DOI 10.31489/2022M1/102-108

UDC 517.9

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The Schrödinger equations generated by q-Bessel operator in quantum calculus

In this paper, we obtain exact solutions of a new modification of the Schrödinger equation related to the Bessel q-operator. The theorem is proved on the existence of this solution in the Sobolev-type space $W_q^2(\mathbb{R}_q^+)$ in the q-calculus. The results on correctness in the corresponding spaces of the Sobolev-type are obtained. For simplicity, we give results involving fractional q-difference equations of real order $\alpha > 0$ and given real numbers in q-calculus. Numerical treatment of fractional q-difference equations is also investigated. The obtained results can be used in this field and be supplement for studies in this field.

Keywords: q-integral, q-Jackson integral, q-difference operator q-derivative, the q-Bessel Fourier transform, the Sobolev type space,the Schrödinger equation, q-Bessel operator.

Introduction

The origin of the q-difference calculus plays an important role due to their numerous applications and its importance in mathematics and other scientific fields. This calculus can be traced back to the works in [1, 2] by F. Jackson and R.D. Carmichael [3] from the beginning of the twentieth century, while basic definitions and properties can be found e.g. in the monographs [4, 5]. Recently, the q-difference calculus has been proposed by W. Al-Salam [6] and R.P. Agarwal [7]. Today, maybe due to the explosion in research within the fractional differential calculus setting, new developments in this theory of fractional q-difference calculus have been addressed extensively by several researchers.

The Schrödinger equation is the fundamental equation of the science of submicroscopic phenomena known as quantum mechanics. This equation studied by the Austrian physicist Erwin Schrödinger in 1926 [8]. Moreover, it is widely used in modern science in such areas as quantum information and econophysics [9, 10].

Nowadays, the several methods and techniques have been developed to study exact and approximate analytical solutions to the modern models of the Schrödinger equation for a better understanding of its dynamical behavior [11, 12]. The Exact solutions of the Schrödinger equation play an important role not only from a pure mathematical point of view but also for the conceptual understanding of the physical phenomena.

The paper is organized as follows: The main results are presented and proved in Section 2. To not disturb these presentations we include some necessary Preliminaries in Section 1.

1 Preliminaries

Throughout this paper, we assume that $0 < q < 1$. We start by recalling some basic notation in the q-calculus, see e.g. the books [4] and [13].

Let $\alpha \in \mathbb{R}$. Then a q-real number $[\alpha]_q$ is defined by

$$[\alpha]_q := \frac{1-q^\alpha}{1-q},$$

where $\lim\limits_{q\to 1} \frac{1-q^\alpha}{1-q} = \alpha$.

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We introduce for $k \in \mathbb{N}$:

$$(a;q)_0 = 1,\ (a;q)_n = \prod_{k=0}^{n-1}\left(1 - q^k a\right),\ (a;q)_\infty = \lim_{n\to\infty}(a,q)_n,\ (a;q)_\alpha = \frac{(a;q)_\infty}{(q^\alpha a;q)_\infty}.$$

The q-analogue of the binomial coefficients $[n]_q!$ are defined by

$$[n]_q! = \begin{cases} 1, & \text{if } n = 0, \\ [1]_q \times [2]_q \times \cdots \times [n]_q, & \text{if } n \in \mathrm{N}. \end{cases}$$

The q^2-differential operator is defined by (see [14] and [15])

$$\partial_q f(x) = \frac{f(q^{-1}x) + f(-q^{-1}x) - f(qx) + f(-qx) - 2f(-x)}{2x(1-q)},$$

where $x \neq 0$.

Note that if f is differentiable at x, then $\lim\limits_{q\to 1} \partial_q f(x) = f'(x)$.

A repeated application of the q^2-analogue differential operator n times is denoted by:

$$\partial_q^0 f = f,\ \partial_q^{n+1} f = \partial_q\left(\partial_q^n f\right).$$

The definite q-integral or the q-Jackson integral of a function f is defined by the formula (see [1] and [2])

$$\int_0^x f(t)d_q t := (1-q)x\sum_{k=0}^{\infty} q^k f(q^k x), \quad x \in (0,b),$$

and the improper q-integral of a function $f(x) : [0,\infty) \to \mathbb{R}$, is defined by the formula

$$\int_0^\infty f(t)d_q t := (1-q)\sum_{k=-\infty}^{\infty} q^k f(q^k).$$

We denote $\mathbb{R}_q^+ = \{q^k, k \in \mathbb{Z}\}$ and define

$$L_{\alpha,q}^p\left(\mathbb{R}_q^+\right) := \{f : \|f\|_{p,\alpha,q} = \left(\int_0^\infty |f(x)|^p x^{2\alpha+1} d_q x\right)^{\frac{1}{p}} < \infty\}.$$

Definition 1. (see [16] and [14], [17]) The q-Bessel Fourier transform is defined for $f \in L_{\alpha,q}^1(\mathbb{R}_q^+)$, by

$$\mathcal{F}_{q,\alpha}(f)(\lambda) = c_{q,\alpha}\int_0^\infty f(x) j_\alpha(\lambda x;q^2) x^{2\alpha+1} d_q x, \quad 0 < x < \infty \tag{1}$$

and its inverse $\mathcal{F}_{q,\alpha}^{-1} g(x)$ is given by

$$\mathcal{F}_{q,\alpha}^{-1} g(x) = c_{q,\alpha}\int_0^\infty F_{q,\alpha}(g)(\lambda) j_\alpha(\lambda x;q^2)\lambda^{2\alpha+1} d_q\lambda, \quad 0 < x < \infty, \tag{2}$$

for $g \in L_{\alpha,q}^1(\mathbb{R}_q^+)$, where $c_{q,\alpha} = \frac{(1+q)^{-\alpha}}{\Gamma_{q^2}(\alpha+1)}$.

Definition 2. (See [18]) For $1 \leq p < \infty$, we define the Sobolev type space associated with the q-Bessel Fourier transform $W_q^p(\mathbb{R}_q^+)$ equipped with the norms

$$\|u\|_{W_q^p(\mathbb{R}_q^+)}^2 := \left(\int_0^\infty \left(1 + |\lambda|^2\right)^{\frac{p}{2}} |\mathcal{F}_{q,\alpha}(u)(\lambda)|^2 d_q\lambda\right)^2.$$

Let $0 < T < \infty$. We also introduce the spaces $C^k\left([0,T]; W_q^p(\mathbb{R}_q^+)\right)$ and $C^k\left([0,T]; L_q^p(\mathbb{R}_q^+)\right)$ defined by the finiteness of the norms

$$\|u\|_{C^k\left([0,T];W_q^s(\mathbb{R}_q^+)\right)} : \sum_{n=0}^{k} \max_{0\leq t\leq T} \|\partial_t^n u(t,\cdot)\|_{W_q^p(\mathbb{R}_q^+)}$$

and

$$\|u\|_{C^k\left([0,T];L_q^p(\mathbb{R}_q^+)\right)} := \sum_{n=0}^{k} \max_{0\leq t\leq T} \|\partial_t^n u(t,\cdot)\|_{L_q^p(\mathbb{R}_q^+)},$$

respectively.

For $\lambda \in \mathbb{C}$, the function $j_\alpha(\lambda x; q^2)$ is the unique even solution of the problem

$$\begin{cases} \Delta_{q,\alpha} f(x) = -\lambda^2 f(x), \\ f(0) = 1, \end{cases}$$

where

$$\Delta_{q,\alpha} f(x) = \frac{1}{|x|^{2\alpha+1}} \partial_q [|x|^{2\alpha+1} \partial_q f(x)].$$

Moreover, if f and $\Delta_{q,\alpha} f$ are in $L^1_{\alpha,q}(\mathbb{R}_{q,+})$, then (see e.g. [14] and [17]):

$$\mathcal{F}_{q,\alpha}(\Delta_{q,\alpha} f)(\lambda) = -\lambda^2 \mathcal{F}_{q,\alpha}(f)(\lambda). \tag{3}$$

Theorem 1. 1) (Plancherel formula [17]) For all $f \in \partial_{q,*}(\mathbb{R})$, we have

$$\|\mathcal{F}_{q,\alpha}(f)\|_{2,\alpha,q} = \|f\|_{2,\alpha,q}. \tag{4}$$

2) (Plancherel theorem) The q-Bessel transform can be uniquely extended to an isometric isomorphism on $L^2_{q,\alpha}(\mathbb{R}_q^+)$ with $\mathcal{F}^{-1}_{q,\alpha} = \mathcal{F}_{q,\alpha}$.

2 Main problem

We consider the Schrödinger equation generated by the q-Bessel operator $\Delta_{q,\alpha} f$ in the following form:

$$\partial_t u(t,x) - i\Delta_{q,\alpha,x} u(t,x) = f(t,x), \quad (t,x) \in [0,T] \times \mathbb{R}_q^+, \tag{5}$$

$$u(0,x) = \varphi(x), \quad x \in \mathbb{R}_q^+. \tag{6}$$

where the function φ is given functions listed above.

Theorem 2. Let $0 < \alpha < 1$. Suppose that $f \in C^1\left([0,T], L^2(\mathbb{R}_q^+)\right)$ and $\varphi \in W_q^2(\mathbb{R}_q^+)$. Then the problem (5)-(6) has a unique solution $u \in C^1\left([0,T], L^2(\mathbb{R}_q^+)\right) \bigcap C\left([0,T], W_q^2(\mathbb{R}_q^+)\right)$ and can be represented by formula

$$\begin{aligned} u(t,x) &= c^2_{q,\alpha} \int_0^\infty \int_0^\infty \exp(-i\lambda^2 t)\varphi(x) j_\alpha(\lambda x; q^2) x^{2\alpha+1} j_\alpha(\lambda x; q^2) \lambda^{2\alpha+1} d_q x d_q \lambda \\ &+ c^2_{q,\alpha} \int_0^t \int_0^\infty \int_0^\infty \exp(-i\lambda^2 (t-s)) f(s,x) j_\alpha(\lambda x; q^2) x^{2\alpha+1} j_\alpha(\lambda x; q^2) \lambda^{2\alpha+1} d_q x d_q \lambda ds. \end{aligned}$$

Proof. We assume that

$$\mathcal{F}_{q,\alpha}\left(u(t,\cdot)\right)(\lambda) = U(t), \quad \mathcal{F}_{q,\alpha}\left(\varphi\right) = \widehat{\varphi}(\lambda), \quad \mathcal{F}_{q,\alpha}\left(f(t,\cdot)\right)(\lambda) = F(t)$$

for fixed λ. Let us prove the existence first. Taking the q-Bessel Fourier transform $\mathcal{F}_{q,\alpha}$ (see (1)) on both sides of (5)–(6) we have a simple initial value problem (IVP) of linear ODE:

$$U'(t) + i\lambda^2 U(t) = F(t) \tag{7}$$

$$U(0) = \widehat{\varphi}(\lambda) \tag{8}$$

and $0 < t < T$. The solution of the problem (7)–(8) is given by

$$U(t) \quad = \quad \widehat{\varphi}(\lambda)\exp(-i\lambda^2 t) + \int\limits_0^t \exp(-i\lambda^2(t-s))F(s)ds. \tag{9}$$

Now by using the inverse q-Bessel Fourier transform $\mathcal{F}_{q,\alpha}^{-1}$ in (2) to (9), we obtain the formula for the solution of the problem (5)–(6), given by

$$\begin{aligned} u(t,x) &= c_{q,\alpha}^2 \int\limits_0^\infty \int\limits_0^\infty \exp(-i\lambda^2 t)\varphi(x) j_\alpha(\lambda x; q^2) x^{2\alpha+1} j_\alpha(\lambda x; q^2)\lambda^{2\alpha+1} d_q x d_q \lambda \\ &+ c_{q,\alpha}^2 \int\limits_0^t \int\limits_0^\infty \int\limits_0^\infty \exp(-i\lambda^2(t-s)) f(s,x) j_\alpha(\lambda x; q^2) x^{2\alpha+1} j_\alpha(\lambda x; q^2)\lambda^{2\alpha+1} d_q x d_q \lambda ds. \end{aligned}$$

Let $\varphi \in W_q^2(R_q^+)$ and $f \in C\left([0,T]; W_q^2(R_q^+)\right)$. Using $|\exp(-z)| < 1$ for $z \in \mathbb{C}$, Parseval's identity(see (4)) and the relation (9) in the following form:

$$\begin{aligned} \left|\mathcal{F}_{q,\alpha}\left(u(t,\cdot)\right)\right|^2 &\leq \left|\widehat{\varphi}(\lambda)\exp(-i\lambda^2 t)\right|^2 + \left|\int\limits_0^t \exp(-i\lambda^2(t-s))F(s)ds\right|^2 \\ &\leq \left|\widehat{\varphi}(\lambda)\right|^2 + t\int\limits_0^t \left|\mathcal{F}_{q,\alpha}\left(f(t,\cdot)\right)(\lambda)\right|^2 ds. \end{aligned}$$

Hence, using the Parseval's identity (4) and (3)

$$\begin{aligned} \|u(t,\cdot)\|_{2,\alpha,q}^2 &= \|\mathcal{F}_{q,\alpha}\left(u(t,\cdot)\right)\|_{2,\alpha,q}^2 \\ &= \int\limits_0^\infty \left|\mathcal{F}_{q,\alpha}\left(u(t,\cdot)\right)(\lambda)\right|^2 \lambda^{2\alpha+1} d_q\lambda \\ &\leq \|\mathcal{F}_{q,\alpha}\left(\varphi\right)\|_{2,\alpha,q}^2 + t\int\limits_0^t \|\mathcal{F}_{q,\alpha}\left(f(s,\cdot)\right)\|_{2,\alpha,q}^2 ds \\ &\leq \|\varphi\|_{2,\alpha,q}^2 + t\int\limits_0^t \max_{0\leq s\leq T}\|f(s,\cdot)\|_{2,\alpha,q}^2 ds \\ &\leq \|\varphi\|_{2,\alpha,q}^2 + T^2\|f\|_{C\left([0,T];L_q^2(\mathbb{R}_q^+)\right)}^2 < \infty. \end{aligned} \tag{10}$$

Then,

$$\begin{aligned}
\|u(t,\cdot)\|_{W_q^2(\mathbb{R}_q^+)} &= \int_0^\infty (1+\lambda^2)\left|\mathcal{F}_{q,\alpha}\left(u(t,\cdot)\right)(\lambda)\right|^2 d_q\lambda \\
&\le \int_0^\infty (1+\lambda^2)\left|\widehat{\varphi}(\lambda)\right|^2 d_q\lambda + t\int_0^t\int_0^\infty (1+\lambda^2)\left|\mathcal{F}_{q,\alpha}\left(f(s,\cdot)\right)(\lambda)\right|^2 d_q\lambda ds \\
&\le \|\varphi\|_{W_q^2(\mathbb{R}_q^+)} + \int_0^t \max_{0\le s\le T}\|\widehat{f}(s,\cdot)\|_{W_q^2(\mathbb{R}_q^+)} ds \\
&\le \|\varphi\|_{W_q^2(\mathbb{R}_q^+)} + \|\widehat{f}\|_{C\left([0,T],W_q^2(R_q^+)\right)} < \infty. \qquad (11)
\end{aligned}$$

From this, $\|u\|_{C\left([0,T],W_q^2(R_q^+)\right)} < \infty$.

Finally, using the relation (9) and the Parseval's identity (4) we have

$$\begin{aligned}
\|\partial_t u(t,\cdot)\|_{W_q^2(\mathbb{R}_q^+)} &\le \int_0^\infty (1+\lambda^2)\left|\mathcal{F}_{q,\alpha}\left(\varphi\right)(\lambda)\right|^2 d_q\lambda + \int_0^\infty \left|\mathcal{F}_{q,\alpha}\left(f(t,\cdot)\right)(\lambda)\right|^2 d_q\lambda \\
&\le \|\varphi\|_{W_q^2(\mathbb{R}_q^+)} + \|\widehat{f}(t,\cdot)\|_{L_q^2(\mathbb{R}_q^+)} \\
&\le \|\varphi\|_{W_q^2(\mathbb{R}_q^+)} + \|\widehat{f}\|_{C\left([0,T],L_q^2(R_q^+)\right)} < \infty. \qquad (12)
\end{aligned}$$

From (10), (11) and (12) we conclude that the solluation $u \in C\left([0,T],W_q^2(R_q^+)\right) \cup C\left([0,T],L_q^2(R_q^+)\right)$ is unique. Assume that there are two different solutions $u(t,x)$ and $v(t,x)$ of Problem (5) and (6) such that

$$\begin{cases} \partial_t u(t,x) - i\Delta_{q,\alpha,x}u(t,x) = f(t,x), & (t,x)\in[0,T]\times\mathbb{R}_q^+, \\ u(0,x) = \varphi(x), & x\in\mathbb{R}_q^+, \end{cases}$$

and

$$\begin{cases} \partial_t \nu(t,x) - i\Delta_{q,\alpha,x}\nu(t,x) = f(t,x), & (t,x)\in[0,T]\times\mathbb{R}_q^+, \\ v(0,x) = \varphi(x), & x\in\mathbb{R}_q^+. \end{cases}$$

Denote $W(t,x)\equiv u(t,x)-v(t,x)$. Then the function $W(t,x)$ is the solution of the following problem.

$$\begin{cases} \partial_t W(t,x) - i\Delta_{q,\alpha,x}W(t,x) = 0, & (t,x)\in[0,T]\times\mathbb{R}_q^+, \\ W(0,x) = 0, & x\in\mathbb{R}_q^+. \end{cases}$$

From (10) it follows that $W(t,x)\equiv 0$. Hence, $u(t,x)\equiv v(t,x)$. This contradiction shows that our assumption is wrong so the solution is unique. The proof is complete.

Acknowledgments

The authors are grateful to the referee for general advices, which improved the final version of the paper. The study was supported by Ministry of Education and Science of the Republic of Kazakhstan, Grant AP08052208.

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Кванттық есептеуде q-Бессель операторымен алынған Шредингер теңдеулері

Мақалада q-Бессель операторымен байланысты Шредингер теңдеуінің жаңа модификациясының нақты шешімдері алынған. Бұл шешімнің Соболев типтес $W_q^2(\mathbb{R}_q^+)$ кеңістігіндегі q-есептеуде бар екендігі туралы теорема дәлелденген. Соболев типтес тиісті кеңістіктердегі дұрыстығы туралы нәтижелер алынды. Қарапайымдылық үшін $a > 0$ нақты ретті бөлшек q-айырымдық теңдеулерін және q-есептеудегі берілген нақты сандарды қамтитын нәтижелер берілген. Бөлшек q-айырымдық теңдеулерін сандық өңдеу де зерттелді. Алынған нәтижелер жаңа және әдебиеттердегі белгілі нәтижелерді жақсартады және толықтырады.

Кілт сөздер: q-интеграл, q-Джексон интегралы, q-айырмашылық операторы, q-туынды, q-Фурье Бессель түрлендіруі, Соболев типтес кеңістік, Шредингер теңдеуі, q-Бессель операторы.

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Уравнения Шредингера, порожденные q-оператором Бесселя в квантовом исчислении

В статье даны получены точные решения новой модификации уравнения Шредингера, связанные с q-оператором Бесселя. Доказана теорема о существовании этого решения в пространстве соболевского типа $W_q^2(\mathbb{R}_q^+)$ в q-исчислении. Получены результаты о корректности в соответствующих пространствах соболевского типа. Для простоты авторами приведены результаты, связанные с дробными q-разностными уравнениями действительного порядка $a > 0$ и заданными вещественными числами в q-исчислении. Исследована численная обработка дробных q-разностных уравнений. Полученные результаты являются новыми и дополняют известные ранее в литературе.

Ключевые слова: q-интеграл, q-интеграл Джексона, q-разностный оператор, q-производная, q-преобразование Бесселя–Фурье, пространство соболевского типа, уравнение Шредингера, q-оператор Бесселя.