https://doi.org/10.31489/2024M4/185-196 *Research article*

Double factorization of the Jonsson spectrum

A.R. Yeshkeyev[1], O.I. Ulbrikht[1,*], M.T. Omarova[1,2]

[1] *Karaganda Buketov University, Karaganda, Kazakhstan;*
[2] *Karaganda University of Kazpotrebsoyuz, Karaganda, Kazakhstan*
(E-mail: aibat.kz@gmail.com, ulbrikht@mail.ru, omarovamt_963@mail.ru)

First of all, we have to note that in this article, we introduced the new concepts of relations between Jonsson theories in the class of cosemanticness for some considered Jonsson spectrum. All consideration of this new approach was done under sufficiently important class of Jonsson theories, which we called as normal Jonsson theories class. The main result, that we obtained, describes the model-theoretical properties of syntactical and semantical similarities inside the fixed cosemanticness class. For all new concepts in the article, we provided classical samples. The main result of this paper is considering normal Jonsson theories class by similarity to some fixed class of polygons (S-acts).

Keywords: Jonsson theory, perfect Jonsson theory, normal Jonsson theory, Jonsson set, almost Jonsson set, Jonsson fragment, syntactic similarity, semantic similarity, Jonsson specrtum, cosemanticness, S-act.

2020 Mathematics Subject Classification: 03C35, 03C48, 03C52, 03C65.

Introduction

The content of this article actually belongs to new approach and studying of generally speaking incomplete theories, and partically more exactly we focused our researches on studying Jonsson theories. Our new approach consists of applying syntactically and semantically similarities inside some considered class of cosemanticness from fixed Jonsson spectrum for some subclass of existentially closed models for considered fixed Jonsson theory. Such new method of studying Jonsson theories by our proposals allowed to penetrate in more details when we have operated with classical settlements of many tasks and problems which appears under considering and researching Jonsson spectra.

The notion of syntactical and semantical similarities was appeared in the works of T.G. Mustafin, for example in [1], when he introduced those notions for studying complete theories under stability consideration terms. By main result of this article it turned out that many concepts from stability theory saved their properties under semantical similarity, starting from basic notions of formulas and types and up to orthogonality, and independence, and forking, and spectral functions, which appears under studying of stability theories and their types. It turned out that, for any complete theories, it follows that there exists syntactically similar elementary theory of some given polygon (S-act) over a fixed monoid S. With the help of such consideration it became clear that all researches in the field of studying Model Theory in complete theories we can operate working with some fixed polygons. And in other side, in the case of incomplete theories after works [2–13] such implementation by using polygons is possible for Jonsson theories. This approach is sufficiently new, moreover not only Jonsson theories and some concepts which linked with Jonsson theories, for example hybrids of Jonsson theories, allowed us to wirk and describe Jonsson theories even for different signatures.

This paper consists of 3 sections. In Section 1, we give some basic information on Jonsson theories and related concepts, and introduce the new notion of normal Jonsson theory. In Section 2, the concepts

*Corresponding author. *E-mail: ulbrikht@mail.ru*

This research was funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP23489523).

Received: 30 June 2024; *Accepted:* 12 September 2024.

of syntactic and semantic similarities for complete theories and for Jonsson theories are described. In Section 3, we present our results obtained for double factorization equivalence class of some fixed Jonsson spectrum and show its syntactic similarity to the class of theories of some polygon.

Note that here and after we will use the termin "S-act" instead of "polygon".

Now let us introduce the notation and determine the frame of our study.

We work in a first-order countable language L. By theory, we mean a consistent set of sentences in the given language.

If T is an L-theory, then E_T denotes a class of existentially closed models of the theory T.

Let A be an L-structure. By $T^0(A)$, we mean the theory $Th_{\forall\exists}(A)$ that is a set of all $\forall\exists$-sentences of L true for the structure A. The theory $T^0(A)$ is called a Kaiser hull of A.

1 *Jonsson theories*

We start with some basic information on Jonsson theories and related concepts. In this section the apparatus of the study of Jonsson theories is described.

Before presenting the concept of Jonsson theories let us remind the definitions of two properties that are essential for studying this class of incomplete theories.

Definition 1. [14; 80] A theory T has the joint embedding property, if, for any models A and B of T, there exists a model M of T and isomorphic embeddings $f : A \to M$, $g : B \to M$.

Definition 2. [14; 80] A theory T has the amalgamation property, if for any models A, B_1, B_2 of T and isomorphic embeddings $f_1 : A \to B_1$, $f_2 : A \to B_2$ there are $M \models T$ and isomorphic embeddings $g_1 : B_1 \to M$, $g_2 : B_2 \to M$, such that $g_1 \circ f_1 = g_2 \circ f_2$.

We write "JEP" and "AP" as shorter forms for the joint embedding and amalgamation properties, correspondingly.

Now let us recall the main definition of this section.

Definition 3. [14; 80] A theory T is called Jonsson, if:
1) the theory T has an infinite model;
2) the theory T is inductive;
3) the theory T has the joint embedding property (JEP);
4) the theory T has the amalgamation property (AP).

There are a lot of classical examples of Jonsson theories:
1) group theory;
2) the theory of abelian groups;
3) the theory of Boolean algebras;
4) the theory of linear orders;
5) field theory of characteristic p, where p is zero or a prime number;
6) the theory of ordered fields;
7) the theory of modules et cetera.

In [6], it is proved that the theory of differentially closed fields of the fixed characteristic is a Jonsson theory as well.

The special properties of Jonsson theories, namely AP and JEP, can be syntactically described by the following two theorems:

Theorem 1. [15] For the first order theory T of the language L (of arbitrary cardinality) the following conditions are equivalent:

1) T has JEP;

2) For all universal sentences α, β of L, if $T \vdash \alpha \vee \beta$ then $T \vdash \alpha$ or $T \vdash \beta$.

If φ and ψ are existential L-sentences such that $T \cup \{\varphi\}$ and $T \cup \{\psi\}$ are consistent then $T \cup \{\varphi, \psi\}$ is consistent.

Theorem 2. [16] The following are equivalent:

1) T has the Amalgamation property;

2) For all universal L-formulas $\alpha_1(\overline{x}), \alpha_2(\overline{x})$ with $T \vdash \forall x(\alpha_1(x) \vee \alpha_2(x))$ there are existential L-sentences $\beta_1(\overline{x}), \beta_2(\overline{x})$ such that

$$T \vdash \forall x(\beta_i(x) \to \alpha_i(x)),\ i = 1, 2,$$

and

$$T \vdash \forall x(\beta_1(x) \vee \beta_2(x)).$$

Another fundamental property of Jonsson theories follows from the theorem of W. Hodges and shows the connection between existentially closed models of such theories:

Theorem 3. [17; 363] Suppose T be an L-theory, and let T admit JEP. Let A and B be existentially closed model of T. Then each $\forall\exists$-sentence that is true in A is true in B as well.

In this paper, we study a special subclass of Jonsson theories, namely perfect Jonsson theories. To describe them we need the following definitions of Mustafin Ye.T.

Definition 4. [18] Let $\kappa \geq \omega$. Model $\mathcal{M}$ of theory T is called:

1) κ-universal for T, if each model of theory T with the power strictly less κ isomorphically imbedded in $\mathcal{M}$;

2) κ-homogeneous for T, if for any two models $\mathcal{A}$ and $\mathcal{A}_1$ of theory T, which are submodels of $\mathcal{M}$ with the power strictly less then κ and for isomorphism $f : \mathcal{A} \to \mathcal{A}_1$ for each extension $\mathcal{B}$ of model $\mathcal{A}$, which is a submodel of $\mathcal{M}$ and is model of T with the power strictly less then κ there exists the extension $\mathcal{B}_1$ of model $\mathcal{A}_1$, which is a submodel of $\mathcal{M}$ and an isomorphism $g : \mathcal{B} \to \mathcal{B}_1$ which extends f.

Definition 5. [18] A model $\mathcal{C}$ of the Jonsson theory T is called a semantic model, if it is ω^+-homogeneous-universal.

Definition 6. [18] The center of Jonsson theory T is an elementary theory of its semantic model $\mathcal{C}$ and denoted through T^*, i.e. $T^* = \mathrm{Th}(\mathcal{C})$.

Definition 7. [19] A Jonsson theory T is called perfect, if a semantic model of T is ω^+-saturated model of T.

The criterion for the perfectness of the Jonsson theory was obtained by Yeshkeyev A.R. and it is as follows:

Theorem 4. [19] For any Jonsson theory T following conditions are equivalent:
1) T is perfect;
2) T^* is the model companion of T.

Let us also demonstrate some properties of a perfect Jonsson theory and its center.

Theorem 5. [20; 1243] Let T be a Jonsson theory. Then for any model $A \in E_T$ theory $T^0(A)$ is Jonsson, where $T^0(A) = Th_{\forall\exists}(A)$.

We can see that in case of perfectness of T its center T^* is also a Jonsson theory.

Proposition 1. [21] Let T be a perfect Jonsson theory, then for every sentence $\varphi \in T^* \backslash T$ the theory $T' = T \cup \{\varphi\}$ is a Jonsson.

The following definition was introduced by Mustafin T.G.

Definition 8. We say that the Jonsson theory T_1 is cosemantic to the Jonsson theory T_2 ($T_1 \bowtie T_2$), if $\mathcal{C}_{T_1} = \mathcal{C}_{T_2}$, where $\mathcal{C}_{T_i}$ are semantic model of T_i, $i = 1, 2$.

The properties of cosemantic Jonsson theories were studied by Mustafin Ye.T. in [18]. This binary relation between two theories is an equivalence relation as it is easy to see. In the framework of study of Jonsson theories, it was introduced as a special tool for comparing Jonsson theories from the point of view of their semantic invariants, i.e. their semantic models. A lot of important model-theoretic properties coincide for Jonsson theories that are cosemantic. In this manner, when considering such properties, we may describe not just a Jonsson theory but the whole class of theories that are cosemantic to it.

In our research, we often apply so-called semantic method, whose essence is to study the properties of L-structures with the help of the theories of these structures. This is why the following notion was introduced by the first author of this paper.

Definition 9. [20] Let K be a class of L-structures. A Jonsson spectrum $JSp(K)$ of K is the followng set of theories

$$JSp(K) = \{T \mid T \text{ is a Jonsson theory and } \forall A \in K\ A \models T\}.$$

The particular case of a Josson spectrum is a Robinson spectrum. Robinsonian theories are $\forall$-axiomatizable Jonsson theories.

Definition 10. [20] Let K be a class of L-structures. A Robinson spectrum $RSp(K)$ of K is the following set of theories

$$RSp(K) = \{T \mid T \text{ is a Robinsonian theory and } \forall A \in K\ A \models T\}.$$

The following proposition is important for studying Robinsonian theories and Robinson spectrum.

Proposition 2. [22] Let K be an arbitrary class of L-structures (possibly, it consists of one structure), $RSp(K)_{/\bowtie}$ be a factor set of the Robinson spectrum of K with respect to cosemanticness. Then every cosemanticness class $[\Delta]$ contains exactly one theory. In other words, for any two Robinsonian L-theories T and T', the relation of cosemanticness is equivalent to the equality (logical equivalence) of theories, i.e. $T \bowtie T' \Leftrightarrow T = T'$.

That is in Robinson spectrum factorized by cosemanticness, each cosemanticness class is single-element.

To study Jonsson theories through their semantic invariants, we often consider specific subsets of the semantic models of these theories. Let us describe them.

Definition 11. Let T be a Jonsson theory, C_T be its semantic model, $X \subseteq C$. X is said to be a Jonsson subset of C_T, if X is an $\exists$-definable set and $cl(X) = M$, where $M \in E_T$.

For each Jonsson subset $X \subseteq C_T$ for the theory T, we always can construct the fragment of X:

Definition 12. The fragment of a Jonsson subset X is a theory $Fr(X) = Th_{\forall\exists}(M)$, where $M = cl(X)$.

Let T be a Jonsson theory, $X \subseteq C_T$ and let $cl(X) = M \in E_T$. That is X is a Jonsson subset of C_T. Then $Fr(X) = Th_{\forall\exists}(M)$ and, moreover, the following lemma is true:

Lemma 1. [19; 299] For any Jonsson set $X \subset C_T$, the fragment $Fr(X)$ is a Jonsson theory.

Obviously, all axioms of T are true in a semantic model of $Fr(X)$, that is $C_{Fr(X)} \in Mod(T)$ and moreover $C_{Fr(X)}$ is existentially closed over T. It means that whenever X is a Jonsson set for T the semantic model of $Fr(X)$ is always embedded in C_T and an existentially closed submodel of C_T for any Jonsson theory T. To generalize this case and refine possible situations in the context of study of Jonsson theories, we introduce the following notions:

Definition 13. Let T be a Jonsson theory, C_T be its semantic model, $X \subseteq C$. X is called an almost Jonsson subset of C_T, if X is an $\exists$-definable set and $cl(X) = M$, where $M \in Mod(T)$, and $Th_{\forall\exists}(M)$ is a Jonsson theory.

By analogy with the concept of a Jonsson set, for an almost Jonsson subset $X \subseteq C_T$ of the theory T, we consider the fragment of X:

Definition 14. The fragment of an almost Jonsson subset X is a theory $Fr(X) = Th_{\forall\exists}(M)$, where $M = cl(X)$.

Thus the following definition refines the class of Jonsson theories whose properties we study in this paper:

Definition 15. A Jonsson theory T is called normal if for each almost Jonsson subset $X \subseteq C_T$, $C_{Fr(X)} \in Mod(T)$ and $C_{Fr(X)}$ is an existentially closed submodel of C_T.

There are natural examples of normal Jonsson theories, let us describe the following.

Example 1. Let T_{AG} be the theory of all abelian groups and let X be a set such as $cl(X) \in M \in Mod(T_{AG})$, i.e. X is an almost Jonsson set and M is an abelian group. It is well-known that $Fr(X) = Th_{\forall\exists}(M)$ is a Jonsson theory. Therefore, T_{AG} is a normal Jonsson theory.

Besides, there are non-normal Jonsson theories. The following example confirms this fact.

Example 2. Let T_V be the theory of all vector spaces. It is known that this theory is Jonsson. Let us consider a vector space that is the semantic model of T and its subspace $V \subseteq C_{T_V}$. The domain of V is a Jonsson set, and, consequently, is an almost Jonsson set. If X is the domain of V, then $cl(X) = V$. However, V is not an existentially closed submodel of C_{T_V}, since V may have another dimension that differs from dimension of C_{T_V}. Dimension of V can be formed by an $\forall\exists$-sentence, and this sentence fails in C_{T_V}. According to Theorem 3, V is not existentially closed in C_{T_V}). Thus, T_V is a Jonsson theory that is not normal.

The following theorem is necessary for our study.

Lemma 2. Let T be a perfect normal Jonsson theory. Then T^* is also a normal Jonsson theory.

Proof. Firstly we should note that it follows from Theorem 5 that T^* is also a perfect Jonsson theory. Moreover, it is easy to see that $T \bowtie T^*$, which means that

$$C_T = C_{T^*}. \tag{1}$$

Let X be an arbitrary almost Jonsson subset of C_{T^*}. Then $cl(X) = M \in Mod(T^*)$, and $C_{Fr(X)} \in Mod(T^*)$. According to Theorem 4, $Mod(T^*) = E_T$, therefore $C_{Fr(X)} \in E_T$, which means that $C_{Fr(X)}$ is an existentially closed submodel of C_T. By (1), $C_{Fr(X)}$ is also an existentially closed submodel of C_{T^*}. Thus T^* is a normal Jonsson theory.

2 *Syntactic and semantic similarities of Jonsson theories*

In this sections, we describe the notions of Mustafin T.G. that he introduced for complete theories, and Jonsson analogies of these notions proposed by the first author of this article.

To study and compare complete theories, especially theories of different languages, Mustafin T.G. [1]. used binary relations, which ha called syntactic similarity and semantic similarity. Let us describe them.

We start with the concept of syntactic similarity of complete theories. Let $F_n(T)$, $n < \omega$ be the Boolean algebra of formulas of T with exactly n free variables $v_1, \ldots, v_n$ and $F(T) = \bigcup_n F_n(T)$.

Definition 16. [1] Complete theories T_1 and T_2 are syntactically similar if and only if there exists a bijection $f : F(T_1) \to F(T_2)$ such that

1) $f \restriction F_n(T_1)$ is an isomorphism of the Boolean algebras $F_n(T_1)$ and $F_n(T_2)$, $n < \omega$;
2) $f(\exists v_{n+1}\varphi) = \exists v_{n+1} f(\varphi)$, $\varphi \in F_{n+1}(T)$, $n < \omega$;
3) $f(v_1 = v_2) = (v_1 = v_2)$.

The following example of syntactic similarity of complete theories was given in [1].

Example 1. The following theories T_1 and T_2 of the signature $\sigma = \langle \varphi, \psi \rangle$ are syntactically similar, where φ, ψ are binary functions:

$$T_1 = \mathrm{Th}(\langle Z; +, \cdot \rangle), \quad T_2 = \mathrm{Th}(\langle Z; \cdot, + \rangle).$$

Now we describe the concept of semantic similarity of complete theories. For this, we need the following definitions.

Definition 17. [1]

1) $\langle A, \Gamma, \mathcal{M} \rangle$ is called the pure triple, where A is not empty, Γ is the permutation group of A and $\mathcal{M}$ is the family of subsets of A such that from $M \in \mathcal{M}$ follows that $g(M) \in \mathcal{M}$ for every $g \in \Gamma$.

2) If $\langle A_1, \Gamma_1, \mathcal{M}_1 \rangle$ and $\langle A_2, \Gamma_2, \mathcal{M}_2 \rangle$ are pure triples and $\psi : A_1 \to A_2$ is a bijection then ψ is an isomorphism, if:

(i) $\Gamma_2 = \{\psi g \psi^{-1} : g \in \Gamma_1\}$;

(ii) $\mathcal{M}_2 = \{\psi(E) : E \in \mathcal{M}_1\}$.

Definition 18. [1] The pure triple $\langle C, Aut(\mathcal{C}), Sub(C) \rangle$ is called the semantic triple of complete theory T, where C is a domain of Monster model $\mathcal{C}$ of theory T, $Aut(\mathcal{C})$ is the automorphism group of $\mathcal{C}$, $Sub(C)$ is a class of all subsets of C each of which is a domain of the corresponding elementary submodel of $\mathcal{C}$.

Definition 19. [1] Complete theories T_1 and T_2 are semantically similar if and only if their semantic triples are isomorphic.

The following example of the semantic similarity of complete theories was given in [1].

Example 2. The following theories T_1 and T_2 are semantically similar, where

$$T_1 = Th(\langle \mathcal{M}_1; P_n, n < \omega; a_{nm}, n, m < \omega \rangle),$$
$$\mathcal{M}_1 = \{a_{nm} : n, m < \omega\},$$
$$P_n(\mathcal{M}_1) = \{a_{nm} : m < \omega\},$$

and

$$T_2 = Th(\langle \mathcal{M}_2; Q_n, n < \omega; Q_{nm}, n, m < \omega; b_{nmk}, n, m, k < \omega \rangle),$$
$$\mathcal{M}_2 = \{b_{nmk} : n, m, k < \omega\},$$
$$Q_n(\mathcal{M}_2) = \{b_{nmk} : m, k < \omega\},$$
$$Q_{nm}(\mathcal{M}_2) = \{b_{nmk} : k < \omega\}.$$

It turned out that the above types of similarity are not equivalent to each other.

Proposition 3. [1] If T_1 and T_2 are syntactically similar, then T_1 and T_2 semantically similar. The converse implication generally fails.

Let us recall the definition of semantic property.

Definition 20. [1] A property (or a notion) of theories (or models, or elements of models) is called semantic if and only if it is invariant relative to semantic similarity.

For example from [1] it is known that:

Proposition 4. The following properties and notions are semantic:

(1) type;

(2) forking;

(3) λ-stability;

(4) Lascar rank;

(5) Strong type;
(6) Morley sequence;
(7) Orthogonality, regularity of types;
(8) $I(\aleph_\alpha, T)$ is the spectrum function.

The following definition was introduced in the frame of Jonsson theories study by first author of this article in [19].

Let T be an arbitrary Jonsson theory, then $E(T) = \bigcup_{n<\omega} E_n(T)$, where $E_n(T)$ is a lattice of $\exists$-formulas with n free variables, T^* is a center of Jonsson theory T, i.e. $T^* = Th(\mathcal{C})$, where $\mathcal{C}$ is semantic model of Jonsson theory T in the sense of [18].

Definition 21. [19] Let T_1 and T_2 are arbitrary Jonsson theories. We say that T_1 and T_2 are Jonsson syntactically similar, if a bijection $f : E(T_1) \longrightarrow E(T_2)$ exists such that:
1) restriction f to $E_n(T_1)$ is isomorphism of lattices $E_n(T_1)$ and $E_n(T_2)$, $n < \omega$;
2) $f(\exists v_{n+1}\varphi) = \exists v_{n+1} f(\varphi)$, $\varphi \in E_{n+1}(T)$, $n < \omega$;
3) $f(v_1 = v_2) = (v_1 = v_2)$.

The examples of syntactic similarities of two Jonsson theories are given in [21].

As in the case of complete theories, the first author of this article defined in [19] a semantic similarity between two Jonsson theories.

Definition 22. [19] The pure triple $\langle C, Aut(\mathcal{C}), Sub(C)\rangle$ is called the Jonsson semantic triple, where C is a domain of semantic model $\mathcal{C}$ of theory T, $Aut(\mathcal{C})$ is the automorphism group of $\mathcal{C}$, $Sub(C)$ is a class of all subsets of C which are domains of the corresponding existentially closed submodels of $\mathcal{C}$.

Definition 23. [19] Two Jonsson theories T_1 and T_2 are called Jonsson semantically similar, if their Jonsson semantic triples are isomorphic as pure triples.

The correctness of this definition follows from the fact that the perfect Jonsson theory has a unique semantic model up to isomorphism. Otherwise, all semantic models are only elementary equivalent to each other.

For the convenience of further exposition we introduce the following notation. The syntactic and semantic similarities of the complete theories T_1 and T_2 will be denoted $T_1 \overset{S}{\bowtie} T_2$ and $T_1 \underset{S}{\bowtie} T_2$ respectively. In the case when we consider Jonsson theories T_1 and T_2, through $T_1 \overset{S}{\rtimes} T_2$ will denote the Jonsson syntactic similarity of theories T_1 and T_2, and through $T_1 \underset{S}{\rtimes} T_2$ Jonsson semantic similarity of theories T_1 and T_2.

Theorem 6. [19] Let T_1 and T_2 are $\exists$-complete perfect Jonsson theories, then following conditions are equivalent:

1) $T_1 \overset{S}{\rtimes} T_2$;
2) $T_1^* \overset{S}{\bowtie} T_2^*$.

An analogous result of Proposition 3 in the case of two Jonsson theories was obtained by Yeshkeyev A.R.

Theorem 7. Let T_1 and T_2 be two Jonsson theories and let T_1 and T_2 be Jonsson syntactically similar. Then T_1 and T_2 are Jonsson semantically similar.

Thus it is true that

$$T_1 \overset{S}{\rtimes} T_2 \Rightarrow T_1 \underset{S}{\rtimes} T_2$$

for any two Jonsson theories T_1 and T_2. That is Jonsson syntactic similarity is a sufficient condition of Jonsson semantic similarity of theories. There are also some cases when this condition is necessary.

In this paper, we consider such specific classes of Jonsson theories for which these two relations are equivalent. We denote this relation by the following:

$$T_1 \overset{SS}{\rtimes} T_2.$$

Lemma 3. [21] Any two cosemantic Jonsson theories are Jonsson semantically similar.

The proof follows from the definition of cosemantic Jonsson theories.

The converse result is also true:

Lemma 4. Let T_1 and T_2 be Jonsson theories and let $T_1 \underset{S}{\rtimes} T_2$. Then $T_1 \bowtie T_2$.

Proof. Let C_{T_1} and C_{T_2} be semantic models of T_1 and T_2, correspondingly. Let $T_1 \underset{S}{\rtimes} T_2$, then Jonsson semantic triples $(C_{T_1}, Aut(C_{T_1}), Sub(C_{T_1})$ and $(C_{T_2}, Aut(C_{T_2}), Sub(C_{T_2})$ are isomorphic as pure triples. Then it is clear that there exists an isomorphism between C_{T_1} and C_{T_1}, which means that $T_1 \bowtie T_2$.

Thus we obtain that, for any two Jonsson theories T_1 and T_2 of language L, it follows that

Corollary 1. $T_1 \overset{S}{\rtimes} T_2 \Rightarrow T_1 \underset{S}{\rtimes} T_2 \Leftrightarrow T_1 \bowtie T_2$.

The definitions of relations of Jonsson semantic and syntactic similarity were also generalized for classes of Jonsson theories in [21]:

Definition 24. [21] Let $\mathcal{A} \in \mathrm{Mod}\sigma_1$, $\mathcal{B} \in \mathrm{Mod}\sigma_2$, $[T]_1 \in JSp(\mathcal{A})/_{\bowtie}$, $[T]_2 \in JSp(\mathcal{B})/_{\bowtie}$. We say that the class $[T]_1$ is Jonsson syntactically similar to class $[T]_2$ and denote $[T]_1 \overset{S}{\rtimes} [T]_2$, if for any theory $\Delta \in [T]_1$ there is theory $\Delta' \in [T]_2$ such that $\Delta \overset{S}{\rtimes} \Delta'$.

Definition 25. [21] The pure triple $\langle C, Aut(\mathcal{C}), \overline{E}_{[T]}\rangle$ is called the Jonsson semantic triple for class $[T] \in JSp(A)/_{\bowtie}$, where C is the semantic model of $[T]$, $AutC$ is the group of all automorphisms of C, $\overline{E}_{[T]}$ is the class of isomorphically images of all existentially closed models of $[T]$.

Definition 26. [21] Let $\mathcal{A} \in \mathrm{Mod}\sigma_1$, $\mathcal{B} \in \mathrm{Mod}\sigma_2$, $[T]_1 \in JSp(\mathcal{A})/_{\bowtie}$, $[T]_2 \in JSp(\mathcal{B})/_{\bowtie}$. We say that the class $[T]_1$ is Jonsson semantically similar to class $[T]_2$ and denote $[T]_1 \underset{S}{\rtimes} [T]_2$, if their semantically triples are isomorphic as pure triples.

3 Properties of classes of S-acts

In this section, we show our main result using the concepts from the previous sections, for the special class of structures, namely for S-acts. Let us shortly describe this class.

Definition 27. [1] By an S-act over a monoid S (sometimes it is called S-acts) we mean a structure with only unary functions $\langle A; f_\alpha : \alpha \in S\rangle$ such that:

1) $f_e(a)$ $\forall a \in A$, where e is the unit of S;

2) $f_{\alpha\beta}(a) = f_\alpha(f_\beta(a))$ $\forall \alpha, \beta \in S$, $\forall a \in A$.

The following results show that any complete theory has some syntactic similar theory.

Theorem 8. [1] For every theory T_2 in a finite signature there is a theory T_1 of S-acts such that some inessential extension of T_1 is an almost envelope of T_2.

Theorem 9. [1] For every theory T_2 in an infinite signature there is a theory T_1 of S-acts such that some inessential extension of T_1 is an envelope of T_2.

Now we present the settlement of our research's problem and the main result of our paper.

Let T be a Jonsson L-theory, E_T be a class of the existentially closed models of T, $K \subseteq E_T$. Let us construct a Jonsson spectrum $JSp(K)$ of the class K and on this spectrum we introduce the following relations: cosemanticness, Jonsson syntactic similarity and Jonsson semantic similarity of Jonsson theories. It is obvious that all these relations are equivalence relations, so we obtain a factor-set of the Jonsson spectrum of K with respect to relations introduced that we denote by $JSp(K)/_{\substack{SS\\ \bowtie}}$. $[T]$ is an equivalence class of a theory T from $JSp(K)/_{\substack{SS\\ \bowtie}}$. The examples of such classes do exist and let us demonstrate some of them. Let us consider $RSp(K)$ which is a partial case of $JSp(K)$. We introduce the relation of Jonsson syntactic similarity of theories on $RSp(K)$. According to Corollary 1, all theories from the equivalence class $[T] \in RSp(K)/_{\substack{S\\ \rtimes}}$ are also Jonsson semantically similar and cosemantic. So we get $RSp(K)/_{\substack{SS\\ \bowtie}}$. According to Theorem 2, in any cosemanticness class in Robinsonian spectrum, there is only one theory with respect to logical equivalence.

Theorem 10. Let T be a Jonsson L-theory, $K \subseteq E_T$, $[T] \in JSp(K)/_{\substack{SS\\ \bowtie}}$ be an $\exists$-complete perfect normal class (i.e. such that all $T_i \in [T]$ are perfect normal Jonsson theories). Then there exists $[T'_\Pi] \in JSp(K')/_{\substack{SS\\ \bowtie}}$, where K' is a class of some S-acts in the corresponding language, such that $[T_\Pi]$ is an $\exists$-complete perfect class and $[T]$ is Jonsson syntactically similar to $[T'_\Pi]$.

Proof. Let $[T]$ be a perfect normal $\exists$-complete equivalence class in $JSp(K)/_{\substack{SS\\ \bowtie}}$. Since the center T^* of this class is a complete theory, according to Theorem 8 in the case of a finite signature and Theorem 9 in the case of an infinite signature, there is a complete theory of the S-act T_Π such that $T^* \overset{S}{\bowtie} T_\Pi$. But then, according to Proposition 3, it follows that $T^* \underset{S}{\bowtie} T_\Pi$. Since the concept of type is a semantic notion (Proposition 4), the concept of a formula is also semantic. It follows from Theorem 1 and Theorem 2 that the properties of JEP and AP are formulated using some L-formulas, i.e. JEP and AP are semantic concepts. It is clear that $\forall\exists$-axiomatizability is also a semantic property, since all axioms are true in the semantic model. This means that the property "to be a Jonsson theory" is a semantic concept, and therefore T_Π is also a Jonsson theory.

Since $[T]$ is a normal class the center T^* is a normal theory as well according to Lemma 2. In this manner, the property of being normal Jonsson theory is also transferred to T_Π, as this notion is semantic. It means that the theory T_Π is a normal Jonsson theory.

Since T^* is a perfect Jonsson theory, then semantic model $\mathcal{C}_T$ of the class $[T]$ is ω^+-saturated. But $T^* \underset{S}{\bowtie} T_\Pi$ and, by definition, the semantic triples of these theories are isomorphic, then $\mathcal{C}_{[T]} \cong \mathcal{C}_{T_\Pi}$, therefore $\mathcal{C}_{T_\Pi}$ is also ω^+-saturated and therefore T_Π is a perfect Jonsson theory. Let K' be a class of S-acts such that $T_\Pi \in JSp(K)$. Then the equivalence class $[T_\Pi] \in JSp(K')/_{\substack{SS\\ \bowtie}}$ is a perfect class, since all theories in this equivalence class has the same semantic model, which is ω^+-saturated.

Consider $JSp(C_{T_\Pi})$. Since the theory T_Π is perfect then $|JSp(C_{T_\Pi})/_{\bowtie}| = 1$ due to the fact that T_Π is normal. Let $\Delta \in JSp(C_{T_\Pi})$, i.e. Δ is Jonsson theory and $\Delta^* = T_\Pi$. We show that Δ is perfect $\exists$-complete Jonsson theory. By virtue of $T^* \underset{S}{\bowtie} \Delta^*$, then from the definition of semantic similarity for complete theories it follows that Δ is a perfect Jonsson theory. If Δ is $\exists$-complete, then we take Δ and then by Theorem 6 it follows that $T \overset{S}{\rtimes} \Delta = T'_\Pi$. If Δ is not $\exists$-complete, then we carry out the following replenishment procedure for this theory. As $\Delta \subset T_\Pi$, then for any existential sentence φ, of the signature language of Δ such that $\Delta \nvdash \varphi$ and $\Delta \nvdash \neg\varphi$, but $\varphi \in T_\Pi$, consider the theory $\Delta' = \Delta \cup \{\varphi\}$. Since $\Delta \subset \Delta' \subset T_\Pi$, and Δ, T_Π are Jonsson theories, it follows from Proposition 1 that Δ' is also a Jonsson theory. If Δ' is not $\exists$-complete, then we continue the procedure of adding existential sentences $\varphi \in T_\Pi$ until Δ' it becomes $\exists$-complete. We make this procedure for each $T' \in JSp(K')/_{\substack{SS\\ \bowtie}}$ and obtain an $\exists$-complete equivalence class.

Let $\overline{\Delta} = \Delta \cup \{\varphi | \varphi \in \Sigma_1, \varphi \in T_\Pi\}$ is the result of replenishment procedure of the theory Δ, i.e. $\overline{\Delta}$ is $\exists$-complete and at the same time $\overline{\Delta}$ is a Jonsson theory. We show that $\overline{\Delta} \in JSp(C_{T_\Pi})$, hence the perfection of the theory of $\overline{\Delta}$ will follow from here. Suppose the contrary, let $\overline{\Delta} \notin JSp(C_{T_\Pi})$, then $C_{T_\Pi} \notin Mod(\overline{\Delta})$, but this is not true since $C_{T_\Pi} \models \Delta$ and for any sentence $\varphi \in \overline{\Delta} \backslash \Delta$, $\varphi \in T_\Pi$. Consequently, $C_{T_\Pi} \models \varphi$ and $C_{T_\Pi} \in \mathrm{Mod}(\overline{\Delta})$. We obtain a contradiction, i.e. $\overline{\Delta} \in JSp(C_{T_\Pi})$. But C_{T_Π} is saturated, therefore, $\overline{\Delta}$ is a perfect Jonsson theory. Then by Theorem 6 we have $T^* \overset{S}{\bowtie} \overline{\Delta}^* \Leftrightarrow T \overset{S}{\rtimes} \overline{\Delta}$, where $\overline{\Delta} = T'_\Pi$. It follows that, for each theory $T \in [T] \in JSp(K)/_{\underset{\bowtie}{SS}}$, there exists such theory $T'_\Pi \in [T'_\Pi] \in JSp(K')/_{\underset{\bowtie}{SS}}$ such that $T \overset{S}{\rtimes} \overline{\Delta}$. Thus, according to Definition 24, the class $[T]$ is Jonsson syntactically similar to the class $[T'_\Pi]$.

It should be noted that, in this manner, the results of [21] are special cases of Theorem 10 considering the theories as single-element equivalence classes in some Jonsson spectrum.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

References

1 Mustafin, T.G. (1990). On similarities of complete theories. *Logic Colloquium '90. Proceedings of the Annual European Summer Meeting of the Association for Symbolic Logic*, 259–265.

2 Yeshkeyev, A. (2020). Method of the rheostat for studying properties of fragments of theoretical sets. *Bulletin of the Karaganda University. Mathematics Series, 4*(100), 152–159. https://doi.org/10.31489/2020m4/152-159

3 Yeshkeyev, A.R. (2010). On Jonsson stability and some of its generalizations. *Journal of Mathematical Sciences, 166*(5), 646–654. https://doi.org/10.1007/s10958-010-9879-z

4 Yeshkeyev, A.R., & Mussina, N.M. (2021). An algebra of the central types of the mutually model-consistent fragments. *Bulletin of the Karaganda University. Mathematics Series, 1*(101), 111–118. https://doi.org/10.31489/2021m1/111-118

5 Yeshkeyev, A.R., & Omarova, M.T. (2021). An essential base of the central types of the convex theory. *Bulletin of the Karaganda University. Mathematics Series, 1*(101), 119–126. https://doi.org/10.31489/2021m1/119-126

6 Yeshkeyev, A.R., Tungushbayeva, I.O., & Kassymetova, M.T. (2022). Connection between the amalgam and joint embedding properties. *Bulletin of the Karaganda University. Mathematics Series, 1*(105), 127–135. https://doi.org/10.31489/2022m1/127-135

7 Yeshkeyev, A.R., Ulbrikht, O.I., & Omarova, M.T. (2023). The Number of Fragments of the Perfect Class of the Jonsson Spectrum. *Lobachevskii Journal of Mathematics, 43*(12), 3658–3673. https://doi.org/10.1134/S199508022215029X

8 Yeshkeyev, A.R.,Yarullina, A.R., Amanbekov, S.M., & Kassymetova M.T. (2023). On Robinson spectrum of the semantic Jonsson quasivariety of unars. *Bulletin of the Karaganda University. Mathematics Series, 2*(110), 169–178. https://doi.org/10.31489/2023m2/169-178

9 Yeshkeyev, A.R., Ulbrikht O.I., & Yarullina A.R. (2022). Existentially positive Mustafin theories of S-acts over a group. *Bulletin of the Karaganda University. Mathematics Series, 2*(106), 172–185. https://doi.org/10.31489/2022m2/172-185

10 Yeshkeyev, A.R., Tungushbayeva, I.O., & Amanbekov, S.M. (2022). Existentially prime Jonsson quasivarieties and their Jonsson spectra. *Bulletin of the Karaganda University. Mathematics Series, 4*(108), 117–124. https://doi.org/10.31489/2022m4/117-124

11 Yeshkeyev, A.R., Ulbrikht O.I., & Issayeva A.K. Algebraically prime and atomic sets. (2023). *Turkic World Mathematical Society (TWMS) Journal of Pure and Applied Mathematics, 14*(2), 232–245. https://doi.org/10.30546/2219-1259.14.2.2023.232

12 Yeshkeyev, A.R., Tungushbayeva, I.O., & Koshekova, A.K. (2024). The cosemanticness of Kaiser hulls of fixed classes of models. *Bulletin of the Karaganda University. Mathematics Series, 1*(113), 208–217. https://doi.org/10.31489/2024M1/208-217

13 Poizat, B., & Yeshkeyev, A. (2022). Back and forth in positive logic. *Studies in Universal Logic*, 603–609. https://doi.org/10.1007/978-3-030-94452-0_31

14 Barwise, J. (1982). *Teoriia modelei: spravochnaia kniga po matematicheskoi logike. Chast 1 [Model theory: Handbook of mathematical logic. Part 1].* Moscow: Izdatelstvo Nauka [in Russian].

15 Ignjatovic, A., & Grulovic, M.Z. (1996). A comment on the joint embedding property. *Periodica Mathematica Hungarica, 33*(1), 45–50. https://doi.org/10.1007/BF02341089

16 Bacsich, P.D., & Hughes, D.R. (1974). Syntactic characterisations of amalgamation, convexity and related properties. *Journal of Symbolic Logic, 39*(3), 433–451. https://doi.org/10.2307/2272886

17 Hodges, W. (1993). *Model Theory. Encyclopedia of Mathematics and its Applications.* Cambridge: Cambridge University Press.

18 Mustafin, Y. (2002). Quelques proprietes des theories de Jonsson. *The Journal of Symbolic Logic, 67*(2), 528–536. https://doi.org/10.2178/jsl/1190150095

19 Yeshkeyev, A.R., & Kassymetova, M.T. (2016). *Ionsonovskie teorii i ikh klassy modelei: monografiia [Jonsson theories and their model classes]*: monograph. Karaganda: Izdatelstvo Karagandinskogo gosudarstvennogo universiteta [in Russian].

20 Yeshkeev, A.R., & Ulbrikht, O.I. (2019). JSp-cosemanticness of R-modules. *Siberian Electronic Methematical Reports, 16,* 1233–1244. https://doi.org/10.33048/semi.2019.16.084

21 Yeshkeyev, A.R., Ulbrikht, O.I., & Urken, G.A. (2023). Similarities of Jonsson spectra's classes. *Bulletin of the Karaganda University. Mathematics Series, 4*(112), 130–143. https://doi.org/10.31489/2023m4/130-143

22 Yeshkeyev, A.R., Tungushbayeva I.O., & Ulbrikht O.I. (2024). On some types of algebras on a Jonsson spectrum. *Siberian Electronic Methematical Reports, 21*(2), 866–881.https://doi.org/10.33048/semi.2024.21.057

Author Information[*]

Aibat Rafhatuly Yeshkeyev — Doctor of physical and mathematical sciences, Professor, Professor of the Department of Algebra, Mathematical Logic and Geometry named after T.G. Mustafin, Karaganda Buketov University, Universitetskaya street 28, Karaganda, Kazakhstan; e-mail: *aibat.kz@gmail.com*; https://orcid.org/00000-0003-0149-6143

Olga Ivanovna Ulbrikht (*corresponding author*) — PhD, Associative Professor of the Department of Algebra, Mathematical Logic and Geometry named after T.G. Mustafin, Karaganda Buketov University, Universitetskaya street 28, Karaganda, Kazakhstan; e-mail: *ulbrikht@mail.ru*; https://orcid.org/0000-0002-3340-2140

Makhabat Toleuovna Omarova — PhD, Head of the Department of Higher Mathematics, Karaganda University of Kazpotrebsoyuz, Akademicheskaya street 9, Karaganda, Kazakhstan; e-mail: *omarovamt_963@mail.ru*; https://orcid.org/0000-0003-4520-7964

[*]The author's name is presented in the order: First, Middle and Last Names.