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On one approximate solution of a nonlocal boundary value problem for the Benjamin-Bona-Mahony equation

The paper investigates a non-local boundary value problem for the Benjamin-Bona-Mahony equation. This equation is a nonlinear pseudoparabolic equation of the third order with a mixed derivative. To find a solution to this problem, an algorithm for finding an approximate solution is proposed. Sufficient conditions for the feasibility and convergence of the proposed algorithm are established, as well as the existence of an isolated solution of a non-local boundary value problem for a nonlinear equation. Estimates are obtained between the exact and approximate solution of this problem.

Keywords: partial differential equation, Benjamin-Bona-Mahony equation, algorithm, approximate solution.

Introduction

The paper considers a nonlocal boundary value problem for a third-order nonlinear partial differential equation or the Benjamin-Bona-Mahony equation. The Benjamin-Bona-Mahony equation or the regularized long-wavelength equation was studied in [1–4]. Modern studies on this topic can be found in [5–13]. In this article, introducing a new function, a non-local boundary value problem for a third-order nonlinear differential equation is reduced to a non-local boundary value problem for a hyperbolic equation. The resulting problem with different conditions was investigated in [14–18]. Similarly to the linear case [19–22], sufficient conditions for the unique solvability of the problem under consideration are established and an algorithm for finding an approximate solution is proposed.

1 Statement of the initial boundary problem

On $\Omega = [0, X] \times [0, Y]$ we consider a nonlocal boundary value problem for the nonlinear equation

$$\frac{\partial^3 w}{\partial x^2 \partial y} = \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial x} + \frac{\partial w}{\partial x}, \quad (x, y) \in \Omega, \quad (1)$$

$$w(x, 0) = \varphi(x), \quad x \in [0, X], \quad (2)$$

$$\frac{\partial w(0, y)}{\partial y} = \alpha(y) \frac{\partial w(X, y)}{\partial y} + \psi(y), \quad y \in [0, Y], \quad (3)$$

$$\frac{\partial w(0, y)}{\partial x} = \theta(y), \quad y \in [0, Y], \quad (4)$$

where the functions $\psi(y), \theta(y)$ are continuously differentiable on $[0, Y]$, the function $\varphi(x)$ is continuously differentiable on $[0, X]$, $\alpha(y) \neq 1$.

Let $C(\Omega, R)$ be the set of functions $w : \Omega \rightarrow R$ continuous on Ω .

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A function $u(x, y) \in C(\Omega, R)$, with partial derivatives $\frac{\partial u(x, y)}{\partial x} \in C(\Omega, R)$, $\frac{\partial u(x, y)}{\partial y} \in C(\Omega, R)$, $\frac{\partial^2 u(x, y)}{\partial x \partial y} \in C(\Omega, R)$, $\frac{\partial^2 u(x, y)}{\partial x^2} \in C(\Omega, R)$, $\frac{\partial^3 u(x, y)}{\partial x^2 \partial y} \in C(\Omega, R)$ is called a solution to problem (1)–(4) if it satisfies equation (1), for all $(x, y) \in \Omega$, and boundary conditions (2)–(4).

To find a solution to problem (1)–(4), we introduce the functions

$$w(0, y) = \lambda(y), \quad \tilde{w}(x, y) = w(x, y) - \lambda(y),$$

the original problem can be written as

$$\begin{aligned} \frac{\partial^3 \tilde{w}(x, y)}{\partial x^2 \partial y} &= \frac{\partial \tilde{w}(x, y)}{\partial y} + \frac{\partial \lambda(y)}{\partial y} + [\tilde{w}(x, y) + \lambda(y)] \frac{\partial \tilde{w}(x, y)}{\partial x} + \frac{\partial \tilde{w}(x, y)}{\partial x}, \\ \tilde{w}(0, y) &= 0, \quad x \in [0, X], \\ \tilde{w}(x, 0) + \lambda(0) &= \varphi(x), \quad \lambda(0) = \varphi(0), \\ \frac{\partial \lambda(y)}{\partial y} &= \alpha(y) \frac{\partial \tilde{w}(X, y)}{\partial y} + \alpha(y) \frac{\partial \lambda(y)}{\partial y} + \psi(y), \quad y \in [0, Y], \\ \frac{\partial \tilde{w}(0, y)}{\partial x} &= \theta(y), \quad y \in [0, Y]. \end{aligned}$$

We introduce a new function $v(x, y) = \frac{\partial \tilde{w}(x, y)}{\partial x}$, then $\tilde{w}(x, y) = \int_0^x v(\xi, y) d\xi$ and the problem goes to the equivalent problem

$$\frac{\partial^2 v(x, y)}{\partial x \partial y} = \int_0^x \frac{\partial v(\xi, y)}{\partial y} d\xi + \frac{\partial \lambda(y)}{\partial y} + \left[\int_0^x v(\xi, y) d\xi + \lambda(y) \right] v(x, y) + v(x, y), \quad (5)$$

$$v(x, 0) = \varphi'(x), \quad x \in [0, X], \quad (6)$$

$$\frac{\partial \lambda(y)}{\partial y} = \frac{\alpha(y)}{1 - \alpha(y)} \int_0^X v(x, y) dx + \frac{\psi(y)}{1 - \alpha(y)}, \quad \lambda(0) = \varphi(0), \quad (7)$$

$$v(0, y) = \theta(y), \quad y \in [0, Y]. \quad (8)$$

Integrating both parts of equation (5) with respect to the variable x and taking into account conditions (8) we get

$$\frac{\partial v(x, y)}{\partial y} = \theta'(y) + \int_0^x \left(\int_0^\xi \frac{\partial v(\xi_1, y)}{\partial y} d\xi_1 + \frac{\partial \lambda(y)}{\partial y} + \left[\int_0^\xi v(\xi_1, y) d\xi_1 + \lambda(y) \right] v(\xi, y) + v(\xi, y) \right) d\xi. \quad (9)$$

Once again integrating over the variable y and using condition (6), we have [23]

$$\begin{aligned} v(x, y) &= \varphi'(x) + \int_0^y \left(\theta'(\eta) + \int_0^x \int_0^\xi \frac{\partial v(\xi_1, \eta)}{\partial \eta} d\xi_1 d\xi + \int_0^x \frac{\partial \lambda(\eta)}{\partial \eta} d\xi + \right. \\ &\quad \left. + \int_0^x \left(\left[\int_0^\xi v(\xi_1, \eta) d\xi_1 + \lambda(\eta) \right] v(\xi, \eta) + v(\xi, \eta) \right) d\xi \right) d\eta. \end{aligned} \quad (10)$$

In addition, from (12) it follows

$$\lambda(y) = \varphi(0) + \int_0^y \left(\frac{\alpha(\eta)}{1 - \alpha(\eta)} \int_0^X v(x, \eta) dx + \frac{\psi(\eta)}{1 - \alpha(\eta)} \right) d\eta. \tag{11}$$

2 Main result

Setting $v(x, y) = \varphi'(x)$, from (7) and (11) we define

$$\frac{\partial \lambda^{(0)}(y)}{\partial y} = \frac{\alpha(y)}{1 - \alpha(y)} \int_0^X \varphi'(x) dx + \frac{\psi(y)}{1 - \alpha(y)} = \frac{\alpha(y)}{1 - \alpha(y)} [\varphi(X) - \varphi(0)] + \frac{\psi(y)}{1 - \alpha(y)},$$

$$\begin{aligned} \lambda^{(0)}(y) &= \varphi(0) + \int_0^y \left(\frac{\alpha(\eta)}{1 - \alpha(\eta)} \int_0^X \varphi'(x) dx + \frac{\psi(\eta)}{1 - \alpha(\eta)} \right) d\eta = \\ &= \varphi(0) + \int_0^y \left(\frac{\alpha(\eta)}{1 - \alpha(\eta)} [\varphi(X) - \varphi(0)] + \frac{\psi(\eta)}{1 - \alpha(\eta)} \right) d\eta. \end{aligned}$$

Using equation (9), with $\lambda(y) = \lambda^{(0)}(y)$, we find

$$\frac{\partial v^{(0)}(x, y)}{\partial y} = \theta'(y) + \int_0^x \left(\frac{\partial \lambda^{(0)}(y)}{\partial y} + \left[\int_0^\xi \varphi'(\xi_1) d\xi_1 + \lambda(y) \right] \varphi'(\xi) + \varphi'(\xi) \right) d\xi.$$

$$v^{(0)}(x, y) = \varphi'(x) + \int_0^y \frac{\partial v^{(0)}(x, \eta)}{\partial \eta} d\eta.$$

Taking the functions $\lambda^{(0)}(y), v^{(0)}(x, y)$, numbers $\rho_1 > 0, \rho_2 > 0$ we construct the sets

$$S(\lambda^{(0)}(y), \rho_1) = \left\{ \lambda(y) \in C([0, Y], R) : \|\lambda(y) - \lambda^{(0)}(y)\| < \rho_1 \right\},$$

$$S(v^{(0)}(x, y), \rho_2) = \left\{ v(x, y) \in C(\Omega, R) : \|v(x, y) - v^{(0)}(x, y)\| < \rho_2, (x, y) \in \Omega \right\},$$

$$\begin{aligned} G^0(\rho_1, \rho_2) &= \left\{ (x, y, w, v) : (x, y) \in \Omega, \left\| w(x, y) - \int_0^x v^{(0)}(\xi, y) d\xi - \lambda^{(0)}(y) \right\| < \rho_1 + \rho_2, \right. \\ &\quad \left. \|v(x, y) - v^{(0)}(x, y)\| < \rho_2 \right\}. \end{aligned}$$

Denote by $U(L_1, L_2, x, y)$ the collection $\left(\lambda^{(0)}(y), v^{(0)}(x, y), \rho_1, \rho_2 \right)$, or which the function $f(x, y, w, v)$ in $G^0(\rho_1, \rho_2)$ has continuous partial derivatives $f'_w(x, y, w, v), f'_v(x, y, w, v)$ and

$$\|f'_w(x, y, w, v)\| \leq L_1, \quad \|f'_v(x, y, w, v)\| \leq L_2, \quad L_1, L_2 - const.$$

Based on the system $\{\lambda(y), \mu(y), v(x, y)\}$ we compose the triple $\{\lambda^{(0)}(y), \mu^{(0)}(y), v^{(0)}(x, y)\}$, which we take as the initial approximation of problem (5)–(8) and build successive approximations according to the following algorithm:

Step 1. Setting $v(x, y) = v^{(0)}(x, y)$, from (7) and (11) we determine $\frac{\partial \lambda^{(1)}(y)}{\partial y}$ and $\lambda^{(1)}(y)$. Using equation (9), with $\lambda(y) = \lambda^{(1)}(y)$, we find $\frac{\partial v^{(1)}(x, y)}{\partial y}$. Next, we find $v^{(1)}(x, y) = \varphi'(x) + \int_0^y \frac{\partial v^{(1)}(x, \eta)}{\partial \eta} d\eta$.

Step 2. Taking $v(x, y) = v^{(1)}(x, y)$, from (7) and (11) we determine $\frac{\partial \lambda^{(2)}(y)}{\partial y}$ and $\lambda^{(2)}(y)$, respectively. Using equation (9), with $\lambda(y) = \lambda^{(2)}(y)$, we find $\frac{\partial v^{(2)}(x, y)}{\partial y}$. Let us find $v^{(2)}(x, y) = \varphi'(x) + \int_0^y \frac{\partial v^{(2)}(x, \eta)}{\partial \eta} d\eta$.

Continuing the process, at the k-th step we obtain the system $\left\{ \frac{\partial \lambda^{(k)}(y)}{\partial y}, \lambda_r^{(k)}(x), \frac{\partial v^{(k)}(x, y)}{\partial y}, v_r^{(k)}(x, t) \right\}$.

The conditions of the following statement ensure the feasibility and convergence of the proposed algorithm, as well as the solvability of problem (5)–(8).

Theorem 1. Let there exist $(\lambda^{(0)}(y), v^{(0)}(x, y), \rho_1, \rho_2) \in U(L_1, L_2, x, y)$, where $(x, y) \in \Omega$, $(\lambda(y), v(x, y)) \in S(\lambda^{(0)}(y), \rho_1) \times S(v^{(0)}(x, y), \rho_2)$ and following conditions are satisfied:

- 1) the function $\varphi(x)$ is continuously differentiable on $[0, X]$,
- 2) the functions $a(y), \psi(y), \theta(y)$ are continuously differentiable on $[0, Y]$, $\alpha(y) \neq 1$,
- 3) $q = X \left(\frac{X}{2} + \frac{X\alpha}{1-\alpha} + \frac{XYL_1}{2} + \frac{XY^2}{2} \frac{L_1\alpha}{1-\alpha} + (L_2 + 1)Y \right) < 1$,
- 4) $\frac{\sigma}{1-q} \frac{\alpha}{1-\alpha} \frac{XY^2}{2} < \rho_1$, $\frac{qY\sigma}{1-q} < \rho_2$,

where $\sigma = \theta' + \frac{X\psi}{1-\alpha} \left(1 + \max_{x \in [0, X]} \|\varphi'(x)\|Y \right) + X \max_{x \in [0, X]} \|\varphi'(x)\| \left(1 + \max_{x \in [0, X]} \|\varphi(x)\| \right)$,

$\alpha = \max_{y \in [0, Y]} \|\alpha(y)\|$, $\psi = \max_{y \in [0, Y]} \|\psi(y)\|$, $\theta = \max_{y \in [0, Y]} \|\theta(y)\|$, then the nonlocal problem (5)–(8) for the nonlinear Benjamin-Bona-Mahony equation has a unique solution belonging to $S(\lambda^{(0)}(y), \rho_1) \times S(v^{(0)}(x, y), \rho_2)$ and estimates are made:

$$a) \|\lambda^*(y) - \lambda^{(k)}(y)\| \leq \frac{\alpha}{1-\alpha} \frac{XY^2}{2} \sum_{i=k}^{\infty} q^i \sigma, \quad b) \|v^*(x, y) - v^{(k)}(x, y)\| \leq Y \sum_{i=k+1}^{\infty} q^i \sigma.$$

Proof. From the zero step of the algorithm, the following estimates hold:

$$\begin{aligned} \|\lambda^{(0)}(y)\| &\leq \varphi(0) + \frac{\psi Y}{1-\alpha}, \quad \left\| \frac{\partial \lambda(y)}{\partial y} \right\| \leq \frac{\psi}{1-\alpha}, \\ \left\| \frac{\partial v^{(0)}(x, y)}{\partial y} \right\| &\leq \theta' + X \frac{\psi}{1-\alpha} + X \max_{x \in [0, X]} \|\varphi(x)\| \max_{x \in [0, X]} \|\varphi'(x)\| + \\ &+ XY \frac{\psi}{1-\alpha} \max_{x \in [0, X]} \|\varphi'(x)\| + X \max_{x \in [0, X]} \|\varphi'(x)\| \leq \\ &\leq \theta' + \frac{X\psi}{1-\alpha} \left(1 + \max_{x \in [0, X]} \|\varphi'(x)\|Y \right) + X \max_{x \in [0, X]} \|\varphi'(x)\| \left(1 + \max_{x \in [0, X]} \|\varphi(x)\| \right) = \sigma, \\ \|v^{(0)}(x, y) - \varphi'(x)\| &\leq \int_0^y \left\| \frac{\partial v^{(0)}(x, \eta)}{\partial \eta} \right\| d\eta < Y\sigma. \end{aligned}$$

From the first step of the algorithm, for $v(x, y) = v^{(0)}(x, y)$, the following inequalities follow

$$\|\lambda^{(1)}(y) - \lambda^{(0)}(y)\| \leq \frac{\alpha}{1-\alpha} \int_0^y \int_0^X \|v^{(0)}(x, \eta) - \varphi'(x)\| dx d\eta < \frac{\alpha}{1-\alpha} \frac{XY^2}{2} \sigma < \rho_1,$$

$$\begin{aligned} \left\| \frac{\partial \lambda^{(1)}(y)}{\partial y} - \frac{\partial \lambda^{(0)}(y)}{\partial y} \right\| &\leq \frac{\alpha}{1-\alpha} \int_0^X \left\| \frac{\partial v^{(0)}(x, y)}{\partial y} \right\| dx, \\ \left\| \frac{\partial v^{(1)}(x, y)}{\partial y} - \frac{\partial v^{(0)}(x, y)}{\partial y} \right\| &\leq \\ &\leq \int_0^x \int_0^\xi \left\| \frac{\partial v^{(0)}(\xi_1, y)}{\partial y} \right\| d\xi_1 d\xi + \int_0^x \left\| \frac{\partial \lambda^{(1)}(y)}{\partial y} - \frac{\partial \lambda^{(0)}(y)}{\partial y} \right\| d\xi + \\ &+ L_1 \int_0^x \int_0^\xi \|v^{(0)}(\xi_1, y) - \varphi'(\xi_1)\| d\xi_1 d\xi + L_1 \|\lambda^{(1)}(y) - \lambda^{(0)}(y)\| \int_0^x d\xi + \\ &+ L_2 \int_0^x \|v^{(0)}(\xi, y) - \varphi'(\xi)\| d\xi + \int_0^x \|v^{(0)}(\xi, y) - \varphi'(\xi)\| d\xi \leq \\ &\leq q \max_{(x,y) \in \Omega} \left\| \frac{\partial v^{(0)}(x, y)}{\partial y} \right\| \leq q\sigma. \end{aligned}$$

$$\|v^{(1)}(x, y) - v^{(0)}(x, y)\| \leq \int_0^y \left\| \frac{\partial v^{(1)}(x, \eta)}{\partial \eta} - \frac{\partial v^{(0)}(x, \eta)}{\partial \eta} \right\| d\eta \leq \int_0^y q\sigma d\eta < \rho_2.$$

At the second step of the algorithm, for $v(x, y) = v^{(1)}(x, y)$, the following estimates hold:

$$\|\lambda^{(2)}(y) - \lambda^{(1)}(y)\| \leq \frac{\alpha}{1-\alpha} \int_0^y \int_0^X \|v^{(1)}(x, \eta) - v^{(0)}(x, \eta)\| dx d\eta \leq \frac{\alpha}{1-\alpha} \frac{XY^2}{2} q\sigma,$$

$$\left\| \frac{\partial v^{(2)}(x, y)}{\partial y} - \frac{\partial v^{(1)}(x, y)}{\partial y} \right\| \leq q \max_{(x,y) \in \Omega} \left\| \frac{\partial v^{(1)}(x, y)}{\partial y} - \frac{\partial v^{(0)}(x, y)}{\partial y} \right\| \leq q^2 \max_{(x,y) \in \Omega} \left\| \frac{\partial v^{(0)}(x, y)}{\partial y} \right\| \leq q^2 \sigma,$$

$$\|v^{(2)}(x, y) - v^{(1)}(x, y)\| \leq \int_0^y \left\| \frac{\partial v^{(2)}(x, \eta)}{\partial \eta} - \frac{\partial v^{(1)}(x, \eta)}{\partial \eta} \right\| d\eta \leq Yq^2\sigma.$$

$$\begin{aligned} \|\lambda^{(2)}(y) - \lambda^{(0)}(y)\| &\leq \|\lambda^{(2)}(y) - \lambda^{(1)}(y)\| + \|\lambda^{(1)}(y) - \lambda^{(0)}(y)\| \leq \\ &\leq \frac{\alpha}{1-\alpha} \frac{XY^2}{2} q\sigma + \frac{\alpha}{1-\alpha} \frac{XY^2}{2} \sigma \leq \frac{\alpha}{1-\alpha} \frac{XY^2}{2} (1+q)\sigma < \rho_1, \end{aligned}$$

$$\left\| \frac{\partial v^{(2)}(x, y)}{\partial y} - \frac{\partial v^{(0)}(x, y)}{\partial y} \right\| \leq (1+q) \max_{(x,y) \in \Omega} \left\| \frac{\partial v^{(1)}(x, y)}{\partial y} - \frac{\partial v^{(0)}(x, y)}{\partial y} \right\| \leq$$

$$\leq (q + q^2) \max_{(x,y) \in \Omega} \left\| \frac{\partial v^{(0)}(x, y)}{\partial y} \right\| \leq (q + q^2)\sigma.$$

$$\|v^{(2)}(x, y) - v^{(0)}(x, y)\| \leq \|v^{(2)}(x, y) - v^{(1)}(x, y)\| + \|v^{(1)}(x, y) - v^{(0)}(x, y)\| \leq Y(q^2 + q)\sigma < \rho_2.$$

At the $(k + 1)$ -th step of the algorithm, for $v(x, y) = v^{(k)}(x, y)$, the following estimates hold:

$$\|\lambda^{(k+1)}(y) - \lambda^{(k)}(y)\| \leq \frac{\alpha}{1 - \alpha} \int_0^y \int_0^X \|v^{(k)}(x, \eta) - v^{(k-1)}(x, \eta)\| dx d\eta. \quad (12)$$

$$\left\| \frac{\partial \lambda^{(k+1)}(y)}{\partial y} - \frac{\partial \lambda^{(k)}(y)}{\partial y} \right\| \leq \frac{\alpha}{1 - \alpha} \int_0^X \left\| \frac{\partial v^{(k)}(x, y)}{\partial y} - \frac{\partial v^{(k-1)}(x, y)}{\partial y} \right\| dx, \quad (13)$$

$$\left\| \frac{\partial v^{(k+1)}(x, y)}{\partial y} - \frac{\partial v^{(k)}(x, y)}{\partial y} \right\| \leq q \max_{(x,y) \in \Omega} \left\| \frac{\partial v^{(k)}(x, y)}{\partial y} - \frac{\partial v^{(k-1)}(x, y)}{\partial y} \right\|, \quad (14)$$

$$\|v^{(k+1)}(x, y) - v^{(k)}(x, y)\| \leq \int_0^y \left\| \frac{\partial v^{(k+1)}(x, \eta)}{\partial \eta} - \frac{\partial v^{(k)}(x, \eta)}{\partial \eta} \right\| d\eta. \quad (15)$$

$$\|\lambda^{(k+1)}(y) - \lambda^{(0)}(y)\| \leq \frac{\alpha}{1 - \alpha} \int_0^y \int_0^X \|v^{(k)}(x, \eta) - \varphi'(x)\| dx d\eta \leq \frac{\alpha}{1 - \alpha} \frac{XY^2}{2} \sum_{i=0}^k q^i \sigma < \rho_1,$$

$$\|v^{(k+1)}(x, y) - v^{(0)}(x, y)\| \leq Y \sum_{i=1}^{k+1} q^i \sigma < \rho_2.$$

Thus, it follows from inequalities (12)–(15) and $q < 1$ that the sequence $\{\lambda^{(k)}(y), v^{(k)}(x, y)\}$ as $k \rightarrow \infty$, converges to $\{\lambda^*(y), v^*(x, y)\}$ to the solution of problem (5)–(8) in $S(\lambda^{(0)}(y), \rho_1) \times S(v^{(0)}(x, y), \rho_2)$.

Let's establish the inequalities

$$\|\lambda^{(k+p)}(y) - \lambda^{(k)}(y)\| \leq \frac{\alpha}{1 - \alpha} \frac{XY^2}{2} \sum_{i=k}^{k+p-1} q^i \sigma, \quad \|v^{(k+p)}(x, y) - v^{(k)}(x, y)\| \leq Y \sum_{i=k+1}^{k+p} q^i \sigma,$$

as $p \rightarrow \infty$ we obtain estimates a), b) of Theorem 1.

Let's prove uniqueness.

Let there be two solutions $(\lambda^*(x), v^*(x, y)), (\lambda^{**}(x), v^{**}(x, y))$ in $S(\lambda^{(0)}(x), \rho_1) \times S(v^{(0)}(x, y), \rho_2)$ of problem (10)–(14). Similar to relations (12)–(15) for the differences $\lambda^{**}(y) - \lambda^*(y)$, $\frac{\partial \lambda^{**}(y)}{\partial y} - \frac{\partial \lambda^*(y)}{\partial y}$, $\frac{\partial v^{**}(x, y)}{\partial y} - \frac{\partial v^*(x, y)}{\partial y}$, $v^{**}(x, y) - v^*(x, y)$, for all $(x, y) \in \Omega$, we get:

$$\|\lambda^{**}(y) - \lambda^*(y)\| \leq \frac{\alpha}{1 - \alpha} \int_0^y \int_0^X \|v^{**}(x, \eta) - v^*(x, \eta)\| dx d\eta,$$

$$\left\| \frac{\partial \lambda^{**}(y)}{\partial y} - \frac{\partial \lambda^*(y)}{\partial y} \right\| \leq \frac{\alpha}{1 - \alpha} \int_0^X \|v^{**}(x, y) - v^*(x, y)\| dx,$$

$$\left\| \frac{\partial v^{**}(x, y)}{\partial y} - \frac{\partial v^*(x, y)}{\partial y} \right\| \leq q \max_{(x, y) \in \Omega} \left\| \frac{\partial v^{**}(x, y)}{\partial y} - \frac{\partial v^*(x, y)}{\partial y} \right\|,$$

$$\|v^{**}(x, y) - v^*(x, y)\| \leq \int_0^y \left\| \frac{\partial v^{**}(x, \eta)}{\partial \eta} - \frac{\partial v^*(x, \eta)}{\partial \eta} \right\| d\eta.$$

Whence it follows that $\lambda^{**}(x) = \lambda^*(x)$, $v^{**}(x, y) = v^*(x, y)$. Theorem 1 is proved.

The function $w^{(k)}(x, y)$, $k = 1, 2, \dots$, is defined by the equality

$$w^{(k)}(x, y) = \int_0^x v^{(k)}(\xi, y) d\xi + \lambda^{(k)}(y)$$

and denote by $S(w^{(0)}(x, y), \rho_1 + \rho_2)$ the set of continuously differentiable with respect to y and twice with respect to x functions $w : \Omega \rightarrow R$ satisfying the inequalities

$$\|w(x, y) - \int_0^x v^{(0)}(\xi, y) d\xi - \lambda^{(0)}(y)\| < \rho_1 + \rho_2.$$

In view of the equivalence of problems (1)–(4) and (5)–(8), Theorem 1 implies.

Theorem 2. If the conditions of Theorem 1 are satisfied, then the sequence of functions $w^{(k)}(x, y)$, $k = 1, 2, \dots$, is contained in $S(w^{(0)}(x, y), \rho_1 + \rho_2)$ converges to the unique solution $w^*(x, y)$ of problem (1)–(4) in $S(w^{(0)}(x, y), \rho_1 + \rho_2)$ and the inequality

$$\|w^*(x, y) - w^{(k)}(x, y)\| \leq \frac{\alpha}{1 - \alpha} \frac{X^2 Y^2}{2} \sum_{i=k+1}^{\infty} q^i \sigma + Y \sum_{i=k}^{\infty} q^i \sigma.$$

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Бенджамин-Бон-Махони тендеуі үшін бейлокал шеттік есептің бір жуық шешімі жайында

Жұмыста Бенджамин-Бона-Махони тендеуі үшін бейлокал шеттік есеп зерттелді. Қарастырылып отырған тендеу аралас туындылы үшінші ретгі сызықтық емес псевдопараболалық тендеу болып табылады. Осы есептің шешімін табу үшін жуық шешімін табу алгоритмі ұсынылған. Ұсынылған алгоритмнің орындалуы мен жинақтылығының жеткілікті шарттары алынды, сондай-ақ, сызықтық емес тендеу үшін бейлокал шеттік есебінің оқшауланған шешімінің бар болуы табылған. Қарастырылған есептің дәл және жуық шешімі арасындағы бағалаулары алынды.

Кілт сөздер: дербес туындылы дифференциалдық тендеу, Бенджамин-Бона-Махони тендеуі, алгоритм, жуық шешім.

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Об одном приближенном решении нелокальной краевой задачи для уравнения Бенджамин-Бона-Махони

В работе исследована нелокальная краевая задача для уравнения Бенджамин-Бона-Махони. Рассматриваемое уравнение является нелинейным псевдопараболическим уравнением третьего порядка со смешанной производной. Для нахождения решения данной задачи предложен алгоритм поиска приближенного решения. Установлены достаточные условия осуществимости и сходимости предложенного алгоритма, а также существование изолированного решения нелокальной краевой задачи для нелинейного уравнения. Получены оценки между точным и приближенным решениями рассматриваемой задачи.

Ключевые слова: дифференциальные уравнения в частных производных, уравнение Бенджамин-Бона-Махони, алгоритм, приближенное решение.