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Solving Problems of Vibrational Processes of Isotropically Homogeneous Elastic Plates

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Abstract—The relevance of the topic is determined by the search for the transcendental frequency equations that are reduced to algebraic and the influence of both boundary conditions at the edges of a rectangular plate and geometric and mechanical characters on the inherent oscillatory frequencies of rectangular flat elements is considered and the previous results for a rectangular plate whose material satisfies a viscoelastic model of Maxwell is generalized. When studying oscillatory processes in a solid deformable body, it is advisable to take the kernel of viscoelastic operators regularly, since only such operators describe the instantaneous elasticity and then the viscous flow, which is characteristic of deformable solids. Integro-differential equations with regular kernels are known to be equivalent to partial differential equations.

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1. INTRODUCTION

Fundamental ideas and approaches in the development of mathematical models, theoretical and experimental studies in the field of dynamic interaction of a plate are associated with the names of such scientists as Akhenbakh, Vlasov, Grigolyuk, Ilyushin, Lentiev, Petrashen, Rakhmatullin, Timoshenko, Filippov [1] and many others. The issues of wave propagation in elastic and viscoelastic media were studied in the works of scientists Kolsky, Grigolyuk, Rabotnova, Rakhmatullina, Akhenbakh, Timoshenko, Filippov [2] and others. Many current scientific and technical problems are associated with the study of oscillatory processes and the propagation of waves in continuous media. The use of the results of these studies is of great benefit when considering unsteady oscillatory and wave processes. However, a number of questions arise related to the reaction of the medium to external influences, the methods of excitation of movements, the kinematic characteristics of the waves, the geometry of the bodies, the solution of which is of practical importance and is achieved using its own methods typical of this field [3–6]. Summarizing the brief review of the works, which is certainly not complete, it can be noted that the solution of the dynamic problems of flat structural elements in the form of plates is far from complete. In the study of most of them, simplifying assumptions about the viscous properties of plate materials were made, and only a limited range of vibration frequencies was considered. When solving applied problems of oscillation of rectangular planar elements, a wide class of oscillation problems arises related to various boundary value problems: approximate oscillation equations, various boundary conditions at the edges of a plane element, and initial conditions [7–9]. In the theory of oscillations, an important point is the determination of the frequencies of natural oscillations, the solution of problems of forced oscillations of a plane element, and the study of the propagation of harmonic waves in them.

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Most of the problems of determining the natural frequencies of flat elements articulated around the edges and on the basis of approximate theories obtained on the basis of hypotheses and assumptions of a mechanical and geometric nature, in particular, on the basis of approximate equations such as the Kirchhoff equation of parabolic type, poorly describing the wave and oscillatory behavior flat element with unsteady external influences. Therefore, in the study of harmonic waves in deformable bodies, the concept of phase velocity as the rate of change of the state of the medium is introduced, while the phase velocity is expressed in terms of the frequencies of natural vibrations and therefore the study of the propagation of harmonic waves is directly related to the problems of determining the natural forms and frequencies of vibration of plane elements bounded. In this paper, we present the results of the study of natural and forced vibrations of flat elements, taking into account the layering of the element material, rheological viscous properties, environmental influences, deformable substrates, anisotropy, etc.

The influence of these factors greatly complicates the study of problems of natural and forced vibrations of a plane element, the propagation of harmonic waves in them. A set of approximate equations, boundary and initial conditions make it possible to formulate and solve various boundary value problems of oscillations and wave processes for a plane element [10].

When studying oscillations and wave processes in a solid deformable body, it is advisable to take the core of viscoelastic operators regular, since only such operators describe instantaneous elasticity and then viscous flow, which is typical for deformable solids. Integro-differential equations with regular kernels are known to be equivalent to partial differential equations.

For other approximate equations of oscillation of a plane element, these equations for regular nuclei can also be reduced to partial differential equations. The next more complex oscillation of a rectangular planar element is the oscillation when two of the opposite edges are pivotally supported, and the other two edges have different types of fastening or are free of stresses. This class of problems leads to transcendental equations for determining the frequencies of natural vibrations, which can be solved both numerically [11] and analytically.

2. FORMULATION OF THE PROBLEM

We consider a flat element as an isotropic homogeneous elastic plate of constant thickness.

We confine ourselves to solving the task on the basis of an approximate equation of transverse oscillations of the fourth order [1, 2]

$$P_0(W) + p\frac{\partial^2 W}{\partial t^2} + \frac{h^2}{6}\left[p^2\left(N^{-1} + 3M^{-1}\right)\frac{\partial^4 W}{\partial t^4} - 4p\left(3 - 2MN^{-1}\right)\Delta\frac{\partial^2 W}{\partial t^2}\right.$$
$$\left. + 8M\left(1 - MN^{-1}\right)\Delta^2 W\right] = \Phi\left(\varphi_Z, f_{\iota z}\right). \tag{1}$$

Since the edges of $(y = 0;\ l_2)$ of the plate are hinged and supported, the solution of equation (1) will be sought in the form

$$W\left(x, y, t\right) = \exp\left(i\frac{b}{h}\xi t\right)\sum_{k=1}^{\infty} W_k\left(x\right)\sin\left(\frac{k\pi y}{l_2}\right). \tag{2}$$

Substituting (2) into equation (1), for W_k we obtain an ordinary differential equation

$$\frac{d^4 W_k}{dx^4} + B_0\frac{d^2 W_k}{dx^2} + B_1 W_k = 0, \tag{3}$$

where the coefficients B_0, B_1 are equal

$$B_0 = \left[\frac{A_1}{A_2}\xi^2\left(\frac{b}{h}\right)^2 - 2\left(\frac{k\pi}{l_2}\right)^2\right];$$

$$B_1 = \left[\left(\frac{k\pi}{l_2}\right)^4 + \frac{A_0}{A_2}\xi^4\left(\frac{b}{h}\right)^4 - \frac{A_1}{A_2}\xi^2\left(\frac{b}{h}\right)^2\left(\frac{k\pi}{l_2}\right)^2 - \frac{1}{A_2}\left(\frac{b}{h}\right)^2\xi^2\right].$$

The general solution of the equation (3) is written as

$$W_k(x) = C_1\left[\frac{\cos(a_0x)}{a_0^n} + \frac{\cos(a_1x)}{a_1^n}\right] + C_2\left[\frac{\cos(a_0x)}{a_0^n} + \frac{\cos(a_1x)}{a_1^n}\right]$$
$$+ C_3\left[\frac{\sin(a_0x)}{a_0^m} + \frac{\sin(a_1x)}{a_1^m}\right] + C_4\left[\frac{\sin(a_0x)}{a_0^m} + \frac{\sin(a_1x)}{a_1^m}\right], \tag{4}$$

where C_j are the integration constants, the a_i, a_j roots of the characteristic equation $a^4 + B_0a^2 + B_1 = 0$ and equal

$$a_{0,1} = \sqrt{\frac{B_0}{2}} \pm \sqrt{\left(\frac{B_0}{2}\right)^2 - B_1}. \tag{5}$$

The integers (n, m) are chosen from the condition of simplifying the solution when the boundary condition on the left edge $x = 0$ is satisfied, and the other boundary conditions on $X = l_1$ lead to a transcendental equation for determining the self-frequencies of the plate.

Let's consider some of the formulated tasks.

Task 1. In this case, at the edges of the plate, we have the boundary conditions

$$W_k = \frac{dW_k}{dx} = 0; \quad (x = 0; l_1). \tag{6}$$

Under boundary conditions (6) in the general solution (4), the numbers $n = 0$; $m = 0$, from the condition on the left end of the constant integration я$_1$, я$_2$ are zero, and from the conditions on the right end we obtain

$$C_2[\cos(a_0l_1) - \cos(a_1l_2)] + C_4\left[\frac{\sin(a_0l_1)}{a_0} - \frac{\sin(a_1l_1)}{a_1}\right] = 0;$$
$$C_2[a_0\sin(a_0l_1) - a_1\sin(a_1l_1)] - C_4[\cos(a_0l_1) - \cos(a_1l_1)] = 0;$$

whence from the condition of non-triviality of the solution we get the transcendental frequency equation

$$2 - \frac{a_0^2 + a_1^2}{a_0a_1}\sin(a_0l_1)\sin(a_1l_1) - 2\cos(a_0l_1)\cos(a_1l_1) = 0. \tag{7}$$

Task 2. The edge $x = 0$ is rigidly fixed, and the edge $X = l_1$ is elastically fixed. In this task the numbers $n = 0$; $m = 0$, and the constants $C_1 = C_3 = C_0$ are the same as in tasks 1 and 3, and in the conditions on the right end we get the frequency equation

$$\left\{2 + \left(a_0^2 + a_1^2\right)\left[\left(a_0^2 - a_1^2\right) - Q + Q_1Q_2\right]\right\}$$
$$\times\left\{-a_0a_1\left[\left(a_0^2 + a_1^2\right) + Q + 2Q_1Q_2\right] + Q\left(\frac{a_0^3}{a_1} + \frac{a_1^3}{a_0}\right) + Q_1Q3\left(\frac{a_0^2 + a_1^2}{a_0a_1}\right)\right\}\sin(a_0l_1)\sin(a_1l_1)$$
$$-\left\{2a_0^2a_1^2 + Q\left(a_0^2 - a_1^2\right) + Q_1Q_2\left(a_0^2 + a_1^2\right) + 2Q_1Q_3\right\}\cos(a_0l_1)\cos(a_1l_1)$$
$$+\left(a_0^2 - a_1^2\right)\left[\left\{\frac{QQ_2}{a_1} + \frac{Q_3}{a_0} - a_0Q_1\right\}\sin(a_0l_1)\sin(a_1l_1)\right.$$
$$\left.\times\left\{a_0Q_1 - \left(\frac{Q_3}{a_1} + \frac{QQ_2}{a_0}\right)\right\}\cos(a_0l_1)\cos(a_1l_1)\right] = 0,$$

where

$$Q_1 = \frac{3p_2h_2(1 - v_0)}{2h_1^2}; \quad Q_2 = \frac{3p_2h_2}{4}\left(\frac{b_1}{h_1}\right)^2;$$
$$Q_3 = \left[\left(\frac{b_1}{h_1b_2}\right)^2 - 2\left(\frac{\pi n}{l_2}\right)^2\right]\frac{3(3 - 4v_1)(1 - v_1)h_2^2p_2b_2^2}{p_1b_1^2h_1^3(1 - v_2)}.$$

Task 3. The edge $x = 0$ is hinged and supported, and the edge $X = l_1$ is free from stresses. In this task $n = 0$; $m = 3$; $C_1 = C_2 = 0; b$, we obtain the frequency equation

$$\cos(\alpha_0 l_1)\sin(\alpha_1 l_1)(Q-\alpha_1^2)\alpha_1^{-3} - \sin(\alpha_0 l_1)\cos(\alpha_1 l_1)(Q-\alpha_0^2)\alpha_0^{-3} = 0.$$

Let us generalize the results of the previous paragraph to the case of an elastic flat rectangular element in the form of a three-layer and transversely isotropic pre-stressed plate.

The equations of oscillation of such flat elements in the case of elastic materials we write in the general form

$$C_0\frac{\partial^2 W}{\partial t^2} + C_1\frac{\partial^4 W}{\partial t^4} - C_2\frac{\partial^2}{\partial t^2}\Delta W + C_3\Delta^2 W = 0, \tag{8}$$

where the coefficients C_j for a three-layer plate are equal

$$C_0 = \left[\frac{\rho_1}{\rho_2} + \frac{(h_2-h_1)}{h_1}\right]; \quad C_2 = -\frac{1}{1-2\nu_2}; \quad D_j = \frac{1}{2(1-\nu_j)}$$

$$C_3 = \frac{2\rho_1 b_1^2}{\rho_2}D_1 h_1\left(h_2 - \frac{h_1}{3}\right) + 2_2 b_2^2(h_2-h_1)^2\frac{2h_2-h_1}{3h_1} - 2D_2 b_2^2(h_2-h_1)(3h_2-h_1) - \frac{\rho_1}{\rho_2}b_1^2(1+2D_1)h_1(h_2-h_1);$$

and for a pre-stressed transversely isotropic plate are equal

$$C_0 = 1; \quad C_1 = \frac{h^2}{6}\left\{\rho\left[\frac{1}{1+C_2}\frac{1}{a_{33}} + \frac{3}{(1+a_0)a_{44}}\right]\right\};$$

$$C_2 = \left[2\frac{1+C_2}{1+a_0} - 2\frac{a_{13}}{a_{33}} - 3(a_{13}^2 - a_{11}a_{33})\frac{1}{a_{33}a_{44}}\right]; \quad C_3 = \left[2(1+C_2)(a_{11}a_{33}-a_{13}^2)\frac{1}{a_{33}\rho}\right].$$

We also look for the solution of equation (8) in the form (2) and for W_k we get the equation

$$\frac{\partial^4 W_k}{\partial x^4} + B_0\frac{\partial^2 W_k}{\partial x^2} + B_1 W_k = 0, \tag{9}$$

where the coefficients B_0, B_1 are equal

$$B_0 = \left[\frac{C_2}{C_3}\left(\frac{b}{h}\right)^2\xi^2 - 2\gamma\right]; \quad B_1 = \left[\gamma^2 + \frac{C_1}{C_3}\left(\frac{b}{h}\right)^4\xi^4 - \frac{C_2}{C_3}\left(\frac{b}{h}\right)^2\gamma\xi^2 - \frac{C_0}{C_3}\left(\frac{b}{h}\right)^2\xi^2\right].$$

As you can see, equation (9) does not differ in form from the equation (3) and the roots of its characteristic equation are also equal to (5) and the general solution of equation (9) also has the form (4). Thus, to find the self-oscillation frequencies for various boundary tasks, it is the same as in (7) and others. The task for an orthotropic plate is solved in the same way.

3. ANALYSIS OF TRANSCENDENTAL EQUATIONS

Let us analyze the transcendental frequency equations of the first point. Firstly, let us consider the simplest transcendental equation [12]

$$\alpha_0\cos(\alpha_0 l_1)\sin(\alpha_1 l_1) - \alpha_1\sin(\alpha_0 l_1)\cos(\alpha_1 l_1) = 0. \tag{10}$$

We introduce the notation

$$l = \frac{l_1}{h}; \quad \alpha_{0.1}^1 = \sqrt{\frac{B_0^1}{2} \pm \sqrt{\left(\frac{B_0^1}{2}\right)^2 - B_1^1};}$$

$$B_0'' = \left[(2-\nu)\xi^2 - 2\gamma\right]; \gamma = \left(\frac{\pi k h}{l_2}\right)^2; \quad B_1^{;} = \left[\gamma^2 + \frac{7-8\nu}{8}\xi^4 - (2-\nu)\gamma\xi^2 - \frac{3}{2}(1-\nu)\xi^2\right], \tag{11}$$

and we will omit the strokes in the future for simplicity. Since the sines and cosines of any argument are equal $\sin z = \sum_{i=0}^{\infty}(-1)^i \frac{z^{2i+1}}{(2i+1)!}$; $\cos z = \sum_{j=0}^{\infty}(-1)^j \frac{z^{2j}}{(2j)!}$. That equation (10) is equivalent to the following

$$\alpha_0\alpha_1 \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{i+j}\frac{\alpha_1^{2i}\alpha_0^{2j} - \alpha_0^{2i}\alpha_1^{2j}}{(2i+1)!(2j)!} l^{2(i+j)} = 0. \tag{12}$$

If we accept the quantity a_0 is determined from the expression (5) with a plus sign under the root, then it follows this root does not vanish for any values of y, v, ξ. Therefore, at the beginning it is possible to $a_1 = 0$ or for ξ we obtain the equation

$$\xi^4 - \frac{8\left[(2-\nu)\gamma + {}^3\!/_2(1-\nu)\right]}{(7-8\nu)}\xi^2 + \frac{8\gamma^2}{(7-8\nu)} = 0;$$

whose roots are equal

$$\begin{aligned}\xi_{1,2} = (7-8\nu)^{-\frac{2}{2}} &\sqrt{4\left[(2-\nu)\gamma + \frac{3}{2}(1-\nu)\right]} \\ \pm &\sqrt{8(1+\nu^2)\gamma^2 + 3\gamma(1-\nu)(2-\nu) + \frac{9}{4}(1-\nu)^2},\end{aligned} \tag{13}$$

since the series in the expressions of trigonometric functions are convergent and the series in equation (11), equivalent to equation (10) is also convergent, in the study of the partial equation (12) it can be limited to a finite number of first terms.

Taking the first three terms in the series (12), we write it in the form

$$\alpha_0\alpha_1(\alpha_1^2 - \alpha_0^2)\left\{\frac{1}{3}l^2 - \frac{1}{30}(\alpha_1^2 + \alpha_0^2)l^4 + \left(\frac{1}{840}(\alpha_1^4 + \alpha_0^2\alpha_1^2 + \alpha_0^4) + \frac{1}{360}\alpha_0^2\alpha_1^2\right) l^6 + \ldots\right\} = 0. \tag{14}$$

The roots of the expression $\alpha 1 = 0$ are equal to (13). The quantity of $(a_1^2 - a_0^2)$ is non-zero for any values of y, v, ξ.

If in the expression (14) we take only the first two terms, we get $(\alpha_1^2 + \alpha_0^2) - 10l^{-2} = 0$ or $B_0 - 10l^{-2} = 0$, and frequency equation $\xi^2 = \frac{2\gamma + 10l^{-2}}{(2-\nu)}$; positive root of which is equal to $\xi = \sqrt{\frac{2\gamma+10l^{-2}}{(2-\nu)}}$. If we take all the first three terms in the expression, we get $\left[(\alpha_1^4 + \alpha_0^4) + \frac{10}{3}\alpha_0^2\alpha_1^2\right] - 28(\alpha_1^2 + \alpha_0^2)l^{-2} + 280l^{-4} = 0$ or $\left[B_0^2 + \frac{4}{3}B_1\right] - 28B_0l^{-2} + 280l^{-4} = 0$, and their corresponding frequency equation

$$\begin{gathered}\left[(2-\nu)^2 + \frac{7+8\nu}{6}\right]\xi^4 - \left[(2-\nu)\left(\frac{16}{3}\gamma + 28l^{-2}\right) + 2(1-\nu)\right]\xi^2 \\ + \left[\frac{16}{3}\gamma^2 + 56\gamma l^{-2} + 280l^{-4}\right] = 0,\end{gathered}$$

which has two positive roots. Similarly, one can take the first four or more terms in expression (12) and obtain a more accurate frequency equation and corresponding frequencies ξ. To find the frequency equation from the series of equation (12), it is necessary to clarify the condition of appropriate retention of a finite number of terms.

Let us apply the d'Alembert principle of series convergence to the series in equation (12). We obtain

$$\left|\frac{\alpha_0^2\alpha_1^2 l^2}{(2i+3)(2j+2)}\right| \le q^2 < 1, \tag{15}$$

where $0 < q < 1$. From the inequality (15) implies that

$$\left|a_0^2 a_1^2\right| \le q_{i,j}^2 = q_{i,j}^2 = q^2\frac{(2i+3)\,(2j+2)}{l^2}. \tag{16}$$

The analysis of inequality (16) shows that it is valid when the solving the inequality

$$-\left(\frac{8}{7-8v}\right)q_{i,j}^2 \leq \xi^4 - 2D\xi^2 + E \leq \left(\frac{8}{7-8v}\right)q_{i.j}^2 = C_{i,j}^2,$$

where the coefficients D, E are equal $D = \frac{4\left[(2-v)\gamma + \frac{3}{2}(1-v)\right]}{(7-8v)}$; $E = \frac{8\gamma^2}{(7-8v)}$, or inequality

$$D^2 - E \leq C_{i,j}^2. \tag{17}$$

By the given parameters of a geometric and mechanical character from the inequality (17) one can determine the necessary number of first terms in series (12) for finding the frequency equation of relative frequencies ξ.

We consider the transcendental equation (7). Like transcendental equation (10), equation (7) is equivalent to the following

$$a_0 a_1 \left\{ 2\left[1 - \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{i+j}\frac{a_0^{2i}a_1^{2j}}{(2i)!(2j)!}J^{2(i+j)}\right] \right.$$
$$\left. -\left(a_0^2 + a_1^2\right)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{i+j}\frac{a_0^{2i}a_1^{2j}}{(2i+1)!(2j+1)!}J^{2(i+j+1)}\right\} = 0. \tag{18}$$

From the equation it follows that, at first, $a_1 = 0$ and we get the frequencies (13).

We write equation (18) by writing the first terms

$$a_0 a_1 \left\{(a_0^2 + a_1^2)J^2 - \frac{1}{6}(5a_0^4 + 5a_1^4 + a_0^2 a_1^2)J^4 + \frac{1}{90}\left[a_0^6 + a_1^6 + 7a_0^2 a_1^2(a_0^2 a_1^2)\right]J^6 + ...\right\} = 0. \tag{19}$$

From (19) it also follows that we can assume $\left(a_0^2 + a_1^2\right) = 0$ and obtain $B_0 = 0$ or $\xi^2 - \frac{2\gamma}{(2-v)} = 0$, whose root is equal to $\xi = \sqrt{\frac{2\gamma}{(2-v)}}$. Similarly, we can suppose approximately $(a_0^2 + a_1^2) - \frac{J^2}{6}(5a_0^4 + 5a_1^4 + a_0^2 a_1^2) = 0$, and obtain the frequency equation $(5B_0^2 - 9B_1) - \frac{6}{J^2}B_0 = 0$ having positive roots. Thus, transcendental frequency equations can be reduced to algebraic and to investigate the influence of both boundary conditions along the edges of a rectangular plate or a rectangular flat element, as well as geometric and mechanical character on the self-oscillation frequencies of rectangular flat elements.

Let us generalize the previous results for a rectangular plate or a flat element, the material of which satisfies the Maxwell viscoelastic model.

Suppose we have a rectangular homogeneous isotropic plate. In this case, the solution of an approximate fourth order equation will be sought in the form

$$W = \exp\left(\frac{b}{h}\xi t\right)\sum_{k=1}^{\infty}W_k \sin\left(\frac{pky}{J_2}\right),$$

where ξ is the complex frequency, the real part of which determines the law of damping of oscillations, and the imaginary part determines the frequencies of self-oscillations. For W_k we get an ordinary differential equation

$$\frac{d^4 W_k}{dx^4} - \overline{B_0}\frac{d^2 W_k}{dx^2} + \overline{B_1}W_k = 0, \tag{20}$$

where $\overline{B_0}, \overline{B_1}$ are equal

$$\overline{B_0} = \left[2\gamma + \frac{A_1}{A_2}\frac{b}{h}\left(\frac{b}{h}\xi^2 + \frac{1}{\tau}\xi\right)\right];$$

$$\overline{B_1} = \left[\frac{A_0}{A_2}\left(\frac{b}{h}\right)^4\xi^4 + 2\frac{A_0}{A_2\tau}\left(\frac{b}{h}\right)^3\xi^3 + \left(\frac{b}{h}\right)^2\left(\frac{1}{A_2} + \frac{1}{\tau^2}\frac{A_0}{A_2} + 2\gamma\frac{A_1}{A_2}\right)\xi^2\right.$$

$$+\frac{1}{A_2\tau}\left(\frac{b}{h}\right)\left(1+2\gamma\frac{A_1}{A_2}\right)\xi+\gamma^2\Big].$$

The coefficients A_j are given in the previous paragraphs.

The general solution of equation (20) we write in the form

$$W_k=C_1\left[\frac{\cosh(a_0x)}{a_1^n}+\frac{\cosh(a_1x)}{a_1^n}\right]+C_2\left[\frac{\cosh(a_0x)}{a_0^n}+\frac{\cosh(a_1x)}{a_1^n}\right]+\left[\frac{\sinh(a_0x)}{a_0^m}+\frac{\sinh(a_1x)}{a_1^m}\right]$$
$$+C_4\left[\frac{\sinh(a_0x)}{a_0^m}+\frac{\sinh(a_1x)}{a_1^m}\right],$$

that is instead of trigonometric functions, we have hyperbolic.

All the boundary value tasks considered above leading to transcendental equations are solved in a similar way. Transcendental equations are obtained from the previous ones, in which values a_j must be replaced by values ia_j, where i is an imaginary unit.

For example, the transcendental equation (10) goes into the equation

$$a_0\cosh(a_0l_1)\sinh(a_1l_1)-a_1\sinh(a_0l_1)\cosh(a_1l_1)=0,$$

which is equivalent to the following

$$a_0a_1\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{a_1^{2i}a_1^{2j}a_0^{2i}}{(2i+1)!\,(2j)!}l^{2(i+j)}=0;\quad a_{0,1}=\sqrt{\frac{\overline{B_0}}{2}\pm\sqrt{\frac{\overline{B_0^2}}{4}-B_1}}. \tag{21}$$

One of the frequency equations of the formula (21) follows from the condition $a_j=0$ and we obtain

$$\xi^4+\frac{2}{\tau_0}\xi^3+\frac{8}{(7-8v)}\left[(2-v)\gamma+\frac{(7-8v)}{8\tau_0^2}+\frac{3}{2}(1-v)\right]\xi^2$$
$$+\frac{12(1-v)}{(7-8v)\tau_0}\left[1+2(2-v)\gamma\right]\xi+\frac{8}{(7-8v)}\gamma^2=0,$$

which coincides with the frequency equation for a rectangular hinged and supported plate on all four sides of the plate, has two complexly conjugate roots.

4. CONCLUSION

Based on theoretical results, a class of boundary value problems on the natural vibrations of rectangular plane elements is formulated and solved.

1. Natural vibrations of rectangular flat elements with a hinged or free description of a rectangular element along all four edges; fourth-order algebraic equations for determining the frequencies of natural vibrations are derived; the numerical results when using approximate fourth-order equations for hyperbolic-type derivatives show that for an elastic plate or a rectangular element, the frequencies of natural vibrations are weakly dependent on the Poisson coefficient, but they are functions of wave harmonics and geometric parameters of a rectangular flat element; taking viscosity into account mainly affects the attenuation coefficient; the influence of the prestressed homogeneous state of the plate material on the frequencies of its natural vibrations is estimated.

2. Natural vibrations of rectangular planar elements, when any two opposite edges of the element are articulated, and the other two can satisfy other boundary conditions or their combinations; an analytical method for solving oscillation problems has been developed, transcendental equations have been obtained for determining the natural frequencies of rectangular plane elements; it is shown that for a wide class of problems, oscillations of the second class of natural frequencies are limited by frequencies from above and below by frequencies for a pivotally supported flat element, i.e. the two frequencies in the articulated description are the boundaries of the variation of frequencies under more complex boundary conditions.

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