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FEYNMAN INTEGRALS, FRACTAL PARADIGM AND NEW POINT OF VIEW ON HYDROACOUSTICS

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In approximation of paraxial acoustic beam pressure of sound wave has been interpreted as amplitude of probability of quasiparticle of hydroacoustic systems. We have called this quasiparticle by 'hidron'. The basis for this interpretation is the representation of Green's function of equation for pressure by path integral. The same arguments have led us to the conclusion about 'natural' fractality of all hydroacoustic systems working with spectrum of signal. It is also proved that field of quantum mechanical averages of momenta of ensemble of hidrons obeys to vector Riemann equation.

Keywords: path integral, fractal tree, quasiparticle, hydroacoustic system, Schrödinger equation, phonon, Green's function, Heisenberg representation, Burgers equation, Gosper curve.

Introduction

In papers [1-4] it is shown that in the whole range of problems of radiowaves propagation, radar-location and radio engineering in equations describing these phenomena there is hidden feynmanon. We call by 'feynmanon' the quantum object moving along fractal trajectories in phase space of dynamical system related with equations of radio system considered by means of Feynman integrals. Methodology of introduction of feynmanons is quite general. In this article this methodology is extended on problems of hydroacoustics.

Main equations and assumptions

In order to consider propagation of sound waves in incompressible fluid let us linearize hydrodynamic equations [5]:

$$\frac{\partial p}{\partial t} + \rho \cdot c^2 \cdot div\vec{v} = 0 \quad \rho \cdot \frac{\partial \vec{v}}{\partial t} + \nabla p = 0\,, \tag{1}$$

where p is pressure of fluid, $\vec{v}$ is velocity in sound wave, ρ is density of fluid and c is velocity of sound in it.

For plane waves $\exp(-i \cdot \omega \cdot t + i \cdot \vec{k} \cdot \vec{r})$ equations (1) can be reduced to the following system of linear algebraic equations:

$$-\omega \cdot p + \rho \cdot c^2 \cdot (\vec{k} \cdot \vec{v}) = 0 \quad -\omega \cdot \rho \cdot \vec{v} + \vec{k} \cdot p = 0\,. \tag{2}$$

Its conditions of consistency give us familiar [5] law of dispersion $\omega^2 = c^2 \cdot \vec{k}^2$.

Choose wave vector in form $\vec{k} = (\vec{k}_\perp, k_0 + \kappa_z)$, where $k_0 = \omega/c$, then taking into account in system (2) only linear on small addition on longitudinal wave vector κ_z terms we obtain:

$$\vec{v}_\perp = \frac{\vec{k}_\perp \cdot p}{\rho \cdot c \cdot k_0} \quad v_z = \frac{(k_0 + \kappa_z) \cdot p}{\rho \cdot c \cdot k_0} \quad (\vec{k}_\perp^2 + 2 \cdot k_0 \cdot \kappa_z) \cdot p = 0\,. \tag{3}$$

Replacing wave vectors on differential operators according to $\vec{k}_\perp \to -i \cdot \nabla_\perp$, $\kappa_z \to -i \cdot \partial/\partial z$ in formulae (3), we find that components of velocity in sound wave are expressed through pressure as:

$$\vec{v}_{\perp} = \frac{-i}{\rho \cdot c \cdot k_0} \cdot \nabla_{\perp} p \quad v_z = \frac{p}{\rho \cdot c} - \frac{i}{\rho \cdot c \cdot k_0} \cdot \frac{\partial p}{\partial z}, \tag{4}$$

and pressure itself obeys to the next equation of parabolic type:

$$i \cdot \frac{\partial p}{\partial z} = -\frac{1}{2 \cdot k_0} \cdot \Delta_{\perp} p. \tag{5}$$

Expression of the solution of equation (5) through the distribution of pressure on plane $z = 0$ is well-known [6]:

$$p(\vec{x}, z) = \int G(\vec{x}, \vec{\xi}; z) \cdot p(\vec{\xi}, 0) \cdot d^2 \xi, \tag{6}$$

where

$$G(\vec{x}, \vec{\xi}; z) = \frac{k_0}{2 \cdot \pi \cdot i \cdot z} \cdot \exp\left[\frac{i \cdot k_0 \cdot (\vec{x} - \vec{\xi})^2}{2 \cdot z}\right] \tag{7}$$

is Green's function (GF) of equation (5).

On the other side equation (5) is two-dimensional nonstationary Schrödinger type equation [7], therefore its GF can be represented by the following Feynman integral [8]:

$$G(\vec{x}, \vec{\xi}; z) = \int_{\vec{Q}(0)=k_0 \cdot \vec{\xi}}^{\vec{Q}(k_0 \cdot z)=k_0 \cdot \vec{x}} \exp\left[i \cdot \int_0^{k_0 \cdot z} \sum_{j=1}^{2} \left(P_j(\tau) \cdot \dot{Q}_j(\tau) - \frac{P_j^2(\tau)}{2}\right) \cdot d\tau\right] \cdot d\mu, \tag{8}$$

where

$$d\mu = \prod_{j=1}^{2} \prod_{\tau} \frac{dP_j(\tau) \cdot dQ_j(\tau)}{2 \cdot \pi}. \tag{9}$$

is Feynman's pseudomeasure [9] in four-dimensional phase space $(\vec{P}, \vec{Q})$.

This representation means that for considered hydroacoustic system one can introduce new quasiparticle. We name this quasiparticle 'hidron'. Properties of hidron are completely similar to properties of free massive quantum mechanical particle in two-dimensional space.

Drawing further analogy with quantum mechanics we see that pressure of fluid (6) one may interpret as hidron's amplitude of probability evolving in dimensionless time $\tau = k_0 \cdot z$, which corresponds to optical length of path in optics.

Distribution of pressure on plane $z = 0$ is initial distribution of amplitude of probability of hidron. It is formed by two-dimensional Fourier integral:

$$p(\vec{x}, 0) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \tilde{p}(\vec{k}_{\perp}) \cdot \exp(i \cdot \vec{k}_{\perp} \cdot \vec{x}) \cdot \frac{d^2 k_{\perp}}{(2 \cdot \pi)^2}, \tag{10}$$

in which also there is Feynman integral of the next kind [2]:

$$\frac{\exp(i \cdot \vec{k}_{\perp} \cdot \vec{x})}{-2 \cdot \pi \cdot i} = \int_{Q(-\pi/2)=\vec{k}_{\perp}/k_0}^{Q(0)=k_0 \cdot \vec{x}} \exp\left[-i \cdot \int_{-\pi/2}^{0} \sum_{j=1}^{2} \left(P_j(\tau) \cdot \dot{Q}_j(\tau) - \frac{P_j^2(\tau) + Q_j^2(\tau)}{2}\right) \cdot d\tau\right] \cdot d\mu. \tag{11}$$

It is easy to see that Hamiltonian in (11) corresponds to two-dimensional symmetric harmonic oscillator.

Comparing path integrals (8) and (11) we come to the conclusion that under $\tau \in [-\pi/2, 0)$ hidron is preparing for its radiation by hydroacoustic system in which it is localized by quadratic potential. At the moment of time $\tau = 0$ this quadratic potential disappears and hidron continues to move in dimensionless time on nondifferentiable trajectory in phase space $(\vec{P}, \vec{Q})$. This movement corresponds to its radiation by hydroacoustic system into fluid. In addition function $\tilde{p}(\vec{k}_\perp)$ from Fourier integral (10) is hidron's amplitude of probability on momenta at initial moment of time.

Fractal structure of trajectories in Feynman integral

An important feature of motion of hidron in four-dimensional phase space $(\vec{P}, \vec{Q})$ is fractality of its phase trajectories. In order to exhibit this fractality let us analyse the procedure of calculation of Feynman integral.

Succeeding [8] to calculate this path integral one ought to approximate functional of action in exponential in (8) by multiple integral namely it is necessary to divide initial time segment $[0, t]$ on N equal segments of width $\Delta\tau = t/N$ and approximate on them coordinates $\vec{Q}(\tau)$ by piecewise linear functions and momenta $\vec{P}(\tau)$ by piecewise constant functions:

$$\vec{Q}(\tau) = \vec{Q}_j + (\vec{Q}_{j+1} - \vec{Q}_j) \cdot (\tau - \tau_j)/\Delta\tau, \quad \vec{P}(\tau) = \vec{P}_j, \quad \tau \in [\tau_j, \tau_{j+1}], \tag{12}$$

where $\vec{P}_j = \vec{P}(\tau_j)$ and $\vec{Q}_j = \vec{Q}(\tau_j)$ are values of canonical variables at points $\tau_j = j \cdot \Delta\tau$, $j = \overline{0, N}$. One obtains GF of equation (6) by means of integration on all coordinates $\vec{Q}_j$ and momenta $\vec{P}_j$ and passage to the limit under $N \to \infty$.

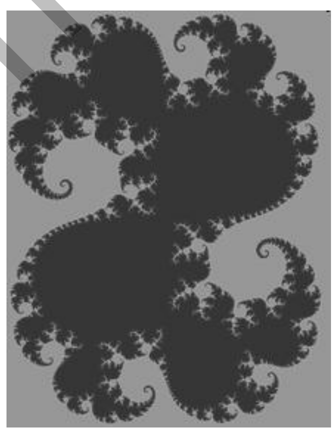

Fig. 1. Fractal dynamics on plane of momenta: Julia set

From formulae (12) one can see that under path integration momenta of hidron range over some set from N points on two-dimensional plane (P_1, P_2). This fact one may interpret as pointwise mapping of this plane into itself. In particular if one introduce complex variable $P = P_1 + i \cdot P_2$ then such mapping can be given by holomorphic function $g(P)$:

$$P_{j+1} = g(P_j), \tag{13}$$

moreover among such functions may be found functions generating fractal sets for instance Julia function [10-13]: $g(P) = P^2 + c$, where c is complex parameter (fig.1).

But Cauchy-Riemann conditions for real and imaginary parts of function $g(P)$ are too strong restrictions. If one requires that mapping $g: R^2 \to R^2$ be only diffeomorphism nevertheless there is set of mappings which demonstrate chaotic behaviour on plane of momenta. One of them is Henon mapping [10] (fig. 2):

$$P_{1,j+1} = 1 - 1{,}4 \cdot P_{1,j}^2 + P_{2,j} \quad P_{2,j+1} = 0{,}3 \cdot P_{1,j} \,. \qquad (14)$$

Thus connection of Feynman integrals with dynamical chaos has been found also.

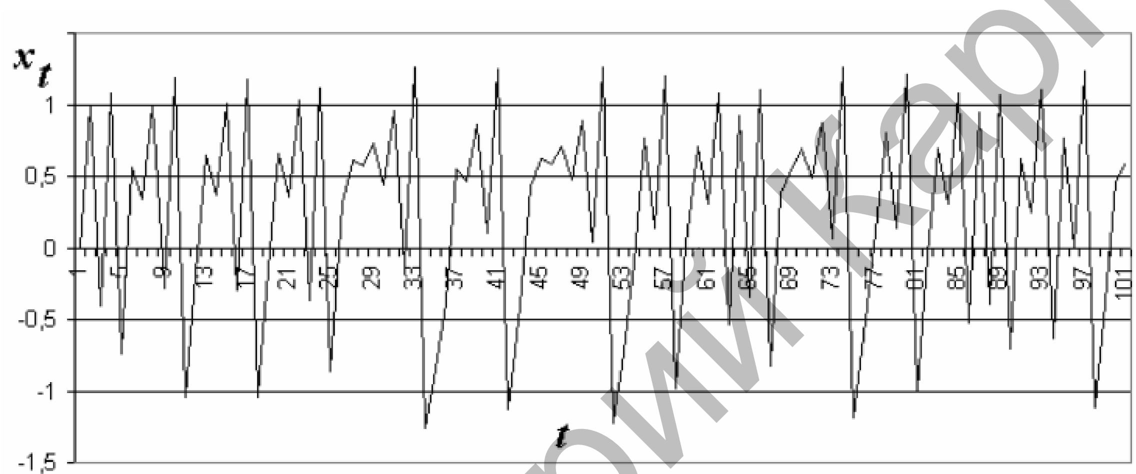

Fig. 2. Time dependence of one of variables in Henon mapping

Now let us consider projections of hidron's trajectories from phase space $(\vec{P}, \vec{Q})$ on two-dimensional plane of coordinates (Q_1, Q_2). From formulae (12) it follows that on each time segment $[\tau_j, \tau_{j+1}]$ trajectory of hidron is segment of straight line with arbitrary angle of inclination to coordinate axes. In particular coordinates of hidron can pass this segment at first at one direction and then can pass it in opposite direction that is coordinates of hidron can return along it into starting-point. Therefore on the whole time segment $[0, t]$ coordinates of hidron can pass such selfsimilar objects as Gosper curve [10-13] or fractal tree [14] (fig. 3).

Consideration of complex variable $Q = Q_1 + i \cdot Q_2$ is also very fruitful. For instance in the framework of approach developed one from trajectories of coordinates of hidron may be part of Koch's curve [10-13] (fig. 4) with fractal dimension $D = \log_3 4 \approx 1{,}26$. On the other side explicit expression for Koch's curve is solution of the following functional equation:

$$Q(\tau) = \begin{cases} \alpha \cdot Q^*(2 \cdot \tau), & 0 \le \tau \le 1/2 \\ (1-\alpha) \cdot Q^*(2 \cdot \tau - 1) + \alpha, & 1/2 \le \tau \le 1 \end{cases} \qquad (15)$$

under $\alpha = 1/2 + i \cdot \sqrt{3}/6$ [10].

If one introduce in phase space $(\vec{P}, \vec{Q})$ four-dimensional lattice with step δ decreasing with growth of N then equations (12) mean that random walk on this lattice is subset of set of phase trajectories of hidron.

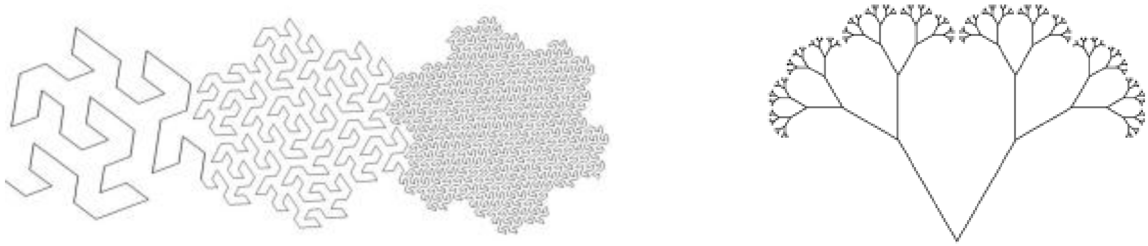

Fig. 3. Fractal dynamics on plane of coordinates:
three iterations of Gosper curve (from the left) and fractal tree (from the right)

Thus construction of Feynman's pseudomeasure introduced in article [9] possesses very nontrivial internal structure described above.

From formula (11) it follows that Fourier transform is fractal operation. And because hydroacoustic systems work with spectra of signals then one immediately may conclude that all existing hydroacoustic systems are fractal.

Mutual relations of hidrons and phonons

It is well-known that linear hydrodynamic equations (1) can be quantized by means the so-called decomposition of field on harmonic oscillators in normalized volume V [15]. This procedure leads to introduction into consideration of quanta of sound in fluid namely phonons describing by the following operator of potential of velocity of fluid [15]:

$$\hat{\varphi}(\vec{r},t) = \sum_{\vec{k}} \sqrt{\frac{c}{2 \cdot V \cdot \rho \cdot k}} \cdot [\hat{a}_{\vec{k}} \cdot \exp(i \cdot \vec{k} \cdot \vec{r} - i \cdot k \cdot c \cdot t) + \hat{a}^{+}_{\vec{k}} \cdot \exp(-i \cdot \vec{k} \cdot \vec{r} + i \cdot k \cdot c \cdot t)], \qquad (16)$$

where operators of creation $\hat{a}^{+}_{\vec{k}}$ and annihilation $\hat{a}_{\vec{k}}$ of phonons obey to Bose canonical commutative relations [15]:

$$[\hat{a}_{\vec{k}}, \hat{a}^{+}_{\vec{k}'}] = \delta_{\vec{k}\vec{k}'} \quad [\hat{a}_{\vec{k}}, \hat{a}_{\vec{k}'}] = 0 \qquad (17)$$

(we use system of units of measurement where Plank constant $\hbar = 1$).

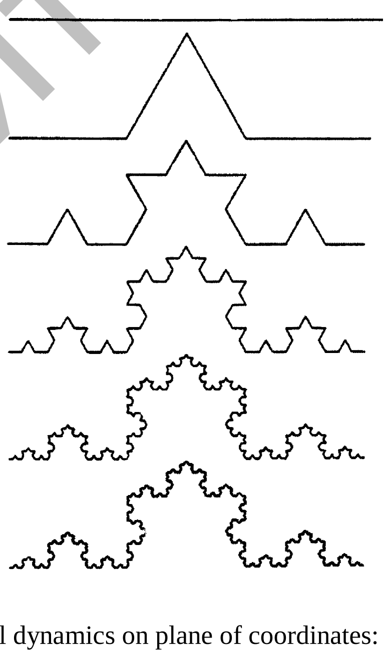

Fig. 4. Fractal dynamics on plane of coordinates: Koch's curve

Quantized field (18) yields the following Hamiltonian of phonons [15]:

$$\hat{H} = \sum_{\vec{k}} c \cdot k \cdot (\hat{a}^{+}_{\vec{k}} \cdot \hat{a}_{\vec{k}} + 1/2) \tag{18}$$

corresponding to a number of quantum mechanical harmonic oscillators.

Operators of creation and annihilation act on eigenstates $|n_{\vec{k}}>$ of Hamiltonian (18) as follows [15]:

$$\hat{a}^{+}_{\vec{k}} \,|\, n_{\vec{k}} >= \sqrt{n_{\vec{k}} + 1} \cdot |\, n_{\vec{k}} + 1 > \quad \hat{a}_{\vec{k}} \,|\, n_{\vec{k}} >= \sqrt{n_{\vec{k}}} \cdot |\, n_{\vec{k}} - 1 > . \tag{19}$$

Further applying familiar formula for generating function of Chebyshev-Hermite polynomials [16] to GF (7) we find the next expansion for it:

$$G(\vec{x}, \vec{\xi}; z) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} G^{z}_{n_1 n_2}(\vec{\xi}) \cdot \frac{\zeta_1^{n_1} \cdot \zeta_2^{n_2}}{\sqrt{n_1! \cdot n_2!}}, \tag{20}$$

where

$$\zeta_{1,2} = x_{1,2} \cdot \sqrt{\frac{2 \cdot k_0}{i \cdot z}} \tag{21}$$

are complex-valued variables and

$$G^{z}_{n_1 n_2}(\vec{\xi}) = \frac{k_0}{2 \cdot \pi \cdot i \cdot z} \cdot \frac{\exp\left[\frac{i \cdot k_0}{2 \cdot z} \cdot (\xi_1^2 + \xi_2^2)\right]}{2^{n_1+n_2} \cdot \sqrt{n_1! \cdot n_2!}} \cdot H_{n_1}\left(\xi_1 \cdot \sqrt{\frac{k_0}{2 \cdot i \cdot z}}\right) \cdot H_{n_2}\left(\xi_2 \cdot \sqrt{\frac{k_0}{2 \cdot i \cdot z}}\right). \tag{22}$$

Substituting series (20) into expression (6) for pressure (or amplitude of probability of hidron) we obtain:

$$p(\vec{x}, z) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} p^{z}_{n_1 n_2} \cdot \frac{\zeta_1^{n_1} \cdot \zeta_2^{n_2}}{\sqrt{n_1! \cdot n_2!}}, \tag{23}$$

where

$$p^{z}_{n_1 n_2} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} G^{z}_{n_1 n_2}(\vec{\xi}) \cdot p(\vec{\xi}, 0) \cdot d^2 \xi . \tag{24}$$

Let us now take into consideration set D_2 of complex function of two variables $\zeta_{1,2} \in C$:

$$f(\zeta_1, \zeta_2) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} C_{n_1 n_2} \cdot \frac{\zeta_1^{n_1} \cdot \zeta_2^{n_2}}{\sqrt{n_1! \cdot n_2!}} . \tag{25}$$

These functions form Hilbert space with respect to the following scalar product [17]:

$$< f \,|\, g >= \int f(\zeta_1, \zeta_2) \cdot g^{*}(\zeta_1, \zeta_2) \cdot \exp(-|\zeta_1|^2 - |\zeta_2|^2) \cdot d\mu(\zeta_1) \cdot d\mu(\zeta_2), \tag{26}$$

where

$$d\mu(\zeta) = \frac{d\zeta^{*} \cdot d\zeta}{2 \cdot \pi \cdot i}, \tag{27}$$

and moreover functions

$$|\, n_1, n_2 > \equiv \frac{\zeta_1^{n_1} \cdot \zeta_2^{n_2}}{\sqrt{n_1! \cdot n_2!}} \tag{28}$$

are orthonormal basis in this Hilbert space [17].

Then let us consider the next operators in D_2:

$$\hat{a}_j = \frac{\partial}{\partial \zeta_j}, \quad \hat{a}_j^+ = \zeta_j, \quad j = 1,2. \tag{29}$$

It is easy to check that operators (29) act on basis (28) as follows:

$$\begin{aligned} &\hat{a}_1 \,| n_1, n_2 >= \sqrt{n_1} \cdot | n_1 - 1, n_2 >, \quad \hat{a}_1^+ \,| n_1, n_2 >= \sqrt{n_1 + 1} \cdot | n_1 + 1, n_2 >, \\ &\hat{a}_2 \,| n_1, n_2 >= \sqrt{n_2} \cdot | n_1, n_2 - 1 >, \quad \hat{a}_2^+ \,| n_1, n_2 >= \sqrt{n_2 + 1} \cdot | n_1, n_2 + 1 > . \end{aligned} \tag{30}$$

Using formulae (30) one can see that operators (29) obey to Bose canonical commutative relations:

$$[\hat{a}_j, \hat{a}_{j'}^+] = \delta_{jj'}, \quad [\hat{a}_j, \hat{a}_{j'}] = 0, \tag{31}$$

moreover states (28) are eigenstates of operator

$$\hat{h} = \hat{a}_1^+ \cdot \hat{a}_1 + \hat{a}_2^+ \cdot \hat{a}_2 + 1 \tag{32}$$

with eigenvalues $n_1 + n_2 + 1$:

$$\hat{h}\,| n_1, n_2 >= (n_1 + n_2 + 1) \cdot | n_1, n_2 > . \tag{33}$$

On the other side operator (32) is Hamiltonian of two-dimensional quantum mechanical symmetric harmonic oscillator written in representation of creation and annihilation operators. Formulae (30)-(33) prove us that basis (28) is isomorphic to eigenfunctions of this oscillator [7]:

$$\psi_{n_1 n_2}(\vec{x}) = \frac{\exp[-(x_1^2 + x_2^2)/2]}{\sqrt{\pi \cdot 2^{n_1+n_2} \cdot n_1! n_2!}} \cdot H_{n_1}(x_1) \cdot H_{n_2}(x_2), \tag{34}$$

and that functional space D_2 (25) is the same as space of states of two-dimensional symmetric harmonic oscillator:

$$\psi(\vec{x}) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} C_{n_1 n_2} \cdot \psi_{n_1 n_2}(\vec{x}). \tag{35}$$

Now let us remind that pressure (23) is subset of space D_2 and therefore amplitude of probability of hidron (23) is subset of space of states of two-dimensional harmonic oscillator (35). Thus we have found correspondence between hidrons and phonons as quantum mechanical harmonic oscillators (18).

But mutual relations of hidrons and phonons prove to be deeper than we have already clarified. Indeed let us consider the next Hermitian operators constructed from operators (29):

$$\hat{M}_1 = -\frac{\hat{a}_1^+ \cdot \hat{a}_2 + \hat{a}_2^+ \cdot \hat{a}_1}{2}, \quad \hat{M}_2 = -i \cdot \frac{\hat{a}_1^+ \cdot \hat{a}_2 - \hat{a}_2^+ \cdot \hat{a}_1}{2}, \quad \hat{M}_3 = \frac{\hat{a}_2^+ \cdot \hat{a}_2 - \hat{a}_1^+ \cdot \hat{a}_1}{2}. \tag{36}$$

Applying for them formulae (30) one can verify that on space D_2 operators (36) obey to commutative relations for Lie algebra of group $SU(2)$ [17]:

$$[\hat{M}_1, \hat{M}_2] = i \cdot \hat{M}_3, \quad [\hat{M}_2, \hat{M}_3] = i \cdot \hat{M}_1, \quad [\hat{M}_3, \hat{M}_1] = i \cdot \hat{M}_2 . \tag{37}$$

Then functions (28) are eigenfunctions of operators $\hat{M}_3$ and $\hat{M}_1^2 + \hat{M}_2^2 + \hat{M}_3^2$ [17]. It means that functions (28) are basises of finite-dimensional representations of group $SU(2)$ [17]. In particular this fact shows hidden $SU(2)-$symmetry of Hamiltonian (32) of two-dimensional quantum mechanical harmonic oscillator in explicit form. Note that the same system occurs in Feynman integral (11).

Thus we have find that amplitude of probability of hidron is formed from subset of basises of finite-dimensional representations of group $SU(2)$. On the other side scalar operator field of phonons (16) ought to be invariant under rotations from group $SO(3)$. At last it is well-known that $SO(3) \approx SU(2)/(\pm 1)$ [7, 16]. This isomorphism gives implicit connection between hidrons and phonons.

Heisenberg representation and Riemann equation

In quantum mechanics time dependence of amplitude of probability one can transfer on operators by means of transition from Schrödinger representation to Heisenberg representation of them [7]. That is why let us introduce instead of Schrödinger operators of coordinate $\hat{\vec{Q}} = \vec{x}$ and momentum $\hat{\vec{P}} = -i \cdot \nabla_{\perp}$ Heisenberg operators:

$$\hat{\vec{Q}}(z) = \hat{U}^{+}(z) \cdot \hat{\vec{Q}} \cdot \hat{U}(z) \quad \hat{\vec{P}}(z) = \hat{U}^{+}(z) \cdot \hat{\vec{P}} \cdot \hat{U}(z), \tag{38}$$

where $\hat{U}(z) = \exp\left[\frac{i \cdot z}{2 \cdot k_0} \cdot \Delta_{\perp}\right]$ is unitary operator of evolution of hidron's state. It's kernel in coordinate representation is GF (7).

Operators (38) obey to the following equations:

$$\frac{d\hat{\vec{Q}}(z)}{dz} = \frac{\hat{\vec{P}}(z)}{k_0} \quad \frac{d\hat{\vec{P}}(z)}{dz} = 0. \tag{39}$$

Respectively equations for mean values of operators (38) on initial state of hidron $p(\vec{x},0)$ are:

$$\frac{d < \hat{\vec{Q}}(z) >}{dz} = \frac{< \hat{\vec{P}}(z) >}{k_0} \quad \frac{d < \hat{\vec{P}}(z) >}{dz} = 0. \tag{40}$$

It is easy to see that system of ordinary differential equations (40) is system of characteristics of the next nonlinear partial differential equation for two-dimensional vector field $\vec{P}(\vec{x}, z)$:

$$k_0 \cdot \frac{\partial \vec{P}}{\partial z} + (\vec{P} \cdot \nabla_{\perp})\vec{P} = 0. \tag{41}$$

To understand physical sense of equation (41) let us consider ensemble of hidrons with initial states in which mean momentum of hidron depends on mean coordinate of hidron that is hidrons with the following initial states:

$$p(\vec{x},0) = \exp[i \cdot \vec{p}_0(\vec{x}_0) \cdot \vec{x}] \cdot f(\vec{x} - \vec{x}_0), \tag{42}$$

where $f(\vec{x}) \in L^2(R^2)$ is real function. Then equation (41) is equation of evolution of field of quantum mechanical averages of momenta of ensemble of hidrons of kind (42) in empty space and also initial condition for this equation is:

$$\vec{P}(\vec{x},0) = \vec{p}_0(\vec{x}). \tag{43}$$

It should be noted that under linearization (1) of hydrodynamic equations we have refused to consider hydrodynamic nonlinearity but in equation (41) it has arisen again.

The simplest classes of solutions of equation (41) one can obtain by choosing the next initial condition:

$$\vec{p}_0(\vec{x}) = (p_{01}(x_1), p_{02}(x_2)). \tag{44}$$

In this case vector equation (system of equations) (41) splits on two independent Riemann equations:

$$k_0 \cdot \frac{\partial P_1}{\partial z} + P_1 \cdot \frac{\partial P_1}{\partial x_1} = 0 \quad k_0 \cdot \frac{\partial P_2}{\partial z} + P_2 \cdot \frac{\partial P_2}{\partial x_2} = 0\,, \tag{45}$$

which give the following solution of input equation:

$$\vec{P}(\vec{x}, z) = (P_1(x_1, z), P_2(x_2, z))\,. \tag{46}$$

For instance if one takes ($a_{1,2}$ and $l_{1,2}$ are constants):

$$\vec{p}_0(\vec{x}) = \left(\frac{a_1}{\sqrt{1 + (x_1/l_1)^2}}, \quad \frac{a_2}{\sqrt{1 + (x_2/l_2)^2}} \right), \tag{47}$$

then one can find the next exact solutions of equations (21) in implicit form:

$$P_{1,2}(x_{1,2}; z) = k_0 \cdot a_{1,2} \cdot \left\{ 1 + \left[\frac{1}{l_{1,2}} \cdot \left(x_{1,2} - \frac{z}{k_0} \cdot P_{1,2}(x_{1,2}; z) \right) \right]^2 \right\}^{-1/2}. \tag{48}$$

Using Cardano formula it is easy to rewrite solutions (48) in explicit form but it would be too inconveniently here. It is well-known that in solutions (48) gradient catastrophe happens namely after start of propagation of nonlinear waves (48) their fronts are tipped over and discontinuities in these solutions appear [6, 18, 19]. Behaviour of solution of vector equation (41) demonstrates us the same picture [18, 19].

Effective tool of overcoming of gradient catastrophe is regularization of equation (41) by vanishing viscosity μ that is reduction it to Burgers equation [18, 19]:

$$k_0 \cdot \frac{\partial \vec{P}}{\partial z} + (\vec{P} \cdot \nabla_\perp)\vec{P} = \mu \cdot \Delta_\perp \vec{P}\,. \tag{49}$$

In works [18, 19] it is shown that in this case fusion of discontinuities of solution of equation (49) obeys to law of inelastic collisions of classical particles. Moreover under special choice of initial condition (44) trajectories of discontinuities on plane (x_1, x_2) form structure of umbrella tree [19]. And this structure coincides with the whole class of fractal trajectories of hidron on plane of its coordinates (Q_1, Q_2) (see fig. 3).

Conclusion

1. Discussing mutual relations of phonons and hidrons in Cartesian coordinates we have found that in the problem investigated there is a number of hidden symmetries. Therefore further perspective of our investigation is analysis of properties of hidron in polar coordinates both in the framework of evolutionary equation (5) and in the framework of Feynman integral with Hamiltonian of two-dimensional free particle:

$$H = \frac{1}{2} \cdot \left(p_r^2 + \frac{p_\varphi^2}{r^2} \right)\,. \tag{50}$$

This way is sure to be fruitful because if we consider familiar [7] expansion of two-dimensional plane wave in polar coordinates:

$$\frac{\exp(i \cdot \vec{p} \cdot \vec{x})}{2 \cdot \pi \cdot i} = \sum_{m=-\infty}^{\infty} \frac{i^m \cdot J_m(p \cdot r)}{2 \cdot \pi \cdot i} \cdot \exp[i \cdot m \cdot (\phi - \phi')]\,, \tag{51}$$

where $\vec{p} = (p \cdot \cos\phi', p \cdot \sin\phi')$ and $\vec{x} = (r \cdot \cos\phi, r \cdot \sin\phi)$, then we observe connection of continuous and discrete in explicit form. Indeed plane wave from the left side of equality (51) is expressed by Feynman integral (11) with discrete fractal dynamics in phase space (see fig. 1, 3, 4). But Bessel functions from the right side of equality (51) are matrix elements of irreducible

representations of total group of motion of two-dimensional Euclidean space [16]. And it is necessary to note here that this group is Lie group [16].
2. It is not complicated to transfer methodology developed on the case of nonhomogeneous fluid.

Let us consider density of fluid and velocity of sound in it as functions of spatial coordinates. Then assuming that dependence of pressure and velocity of fluid on time is equal to $\exp(-i \cdot \omega \cdot t)$ one can reduce equations (1) by means of the following substitution [20]:

$$p(\vec{x}, z) = \sqrt{\rho(\vec{x}, z)} \cdot \Psi(\vec{x}, z) \tag{52}$$

to Helmholtz equation [20]

$$\Delta\Psi + k^2(\vec{x}, z) \cdot \Psi = 0, \tag{53}$$

where

$$k^2(\vec{x}, z) = \frac{\omega^2}{c^2(\vec{x}, z)} + \frac{\Delta\rho(\vec{x}, z)}{2 \cdot \rho(\vec{x}, z)} - \frac{3}{4} \cdot [\nabla \ln \rho(\vec{x}, z)]^2. \tag{54}$$

Further let us denote $k_0(z) = k(\vec{0}, z)$ and introduce new function $\psi(\vec{x}, z)$ as follows [21]:

$$\Psi(\vec{x}, z) = \frac{1}{\sqrt{k_0(z)}} \cdot \psi(\vec{x}, z) \cdot \exp\left[i \cdot \int_0^z k_0(\xi) \cdot d\xi \right]. \tag{55}$$

Then under some assumptions [21] we obtain the so-called Leontovich-Fock parabolic equation instead of evolutionary equation (5):

$$i \cdot \frac{\partial \psi}{\partial z} = -\frac{1}{2 \cdot k_0(z)} \cdot \Delta_\perp \psi + \frac{k_0^2(z) - k^2(\vec{x}, z)}{k_0^2(z)} \cdot \psi. \tag{56}$$

Thus it has become clear that hidrons exist in nonhomogeneous fluid too. Fractality of hidron's trajectories in phase space $(\vec{P}, \vec{Q})$ also takes place. Only instead of Hamiltonian of two-dimensional free particle in Feynman integral one ought to use the following Hamiltonian:

$$H(\vec{P}, \vec{Q}; z) = \frac{\vec{P}^2}{2 \cdot k_0(z)} + \frac{k_0^2(z) - k^2(\vec{Q}, z)}{k_0^2(z)}. \tag{57}$$

At last in order to clarify mutual relations of hidrons and phonons in this system for quantization of potential of velocity of nonhomogeneous fluid one ought to apply some analog of representation of Furry in quantum electrodynamics [22].
3. In this article it is shown that all existing now hydroacoustic systems are fractal. Thus one of the authors of this article namely A. A. Potapov proves to be in the right in his foreseeing [23] of steady growth of the role of fractal paradigm both in radio physics and in modern natural science as a whole.
4. Results of this work have been reported on 9th International Scientific Conference 'Chaos and Structures in Nonlinear Systems. Theory and Experiment' devoted to 90th anniversary of academician E. A. Buketov (Karaganda, Kazakhstan, June 18 – 20, 2015) [24].

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