:

$$M_1 = \max_{x \in \overline{\Omega}} \varphi(x) < \infty, \quad M_2 = \max_{x \in \overline{\Omega}} \psi(x) < \infty,$$

$$\varphi(x), \psi(x) > 1 \text{ in } \Omega, \quad \nabla\varphi\nu < 0, \ \nabla\psi\nu < 0 \text{ on } \partial\Omega.$$

To simplify notation, we also let $\mu_1 = \max\{m_1 + k_1(p_1 - 2), r_1\}$, $\mu_2 = \max\{m_2 + k_2(p_2 - 2), r_2\}$.

Theorem 1. Let $n_1 n_2 < p_1 p_2$. If $q_1 q_2 < \mu_1 \mu_2$, then all nonnegative solutions of problem (1) are global.

Theorem 2. Let $n_1 n_2 < p_1 p_2$ and $q_1 q_2 = \mu_1 \mu_2$, then:

1. if $r_1 > m_1 + k_1(p_1 - 2)$, $r_2 > m_2 + k_2(p_2 - 2)$, which is $q_1 q_2 = r_1 r_2$ and if α_1, α_2 are sufficiently large, then nonnegative solutions of problem (1) blows up in finite time, and exists globally for small initial values;
2. if $r_1 < m_1 + k_1(p_1 - 2)$, $r_2 < m_2 + k_2(p_2 - 2)$, thus $q_1 q_2 = (m_1 + k_1(p_1 - 2))(m_2 + k_2(p_2 - 2))$, then all nonnegative solutions of problem (1) are global for small initial values;
3. suppose $r_1 < m_1 + k_1(p_1 - 2)$, $r_2 > m_2 + k_2(p_2 - 2)$, thus $q_1 q_2 = (m_1 + k_1(p_1 - 2))r_2$, then there is a non-negative blow-up in finite time solution of problem (1) for large initial data;
4. suppose $r_1 > m_1 + k_1(p_1 - 2)$, $r_2 < m_2 + k_2(p_2 - 2)$, thus $q_1 q_2 = r_1(m_2 + k_2(p_2 - 2))$, then there is a non-negative blow-up in finite time solution of problem (1) for large initial data.

Theorem 3. Suppose $n_1 n_2 < p_1 p_2$. If $q_1 q_2 > \mu_1 \mu_2$, then nonnegative solutions of problem (1) blows up in finite time for sufficiently large initial data and exists globally for small initial values.

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2 Global existence and Blow-up

Here we give the proof of global existence and blow-up solutions using the comparison principle. Self-similar approach and nonlinear splitting methods are used to construct comparable solutions. We start with Theorem 1.

Proof of Theorem 1. We divide the proof of Theorem 1 into 4 cases:

Case 1: When $\mu_1 = m_1 + k_1(p_2-2)$, $\mu_2 = m_2 + k_2(p_2-2)$, thus $q_1q_2 < (m_1 + k_1(p_2-2)) \times \times (m_2 + k_2(p_2-2))$ and $n_1n_2 < p_1p_2$. We have $u \leq w$ and $v \leq z$, where (w,z) satisfies

$$\begin{aligned}
&\frac{\partial w}{\partial t} = \nabla\left(|x|^{n_1} w^{m_1-1}\left|\nabla w^{k_1}\right|^{p_1-2}\nabla w\right) + z^{q_1}, \quad x\in\Omega, t>0,\\
&\frac{\partial z}{\partial t} = \nabla\left(|x|^{n_2} z^{m_2-1}\left|\nabla z^{k_2}\right|^{p_2-2}\nabla z\right) + w^{q_2}, \quad x\in\Omega, t>0,\\
&w(x,t) = z(x,t) = 0, \quad x\in\partial\Omega, t>0,\\
&w(x,0) = w_0(x), \quad z(x,0) = z_0(x), \quad x\in\Omega
\end{aligned}$$

by comparison principle and from [3, 20], it follows that (w,z) is global and so it is (u,v).

Case 2: When $\mu_1 = r_1$, $\mu_2 = r_2$, thus $q_1q_2 < r_1r_2$ and $n_1n_2 < p_1p_2$. Let $(\overline{u},\overline{v}) = (A_1, A_2)$, where $A_1 \geq \max_{x\in\overline{\Omega}} u_0(x)$, $A_2 \geq \max_{x\in\overline{\Omega}} v_0(x)$ and A_1, A_2 will be determined later. After some calculations, we have

$$\begin{aligned}
&\frac{\partial \overline{u}}{\partial t} - \nabla\left(|x|^{n_1}\overline{u}^{m_1-1}\left|\nabla\overline{u}^{k_1}\right|^{p_1-2}\nabla\overline{u}\right) - \overline{v}^{q_1} + \alpha_1\overline{u}^{r_1} = \alpha_1 A_1^{r_1} - A_2^{q_1},\\
&\frac{\partial \overline{v}}{\partial t} - \nabla\left(|x|^{n_2}\overline{v}^{m_2-1}\left|\nabla\overline{v}^{k_2}\right|^{p_2-2}\nabla\overline{v}\right) - \overline{u}^{q_2} + \alpha_2\overline{v}^{r_2} = \alpha_2 A_2^{r_2} - A_1^{q_2}.
\end{aligned}$$

$(\overline{u},\overline{v}) = (A_1, A_2)$ is a time-independent upper solution of problem (1) if

$$\alpha_1 A_1^{r_1} \geq A_2^{q_1} \quad \textit{and} \quad \alpha_2 A_2^{r_2} \geq A_1^{q_2},$$

i.e.

$$A_2^{\frac{q_1}{r_1}}\alpha_1^{-\frac{1}{r_1}} \leq A_1 \leq A_2^{\frac{r_2}{q_2}}\alpha_2^{\frac{1}{q_2}}. \tag{6}$$

Since $q_1q_2 < r_1r_2$, then there exist A_1, A_2 satisfying (6).

Case 3: When $\mu_1 = r_1$, $\mu_2 = m_2 + k_2(p_2-2)$ and $n_1n_2 < p_1p_2$, we have $q_1q_2 < r_1(m_2 + k_2(p_2-2)))$. Let $(\overline{u},\overline{v}) = (A_1, A_2\psi(x))$, where $\psi(x)$ from (5) and we can choose $A_1 \geq \max_{x\in\overline{\Omega}} u_0(x)$ and $A_2 \geq \max_{x\in\overline{\Omega}} v_0(x)$ satisfying

$$A_1^{\frac{q_2}{m_2+k_2(p_2-2)}} \leq A_2 \leq \frac{1}{M_2}\left(\alpha_1 A_1^{r_1}\right)^{\frac{1}{q_1}}.$$

After direct computation, we have

$$\begin{aligned}
&\frac{\partial \overline{u}}{\partial t} - \nabla\left(|x|^{n_1}\overline{u}^{m_1-1}\left|\nabla\overline{u}^{k_1}\right|^{p_1-2}\nabla\overline{u}\right) - \overline{v}^{q_1} + \alpha_1\overline{u}^{r_1} \geq 0,\\
&\frac{\partial \overline{v}}{\partial t} - \nabla\left(|x|^{n_2}\overline{v}^{m_2-1}\left|\nabla\overline{v}^{k_2}\right|^{p_2-2}\nabla\overline{v}\right) - \overline{u}^{q_2} + \alpha_2\overline{v}^{r_2} \geq 0.
\end{aligned} \tag{7}$$

So, $(\overline{u},\overline{v}) = (A, B\psi(x))$ is a upper solution for system (1)

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Case 4: If $\mu_1 = m_1 + k_1(p_1 - 2)$, $\mu_2 = r_2$, we have $q_1 q_2 < r_2(m_1 + k_1(p_1 - 2))$ and $n_1 n_2 < p_1 p_2$, we let $(\overline{u}, \overline{v}) = (A_1(\varphi(x) + 1), A_2)$, where $\varphi(x)$ from (4) and choose such $A_1 \geq \max_{x\in\overline{\Omega}} u_0(x)$ and $A_2 \geq \max_{x\in\overline{\Omega}} v_0(x)$ satisfying

$$A_2^{\frac{q_1}{m_1+k_1(p_1-2)}} \leq A_1 \leq \frac{1}{M_1}\left(\alpha_2 A_2^{r_2}\right)^{\frac{1}{q_2}}.$$

Then system of inequalities (7) is fulfilled.

Proof of Theorem 1 is completed.

Proof of Theorem 2. We consider 4 cases to prove Theorem 2:

Case 1: When $r_1 > m_1 + k_1(p_1 - 2)$, $r_2 > m_2 + k_2(p_2 - 2)$ and $n_1 n_2 < p_1 p_2$, we know $q_1 q_2 = r_1 r_2$. We can choose A_1 and A_2 sufficiently large, fulfilling $A_1 \geq \max_{x\in\overline{\Omega}} u_0(x)$, $A_2 \geq \max_{x\in\overline{\Omega}} v_0(x)$ and

$$\alpha_1^{-\frac{1}{r_1}} A_2^{\frac{q_1}{r_1}} \leq A_1 \leq \alpha_2^{\frac{1}{q_2}} A_2^{\frac{r_2}{q_2}}.$$

It is clear, that $(\overline{u}, \overline{v}) = (A_1, A_2)$ is a weak upper solution of problem (1). It is easy to check, by comparison principle that the solution $(\overline{u}, \overline{v}) = (A_1, A_2)$ of problem (1) is global.

Now we need to prove our blow-up conclusion. Assume that Ω contains the origin. Denote

$$\underline{u}(x,t) = (T-t)^{-\gamma_1} U(\xi_1), \quad \underline{v}(x,t) = (T-t)^{-\gamma_2} V(\xi_2),$$

where $U(\xi_1) = \left(A^{\frac{p_1-n_1}{p_1-1}} - \xi_1^{\frac{p_1-n_1}{p_1-1}}\right)^{\frac{p_1-1}{m_1-1+k_1(p_1-2)}}$, $V(\xi_2) = \left(A^{\frac{p_2-n_2}{p_2-1}} - \xi_2^{\frac{p_2-n_2}{p_2-1}}\right)^{\frac{p_2-1}{m_2-1+k_2(p_2-2)}}$, $\xi_1 = \frac{|x|}{(T-t)^{\beta_1}}$, $\xi_2 = \frac{|x|}{(T-t)^{\beta_2}}$, $\beta_1 = \frac{1-\gamma_1(m_1+k_1(p_1-2)-1)}{p_1} > 0$, $\beta_2 = \frac{1-\gamma_2(m_2+k_2(p_2-2)-1)}{p_2} > 0$ and $\gamma_1, \gamma_2, A, T > 0$ are to be determined later. Note that $B_{AT^\beta}(0)$ contains the support of $\underline{u}(x,t)$ and $\underline{v}(x,t)$, where $\beta = \max \beta_1, \beta_2$ if $T > 1$; $\beta = \min \beta_1, \beta_2$ if $T \leq 1$, which is included in Ω if T is sufficiently small.

After a direct computation, we obtain

$$\frac{\partial \underline{u}}{\partial t} = (T-t)^{-\gamma_1-1}\left(\gamma_1 U(\xi_1) + \beta_1 \xi_1 \frac{dU}{d\xi_1}\right)$$

$$\nabla\left(|x|^{n_1} \underline{u}^{m_1-1}\left|\nabla \underline{u}^{k_1}\right|^{p-2} \nabla \underline{u}\right) = (T-t)^{-\gamma_1(m_1+k_1(p_1-2))-\beta_1 p_1}$$

$$\xi_1^{n_1}\left(k_1^{p_1-2} N b_1^{p_1-1} U(\xi_1) + k_1^{p_1-2} b_1^{p_1-1} \xi_1 \frac{dU}{d\xi_1}\right),$$

$$\frac{\partial \underline{v}}{\partial t} = (T-t)^{-\gamma_2-1}\left(\gamma_2 V(\xi_2) + \beta_2 \xi_2 \frac{dV}{d\xi_2}\right)$$

$$\nabla\left(|x|^{n_2} \underline{v}^{m_2-1}\left|\nabla \underline{v}^{k_2}\right|^{p-2} \nabla \underline{v}\right) = (T-t)^{-\gamma_2(m_2+k_2(p_2-2))-\beta_2 p_2}$$

$$\xi_2^{n_2}\left(k_2^{p_2-2} N b_2^{p_2-1} V(\xi_2) + k_2^{p_2-2} b_2^{p_2-1} \xi_2 \frac{dV}{d\xi_2}\right),$$

where $b_1 = \frac{p_1}{m_1+k_1(p_1-2)-1}$ and $b_2 = \frac{p_2}{m_2+k_2(p_2-2)-1}$.

We need to find suitable parameters such that

$$(T-t)^{-\gamma_1-1}\left(\gamma_1 U(\xi_1) + \beta_1 \xi_1 \frac{dU}{d\xi_1} - k_1^{p_1-2} N b_1^{p_1-1} \xi_1^{n_1} U(\xi_1) - k_1^{p_1-2} b_1^{p_1-1} \xi_1^{n_1+1} \frac{dU}{d\xi_1}\right) + \\ + (T-t)^{-r_1\gamma_1} U^{r_1}(\xi_1) \leq (T-t)^{-q_1\gamma_2} V^{q_1}(\xi_2) \tag{8}$$

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$$(T-t)^{-\gamma_2-1}\left(\gamma_2 V(\xi_2)+\beta_2\xi_2\frac{dV}{d\xi_2}-k_2^{p_2-2}Nb_2^{p_2-1}\xi_2^{n_2}V(\xi_2)-k_2^{p_2-2}b_2^{p_2-1}\xi_2^{n_2+1}\frac{dV}{d\xi_2}\right)+ \\ +(T-t)^{-r_2\gamma_2}V^{r_2}(\xi_2)\le(T-t)^{-q_2\gamma_1}V^{q_2}(\xi_1). \tag{9}$$

Note that U, V are continuous for C^2 except for $\xi_1 = A$, $\xi_2 = A$ where U', V' has a positive jump. Therefore, to obtain a lower solution of (1), we will prove (8) and (9) pointwise for $\xi_1 > 0$, with $\xi_1 \neq A$. It is easy to see that

$$\frac{dU}{d\xi_1}=-\frac{p_1}{m_1+k_1(p_1-2)-1}\xi_1^{\frac{p_1+n_1}{p_1-1}-1}\left(A^{\frac{p_1+n_1}{p_1-1}}-\xi_1^{\frac{p_1+n_1}{p_1-1}}\right)^{\frac{p_1-1}{m_1+k_1(p_1-2)-1}-1},$$

$$\frac{dV}{d\xi_2}=-\frac{p_2}{m_2+k_2(p_2-2)-1}\xi_2^{\frac{p_2+n_2}{p_2-1}-1}\left(A^{\frac{p_2+n_2}{p_2-1}}-\xi_2^{\frac{p_2+n_2}{p_2-1}}\right)^{\frac{p_2-1}{m_2+k_2(p_2-2)-1}-1}$$

and (8) is trivial for $\xi_1 \geq A$. A simple computation shows that (8) is satisfied. We distinguish two steps for $0 < \xi_1 < \theta_1 A$ and $\theta_1 A < \xi_1 < A$, where

$$\theta_1=\left(\frac{\gamma_1+k_1^{p_1-2}Nb_1^{p_1-1}}{\gamma_1+k_1^{p_1-2}Nb_1^{p_1-1}+\beta_1 b_1+k_1^{p_1-2}b_1^{p_1}}\right)^{\frac{p_1-1}{p_1+n_1}}<1.$$

Step 1. For $\theta_1 A < \xi_1 < A$, we have

$$\gamma_1 U(\xi_1)+\beta\xi_1\frac{dU}{d\xi_1}-k_1^{p_1-2}Nb_1^{p_1-1}\xi_1^{n_1}U(\xi_1)-k_1^{p_1-2}b_1^{p_1-1}\xi_1^{n_1+1}\frac{dU}{d\xi_1}$$
$$=\left(A^{\frac{p_1+n_1}{p_1-1}}-\xi_1^{\frac{p_1+n_1}{p_1-1}}\right)^{\frac{p_1-1}{m_1+k_1(p_1-2)-1}-1}$$
$$\left(\left(\gamma_1+k_1^{p_1-2}Nb_1^{p_1-1}\right)A^{\frac{p_1+n_1}{p_1-1}}-\left(\gamma_1+k^{p_1-2}Nb_1^{p_1-1}+\beta_1 b_1+k_1^{p_1-2}b_1^{p_1}\right)\xi_1^{\frac{p_1+n_1}{p_1-1}}\right)\le$$
$$\le\left(A^{\frac{p_1+n_1}{p_1-1}}-\xi_1^{\frac{p_1+n_1}{p_1-1}}\right)^{\frac{p_1-1}{m_1+k_1(p_1-2)-1}-1}$$
$$\left(\left(\gamma_1+k_1^{p_1-2}Nb_1^{p_1-1}\right)A^{\frac{p_1+n_1}{p_1-1}}-\left(\gamma_1+k_1^{p_1-2}Nb_1^{p_1-1}+\beta_1 b_1+k_1^{p_1-2}b_1^{p_1}\right)(\theta_1 A)^{\frac{p_1+n_1}{p_1-1}}\right)$$
$$\le-\beta_1 b_1(\theta_1 A)^{\frac{p_1+n_1}{p_1-1}}\left(A^{\frac{p_1+n_1}{p_1-1}}-\xi_1^{\frac{p_1+n_1}{p_1-1}}\right)_+^{\frac{p_1-1}{m_1+k_1(p_1-2)-1}-1}.$$

Step 2. For $0 < \xi_1 \leq \theta_1 A$, the inequality

$$U(\xi_1)\ge\left(1-\theta_1^{\frac{p_1+n_1}{p_1-1}}\right)^{\frac{p_1-1}{m_1+k_1(p_1-2)-1}}A^{\frac{p_1}{m_1+k_1(p_1-2)-1}}>0,$$

$$V(\xi_2)\ge\left(1-\theta_1^{\frac{p_2+n_2}{p_2-1}}\right)^{\frac{p_2-1}{m_2+k_2(p_2-2)-1}}A^{\frac{p_2}{m_2+k_2(p_2-2)-1}}>0$$

holds. It follows from $\gamma_2 q_1 > \gamma_1 + 1$ that

$$(T-t)^{-\gamma_1-1}\left(\gamma_1 U(\xi_1)+\beta_1\xi_1\frac{dU}{d\xi_1}-k^{p_1-2}Nb_1^{p_1-1}\xi_1^{n_1}U(\xi_1)-k_1^{p_1-2}b_1^{p_1-1}\xi_1^{n_1+1}\frac{dU}{d\xi_1}\right)$$
$$\le\frac{p_1-1}{p_1}(T-t)^{-q_1\gamma_2}U^{q_1}(\xi_1)$$

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if T is sufficiently small. If

$$\alpha_1 (T-t)^{-r_1\gamma_1} U^{r_1}(\xi_1) \le \frac{p_1-1}{p_1}(T-t)^{-q_1\gamma_2} V^{\gamma_2}(\xi_2), \tag{10}$$

then (8) holds.
Similarly, if

$$\alpha_2 (T-t)^{-r_2\gamma_2} V^{r_2}(\xi_2) \le \frac{p_2-1}{p_2}(T-t)^{-q_2\gamma_1} U^{q_2}(\xi_1), \tag{11}$$

$\gamma_1 q_2 > \gamma_2 + 1$ and T is sufficiently small, then (9) holds.

Next, we choose suitable γ_1 γ_2 to satisfy (10) and (11). It is easy to see that there is $\gamma_1 > \frac{q_1+1}{q_1q_2-1}$ and $\gamma_2 > \frac{q_2+1}{q_1q_2-1}$, fulfilling the inequalities

$$\gamma_2 q_1 > \gamma_1 + 1, \quad \gamma_1 q_2 > \gamma_2 + 1. \tag{12}$$

If $q_1q_2 = r_1r_2$, then we choose some large γ_1 and γ_2 satisfying (12), and $q_1 = \frac{\gamma_1}{\gamma_2} r_1$, hence $q_2 = \frac{\gamma_2}{\gamma_1} r_2$. Consequently, by (10) and (11), for sufficiently small α_1 and α_2 the following hold

$$\alpha_1 U^{r_1} \le \frac{p_1-1}{p_1} V^{q_1}, \quad \alpha_2 V^{r_2} \le \frac{p_2-1}{p_2} U^{q_2}. \tag{13}$$

Hence, $(\underline{u}, \underline{v})$ is a blow-up lower solution of problem (1) with sufficiently large initial data $(\underline{u}_0, \underline{v}_0)$.

Case 2: When $r_1 < m_1 + k_1(p_1-2)$, $r_2 < m_2 + k_2(p_2-2)$ and $n_1n_2 < p_1p_2$, we have $q_1q_2 = (m_1 + k_1(p_1-2))(m_2 + k_2(p_2-2))$. We choose A_1 and A_2 satisfying

$$(A_2\psi(x))^{\frac{q_1}{m_1+k_1(p_1-2)}} \le A_1 \le \left(A_2\varphi^{-q_2}(x)\right)^{\frac{m_2+k_2(p_2-2)}{q_2}}.$$

The solution is global for small initial data, because $(\overline{u}, \overline{v}) = (A_1\varphi(x), A_2\psi(x))$, where $\varphi(x)$, $\psi(x)$ satisfy (4), (5) respectively, is a global upper solution of problem (1).

Cases 3 and 4: Cases 3 and 4 are proved similarly to Case 2.

Proof of Theorem 3. We consider two main cases: large initial values and small initial data for theorem. First, we consider the case of large initial values and prove that the solution blows up in finite time. Then, we consider the case of small initial data and show that the solutions exist globally. For each, we break into four subcases based on different conditions of μ_1 and μ_2.

Case 1: 1. When $\mu_1 = m_1 + k_1(p_1-2)$, $\mu_2 = m_2 + k_2(p_2-2)$, that is $q_1q_2 > (m_1 + k_1(p_1-2))(m_2 + k_2(p_2-2))$ and $n_1n_2 < p_1p_2$. Let $(\overline{u}, \overline{v}) = (A_1\varphi(x), A_2\psi(x))$, where $\varphi(x)$, $\psi(x)$ satisfy (4), (5) respectively. Choosing then

$$A_1 = \frac{1}{2}\left((A_2M_2)^{\frac{q_1}{m_1+k_1(p_1-2)}} + A_2^{\frac{m_2+k_2(p_2-1)}{q_2}} M_1^{-1}\right),$$

$$A_2 = \left(M_1^{m_1+k_1(p_1-2)} M_2^{q_1}\right)^{-\frac{q_2}{q_1q_2-(m_1+k_1(p_1-2))(m_2+k_2(p_2-1))}}.$$

Therefore, $(\overline{u}, \overline{v})$ is a global upper solution for problem (1) if $A_1 \ge \max_{x\in\overline{\Omega}} u_0(0)$ and $A_2 \ge \max_{x\in\overline{\Omega}} v_0(0)$ for small initial values.

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2. When $\mu_1 = r_1$, $\mu_2 = r_2$ and $n_1 n_2 < p_1 p_2$, that is $q_1 q_2 > r_1 r_2$ choosing
$$A_1 = \frac{1}{2}\left(\alpha_1^{-\frac{1}{r_1}} A_2^{\frac{q_1}{r_1}} + \alpha_2^{\frac{1}{q_2}} A_2^{\frac{r_2}{q_2}}\right) \quad \text{and} \quad A_2 = \left(\alpha_1^{q_2}\alpha_2^{r_1}\right)^{\frac{1}{q_1 q_2 - r_1 r_2}},$$
then $(\overline{u}, \overline{v}) = (A_1, A_2)$ is a global upper solution for problem (1), if $A_1 \geq \max_{x\in\overline{\Omega}} u_0(0)$ and $A_2 \geq \max_{x\in\overline{\Omega}} v_0(0)$ for small initial values.
3. When $\mu_1 = r_1$, $\mu_2 = m_2 + k_2(p_2 - 2)$ and $n_1 n_2 < p_1 p_2$, that is $q_1 q_2 > r_1(m_2 + k_2(p_2 - 2))$. Let $(\overline{u}, \overline{v}) = (A_1, A_2\psi(x))$, where $\psi(x)$ satisfies (5). Then we choose
$$A_1 = \left(\alpha_1 M_2^{q_1}\right)^{-\frac{m_2 + k_2(p_2-2)}{q_1 q_2 - r_1(m_2 + k_2(p_2 - 1))}}, \quad A_2 = \frac{1}{2}\left(A_1^{\frac{q_2}{m_2 + k_2(p_2-2)}} + \left(\alpha_1 A_1^{r_1}\right)^{\frac{1}{q_1}} \frac{1}{M_2}\right).$$
Therefore, $(\overline{u}, \overline{v})$ is a global upper solution for problem (1), if $A_1 \geq \max_{x\in\overline{\Omega}} u_0(0)$ and $A_2 \geq \max_{x\in\overline{\Omega}} v_0(0)$.
4. When $\mu_1 = m_1 + k_1(p_1 - 2)$, $\mu_2 = r_2$ and $n_1 n_2 < p_1 p_2$, that is $q_1 q_2 > r_2(m_1 + k_1(p_1 - 2))$. Let $(\overline{u}, \overline{v}) = (A_1\varphi(x), A_2)$, where $\varphi(x)$ satisfies (4). We choose
$$A_1 = \frac{1}{2}\left(A_2^{\frac{q_1}{m_1 + k_1(p_1-2)}} + \left(\alpha_2 A_2^{r_2}\right)^{\frac{1}{q_2}} \frac{1}{M_1}\right), \quad A_2 = \left(\alpha_2 M_1^{q_2}\right)^{-\frac{m_1 + k_1(p_1-2)}{q_1 q_2 - r_2(m_1 + k_1(p_1 - 2))}}.$$
Therefore, $(\overline{u}, \overline{v})$ is a global upper solution for problem (1), if $A_1 \geq \max_{x\in\overline{\Omega}} u_0(0)$ and $A_2 \geq \max_{x\in\overline{\Omega}} v_0(0)$.

Case 2: Next, consider large initial values. We construct a blow-up lower solution and use the comparison principle. Let $w(x) > 0$ be a continuous function and $w(x)|_{\partial\Omega} = 0$. We assume, that $0 \in \Omega$ and $w(0) > 0$.

1. Let $r_1 \geq m_1 + k_1(p_1 - 2)$ and $r_2 \geq m_2 + k_2(p_2 - 2)$ and $n_1 n_2 < p_1 p_2$, hence $q_1 q_1 > r_1 r_2$. The proof is similar to that in [1]. However, we give more details. In order to fulfill (13), we require
$$\begin{aligned} &\gamma_2 q_1 > \gamma_1 + 1, \quad \gamma_2 q_1 > \gamma_1 r_1, \\ &\gamma_1 q_2 > \gamma_2 + 1 > \gamma_2 r_2. \end{aligned} \tag{14}$$
We set $\lambda = \gamma_1/\gamma_2$, then by (14),
$$\frac{r_2}{q_2} < \lambda < \frac{q_1}{r_1}, \quad r_2 - 1 < \frac{1}{\gamma_2} < \min\{q_1 - \lambda, \lambda q_2 - 1\}.$$
If $\lambda \leq \frac{q_1+1}{q_2+1}$, then $\min\{q_1 - \lambda, \lambda q_2 - 1\} = \lambda q_2 - 1$. We assume, that
$$\frac{r_2}{q_2} < \frac{q_1 + 1}{q_2 + 1}. \tag{15}$$
Since $q_1 q_2 > r_1 r_2$, (15) holds or
$$\frac{r_1}{q_1} < \frac{q_2 + 1}{q_1 + 1}. \tag{16}$$
If (16) holds, we just exchange the roles of functions u and v in problem (1). Therefore, we need to guarantee, that (15) holds.

To fulfill (14), we have to find a suitable λ from
$$\frac{r_2}{q_2} < \lambda < \min\left\{\frac{q_1 + 1}{q_2 + 1}, \frac{q_1}{r_1}\right\}.$$

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It is possible since $\frac{r_2}{q_2} < \frac{q_1}{r_1}$ and $\alpha_2 > 0$, such that

$$0 < r_2 - 1 < \frac{1}{\gamma_2} < \lambda q_2 - 1.$$

Thus, (13) holds. Therefore, $(\underline{u}, \underline{v})$ is a lower solution of (1) for $r_1 \geq m_1 + k_1(p_1 - 2)$ and $r_2 > m_2 + k_2(p_2 - 2)$.

2. If $r_1 < m_1 + k_1(p_1 - 2)$ and $r_2 \geq m_2 + k_2(p_2 - 2)$ and $n_1 n_2 < p_1 p_2$, hence $q_1 q_2 > r_2 (m_1 + k_1(p_1 - 2))$. Any solution of (1) is an upper solution to the following homogeneous Dirichlet problem

$$\begin{aligned} &\frac{\partial u}{\partial t} \geq \nabla \left(|x|^{n_1} u^{m_1 - 1} \left| \nabla u^{k_1} \right|^{p_1 - 2} \nabla u \right) + v^{q_1} - \alpha_1 u^{r_1} - \alpha_1, \quad x \in \Omega, t > 0, \\ &\frac{\partial v}{\partial t} \geq \nabla \left(|x|^{n_2} v^{m_2 - 1} \left| \nabla v^{k_2} \right|^{p_2 - 2} \nabla v \right) + u^{q_2} - \alpha_2 v^{r_2}, \quad x \in \Omega, t > 0. \end{aligned} \tag{17}$$

Following $q_1 q_2 > r_2 (m_1 + k_1(p_1 - 2))$, similarly to the above proof, one can see that $(\underline{u}, \underline{v})$ is still a lower solution of (17) for appropriate u_0 and v_0. It means that (u, v) blows up.

The subcases (3) $r_1 \geq m_1 + k_1(p_1 - 2)$, $r_2 < m_2 + k_2(p_2 - 2)$ and $n_1 n_2 < p_1 p_2$ and (4) $r_1 < m_1 + k_1(p_1 - 2)$, $r_2 < m_2 + k_2(p_2 - 2)$ and $n_1 n_2 < p_1 p_2$ can be treated in a similar way.

Proof of Theorem 3 is completed.

Conclusion

This paper considers the problem of a doubly nonlinear degenerate parabolic system with nonlinear source and absorption terms with variable density. We extend existing results on the global existence and blow-up of solutions to the case with variable density in the diffusion term. Since the global existence and blow-up property of solutions allow us to predict or control the future of processes, the obtained results are significant and contribute to a concise understanding of doubly nonlinear degenerate parabolic problems. Future directions of research include investigating the problem in wider settings and studying other qualitative properties of the solutions.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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