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## Order of the trigonometric widths of the Nikol'skii-Besov classes with mixed metric in the metric of anisotropic Lorentz spaces

In this paper we estimate the order of the trigonometric width of the Nikol'skii-Besov classes  $B_{\mathbf{p}}^{\alpha\tau}(\mathbb{T}^n)$  with mixed metric in the anisotropic Lorentz space  $L_{\mathbf{q}\theta}(\mathbb{T}^n)$  when  $\mathbf{1} < \mathbf{p} = (p_1, \dots, p_n) < \mathbf{2} < \mathbf{q} = (q_1, \dots, q_n)$ . The concept of a trigonometric width in the one-dimensional case was first introduced by R.S. Ismagilov and he established his estimates for certain classes in the space of continuous functions. For a function of several variables exact orders of trigonometric width of Sobolev class  $W_p^r$ , Nikol'skii class  $H_p^r$  in the space  $L_q$  are established by E.S. Belinsky, V.E. Majorov, Yu. Makovoz, G.G. Magaril-Ilyayev, V.N. Temlyakov. This problem for the Besov class  $B_{pq}^r$  was investigated by A.S. Romanyuk, D.B. Bazarkhanov. The trigonometric width for the anisotropic Nikol'skii-Besov classes  $B_{\mathbf{p}\mathbf{r}}^{\alpha\tau}(\mathbb{T}^n)$  in the metric of the anisotropic Lorentz spaces  $L_{\mathbf{q}\theta}(\mathbb{T}^n)$  was found by K.A. Bekmaganbetov and Ye. Toleugazy.

*Keywords:* trigonometric widths, anisotropic Lorentz space, Nikol'skii-Besov class with mixed metric.

### Introduction

Let  $V \subset L_1(\mathbb{T}^n)$  be the normed space and  $F \subset V$  be some functional class. The trigonometric width of the class  $F$  in the space  $V$  is defined as follows (see [1])

$$d_M^T(F, V) = \inf_{\Omega_M} \sup_{f \in F} \inf_{t(\Omega_M; \mathbf{x})} \|f(\cdot) - t(\Omega_M; \cdot)\|_V,$$

where  $t(\Omega_M; \mathbf{x}) = \sum_{j=1}^M c_j e^{i(\mathbf{k}_j, \mathbf{x})}$ ,  $\Omega_M = \{\mathbf{k}_1, \dots, \mathbf{k}_M\}$  is the set of vectors  $\mathbf{k}_j = (k_1^j, \dots, k_n^j)$  from the integer lattice  $\mathbb{Z}^n$  and  $c_j$  are some numbers ( $j = 1, \dots, M$ ).

The concept of a trigonometric width in the one-dimensional case was first introduced by R.S. Ismagilov [1] and he established its estimates for certain classes in the space of continuous functions. For a function of several variables exact orders of trigonometric widths of Sobolev class  $W_p^r$ , Nikol'skii class  $H_p^r$  in the space  $L_q$  are established by E.S. Belinsky [2], V.E. Majorov [3], Yu. Makovoz [4], G.G. Magaril-Ilyayev [5], V.N. Temlyakov [6]. This problem for the Besov class  $B_{pq}^r$  was investigated by A.S. Romanyuk [7], D.B. Bazarkhanov [8]. The trigonometric width for the anisotropic Nikol'skii-Besov classes  $B_{\mathbf{p}\mathbf{r}}^{\alpha\tau}(\mathbb{T}^n)$  in the metric of the anisotropic Lorentz spaces  $L_{\mathbf{q}\theta}(\mathbb{T}^n)$  was found by K.A. Bekmaganbetov and Ye. Toleugazy [9].

We study the problem of estimating the order of the trigonometric width of the Nikol'skii-Besov classes  $B_{\mathbf{p}}^{\alpha\tau}(\mathbb{T}^n)$  with a mixed metric in the metric of anisotropic Lorentz spaces  $L_{\mathbf{q}\theta}(\mathbb{T}^n)$ .

### Preliminaries and auxiliary results

Let  $f(\mathbf{x}) = f(x_1, \dots, x_n)$  be a measurable function defined by  $\mathbb{T}^n$ . Let multiindexes  $\mathbf{1} \leq \mathbf{p} = (p_1, \dots, p_n) < \infty$ . A Lebesgue space  $L_{\mathbf{p}}(\mathbb{T}^n)$  with mixed metric is the set of functions for which the following quantity is finite

$$\|f\|_{L_{\mathbf{p}}(\mathbb{T}^n)} = \left( \int_0^{2\pi} \left( \dots \left( \int_0^{2\pi} |f(x_1, \dots, x_n)|^{p_1} dx_1 \right)^{p_2/p_1} \dots \right)^{p_n/p_{n-1}} dx_n \right)^{1/p_n}.$$

Here, the expression  $\left( \int_0^{2\pi} |f(t)|^p dt \right)^{1/p}$  for  $p = \infty$  is understood as  $\sup_{0 \leq t < 2\pi} |f(t)|$ .

For the function  $f \in L_{\mathbf{p}}(\mathbb{T}^n)$  we denote

$$\Delta_{\mathbf{s}}(f, \mathbf{x}) = \sum_{\mathbf{k} \in \rho(\mathbf{s})} a_{\mathbf{k}}(f) e^{i(\mathbf{k}, \mathbf{x})},$$

where  $\{a_{\mathbf{k}}(f)\}_{\mathbf{k} \in \mathbb{Z}^n}$  are Fourier coefficients of the function  $f$  with respect to the multiple trigonometric system  $\rho(\mathbf{s}) = \{\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{Z}^n : 2^{s_i-1} \leq |k_i| < 2^{s_i}, i = 1, \dots, n\}$ ,  $(\mathbf{k}, \mathbf{x}) = \sum_{j=1}^n k_j x_j$  – inner product.

Let  $\mathbf{0} < \alpha = (\alpha_1, \dots, \alpha_n) < \infty$ ,  $\mathbf{0} < \tau = (\tau_1, \dots, \tau_n) \leq \infty$ . The class of Nikol'skii-Besov  $B_{\mathbf{p}}^{\alpha\tau}(\mathbb{T}^n)$  with a mixed metric is the set of functions  $f$  from  $L_{\mathbf{p}}(\mathbb{T}^n)$  for which the following inequality holds

$$\|f\|_{B_{\mathbf{p}}^{\alpha\tau}(\mathbb{T}^n)} = \left\| \left\{ \mathbf{2}^{(\alpha, \mathbf{s})} \|\Delta_{\mathbf{s}}(f, \cdot)\|_{L_{\mathbf{p}}(\mathbb{T}^n)} \right\}_{\mathbf{s} \in \mathbb{Z}_+^n} \right\|_{l_{\tau}} \leq 1,$$

where  $\|\cdot\|_{l_{\tau}}$  is the norm of discrete Lebesgue space  $l_{\tau}$  with a mixed metric.

Let  $f(\mathbf{x}) = f(x_1, \dots, x_n)$  be a measurable function defined on  $\mathbb{T}^n$ . We denote by  $f^*(\mathbf{t}) = f^{*1, \dots, *n}(t_1, \dots, t_n)$  the function obtained by applying to the first nonincreasing permutation, successively with respect to the variables  $x_1, \dots, x_n$  for fixed other variables.

Let multiindexes  $\mathbf{q} = (q_1, \dots, q_n)$ ,  $\theta = (\theta_1, \dots, \theta_n)$  satisfy the conditions: if  $0 < q_j < \infty$ , then  $0 < \theta_j \leq \infty$ , if  $q_j = \infty$ , then  $\theta_j = \infty$  for every  $j = 1, \dots, n$ . An anisotropic Lorentz space  $L_{\mathbf{q}\theta}(\mathbb{T}^n)$  is the set of functions for which the following quantity is finite

$$\|f\|_{L_{\mathbf{q}\theta}(\mathbb{T}^n)} = \left( \int_0^{2\pi} \dots \left( \int_0^{2\pi} \left( t_1^{1/q_1} \dots t_n^{1/q_n} f^{*1, \dots, *n}(t_1, \dots, t_n) \right)^{\theta_1} \frac{dt_1}{t_1} \right)^{\theta_2/\theta_1} \dots \frac{dt_n}{t_n} \right)^{1/\theta_n}.$$

Let  $\Omega_M$  be a set containing at most  $M$  vectors  $\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{Z}^n$ .

*Lemma 1* [10]. Let  $\mathbf{2} \leq \mathbf{q} < \infty$ . Then for any trigonometric polynomial

$$P(\Omega_M, \mathbf{x}) = \sum_{j=1}^M e^{i(\mathbf{k}^j, \mathbf{x})}$$

and for any number  $N \leq M$  there exists a trigonometric polynomial  $P(\Omega_N, \mathbf{x})$  containing at most  $N$  harmonics and such that

$$\|P(\Omega_M, \cdot) - P(\Omega_N, \cdot)\|_{L_{\mathbf{q}}(\mathbb{T}^n)} \leq CMN^{-1/2},$$

moreover  $\Omega_N \subset \Omega_M$  and all coefficients  $P(\Omega_N, \mathbf{x})$  are the same and do not exceed  $MN^{-1}$ .

*Corollary 1* [11]. Let  $\mathbf{2} < \mathbf{q} = (q_1, \dots, q_n) \leq \infty$ ,  $\mathbf{0} < \theta = (\theta_1, \dots, \theta_n) \leq \infty$ . Then for any trigonometric polynomial

$$P(\Omega_M, \mathbf{x}) = \sum_{j=1}^M e^{i(\mathbf{k}^j, \mathbf{x})}$$

and for any number  $N \leq M$  there exists a trigonometric polynomial  $P(\Omega_N, \mathbf{x})$  containing at most  $N$  harmonics and such that

$$\|P(\Omega_M, \cdot) - P(\Omega_N, \cdot)\|_{L_{\mathbf{q}\theta}(\mathbb{T}^n)} \leq CMN^{-1/2},$$

moreover  $\Omega_N \subset \Omega_M$  and all coefficients  $P(\Omega_N, \mathbf{x})$  are the same and do not exceed  $MN^{-1}$ .

For any  $\mathbf{s} \in \mathbb{Z}_+^n$  we consider a linear operator

$$(T_{N_{\mathbf{s}}} f)(\mathbf{x}) = f(\mathbf{x}) * \left( \sum_{\mathbf{k} \in \rho(\mathbf{s})} e^{i(\mathbf{k}, \mathbf{x})} - t(\Omega_{N_{\mathbf{s}}}, \mathbf{x}) \right),$$

where  $t(\Omega_{N_{\mathbf{s}}}, \mathbf{x})$  is a trigonometric polynomial from Corollary 1, which is approaching the «block»  $t_{\mathbf{s}}(\mathbf{x}) = \sum_{\mathbf{k} \in \rho(\mathbf{s})} e^{i(\mathbf{k}, \mathbf{x})}$ .

*Lemma 2.* Let  $\mathbf{1} < \mathbf{p} < \mathbf{2}$ , the multiindex  $\mathbf{q} = (q_1, \dots, q_n)$  be such that  $2 < q_j < p'$  for all  $j = 1, \dots, n$  and  $\mathbf{0} < \theta = (\theta_1, \dots, \theta_n) \leq \infty$ . Then the norm operator  $T_{N_{\mathbf{s}}}$  acting from  $L_{\mathbf{p}}(\mathbb{T}^n)$  to  $L_{\mathbf{q}\theta}(\mathbb{T}^n)$  satisfies the following inequality

$$\|T_{N_{\mathbf{s}}}\|_{L_{\mathbf{p}}(\mathbb{T}^n) \rightarrow L_{\mathbf{q}\theta}(\mathbb{T}^n)} \leq C_1 2^{(\mathbf{1}, \mathbf{s})} N_{\mathbf{s}}^{-(1/2+1/\mathbf{p}')}.$$

*Proof.* Taking into account that the coefficients of the polynomial  $t(\Omega_{N_s}, \mathbf{x})$  are the same and do not exceed  $2^{(1,s)}N_s^{-1}$  by Parseval's equality we have

$$\|T_{N_s}\|_{L_2(\mathbb{T}^n) \rightarrow L_2(\mathbb{T}^n)} \leq C_1 2^{(1,s)} N_s^{-1}. \tag{1}$$

Further, using the generalized Minkowski's inequalities and Corollary 1 we can write

$$\|T_{N_s} f\|_{L_{\mathbf{q}^* \theta^*}(\mathbb{T}^n)} \leq \|f\|_{L_1(\mathbb{T}^n)} \left\| \sum_{\mathbf{k} \in \rho(\mathbf{s})} e^{i(\mathbf{k}, \cdot)} - t(\Omega_{N_s}, \cdot) \right\|_{L_{\mathbf{q}^* \theta^*}(\mathbb{T}^n)} \leq C_2 2^{(1,s)} N_s^{-1/2} \|f\|_{L_1(\mathbb{T}^n)}.$$

Therefore, by definition,  $\|T_{N_s}\|_{L_1(\mathbb{T}^n) \rightarrow L_{\mathbf{q}^* \theta^*}(\mathbb{T}^n)}$  we find

$$\|T_{N_s}\|_{L_1(\mathbb{T}^n) \rightarrow L_{\mathbf{q}^* \theta^*}(\mathbb{T}^n)} \leq C_2 2^{(1,s)} N_s^{-1/2}. \tag{2}$$

Further, using the Riesz-Thorin interpolation theorem for Lebesgue spaces and anisotropic Lorentz spaces, we obtain

$$\|T_{N_s}\|_{L_1(\mathbb{T}^n) \rightarrow L_{\mathbf{q}^* \theta^*}(\mathbb{T}^n)} \leq \|T_{N_s}\|_{L_2(\mathbb{T}^n) \rightarrow L_2(\mathbb{T}^n)}^{1-\lambda} \|T_{N_s}\|_{L_1(\mathbb{T}^n) \rightarrow L_{\mathbf{q}^* \theta^*}(\mathbb{T}^n)}^\lambda, \tag{3}$$

where  $0 < \lambda < 1$  and  $1/\mathbf{p} = (1-\lambda)/2 + \lambda/\mathbf{1}$ ,  $1/\mathbf{q} = (1-\lambda)/2 + \lambda/\mathbf{q}^*$  and  $1/\theta = (1-\lambda)/2 + \lambda/\theta^*$ .

By substituting (1) and (2) to (3) and performing elementary transformations, we receive at the required estimate with the additional condition  $0 < \theta = (\theta_1, \dots, \theta_n) < \mathbf{p}' = (p', \dots, p')$ . For the remaining values of the parameters  $\theta = (\theta_1, \dots, \theta_n)$  the validity of the assertion follows from the embedding  $L_{\mathbf{q}\theta_1}(\mathbb{T}^n) \hookrightarrow L_{\mathbf{q}\theta_2}(\mathbb{T}^n)$  for  $0 < \theta_1 = (\theta_1^1, \dots, \theta_n^1) \leq \theta_2 = (\theta_1^2, \dots, \theta_n^2) \leq \infty$ .

Let us formulate a special case of the embedding theorem from E.D. Nursultanov's paper ([12]) as a Lemma.

*Lemma 3* [12]. Let  $\mathbf{1} \leq \mathbf{p} = (p_1, \dots, p_n) < \mathbf{q} = (q_1, \dots, q_n) < \infty$ ,  $0 < \tau = (\tau_1, \dots, \tau_n) \leq \infty$  and  $\alpha = 1/\mathbf{p} - 1/\mathbf{q}$ , then

$$B_{\mathbf{p}}^{\alpha\tau}(\mathbb{T}^n) \hookrightarrow L_{\mathbf{q}\tau}(\mathbb{T}^n).$$

Furthermore we need the following sets

$$Y^n(N, \gamma) = \left\{ \mathbf{s} = (s_1, \dots, s_n) \in \mathbb{Z}_+^n : \sum_{j=1}^n \gamma_j s_j \geq N \right\},$$

$$\mathbb{N}^n(N, \gamma) = \left\{ \mathbf{s} = (s_1, \dots, s_n) \in \mathbb{Z}_+^n : \sum_{j=1}^n \gamma_j s_j = N \right\}.$$

*Lemma 4* [13]. Let  $n \in \mathbb{N}$ ,  $n \geq 2$ ,  $0 < \gamma' = (\gamma'_1, \dots, \gamma'_n) \leq \gamma = (\gamma_1, \dots, \gamma_n) < \infty$ ,  $\delta > 0$  and  $0 < \varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \leq \infty$ . Then

$$\left\| \left\{ 2^{-\delta(\gamma, \mathbf{s})} \right\}_{\mathbf{s} \in Y^n(N, \gamma')} \right\|_{l_\varepsilon(\mathbb{Z}_+^n)} \leq C 2^{-\delta\eta N} N^{\sum_{j \in A \setminus \{j_1\}} 1/\varepsilon_j},$$

where  $\eta = \min\{\gamma_j/\gamma'_j : j = 1, \dots, n\}$ ,  $A = \{j : \gamma_j/\gamma'_j = \eta, j = 1, \dots, n\}$ ,  $j_1 = \min\{j : j \in A\}$ .

*Lemma 5* [13]. Let  $n \in \mathbb{N}$ ,  $n \geq 2$ ,  $0 < \gamma = (\gamma_1, \dots, \gamma_n) < \infty$ ,  $\delta \in \mathbb{R}$  and  $0 < \varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \leq \infty$ . Then

$$\left\| \left\{ 2^{-\delta(\gamma, \mathbf{s})} \right\}_{\mathbf{s} \in \mathbb{N}^n(N, \gamma')} \right\|_{l_\varepsilon(\mathbb{Z}_+^n)} \asymp 2^{-\delta N} N^{\sum_{j=2}^n 1/\varepsilon_j}.$$

### Main result

The main result of this paper includes:

*Theorem 1.* Let  $\mathbf{1} < \mathbf{p} = (p_1, \dots, p_n) < \mathbf{2} < \mathbf{q} = (q_1, \dots, q_n) < \mathbf{p}'_0 = (p'_0, \dots, p'_0)$ ,  $p_0 = \max\{p_j : j = 1, \dots, n\}$ ,  $\mathbf{1} \leq \tau = (\tau_1, \dots, \tau_n)$ ,  $\theta = (\theta_1, \dots, \theta_n)$  and  $\alpha = (\alpha_1, \dots, \alpha_n)$  be such that  $\alpha_j > 1 + 1/p_j - 1/p_0$  for all  $j = 1, \dots, n$ . Let  $\zeta = \min\{\alpha_j - 1/p_j + 1/q_j : j = 1, \dots, n\}$ ,  $D = \{j = 1, \dots, n : \alpha_j - 1/p_j + 1/q_j = \zeta\}$ ,  $j_1 = \min\{j : j \in D\}$ ,  $q_j = q_{j_1}$  for all  $j \in D$  and  $q_j \geq q_{j_1}$  for all  $j \notin D$ .

Then the following relation holds

$$d_M^T(B_{\mathbf{p}}^{\alpha\tau}(\mathbb{T}^n), L_{\mathbf{q}\theta}(\mathbb{T}^n)) \asymp M^{-(\alpha_{j_1-1/p_{j_1}+1/2})} (\log M)^{(|D|-1)(\alpha_{j_1-1/p_{j_1}+1/2}) + \sum_{j \in D \setminus \{j_1\}} (1/2-1/\tau_j)_+}, \quad (4)$$

where  $|D|$  is amount of elements of the set  $D$ ,  $a_+ = \max\{a; 0\}$ .

*Proof.* Let  $f \in B_{\mathbf{p}}^{\alpha\tau}(\mathbb{T}^n)$ . For any natural number  $M$  there exists the natural number  $m$  such that  $M \asymp 2^m m^{(|D|-1)}$ . We will seek an approximating polynomial  $P(\Omega_M; \mathbf{x})$  in the following form

$$P(\Omega_M; \mathbf{x}) = \sum_{(\gamma', \mathbf{s})} \Delta_{\mathbf{s}}(f, \mathbf{x}) + \sum_{m \leq (\gamma', \mathbf{s}) < \beta m} t(\Omega_{N_{\mathbf{s}}}; \mathbf{x}) * \Delta_{\mathbf{s}}(f, \mathbf{x}), \quad (5)$$

where

$$\beta = \left( \alpha_{j_1-1/p_{j_1}+1/2} - \frac{\log m}{m} \sum_{j \in D \setminus \{j_1\}} \left( (1/2-1/\tau_j)_+ - (1/\theta_j-1/\tau_j)_+ \right) \right) / (\alpha_{j_1-1/p_{j_1}+1/q_{j_1}}),$$

$\gamma_j = (\alpha_j - 1/p_j + 1/q_j) / (\alpha_{j_1-1/p_{j_1}+1/q_{j_1}})$ ,  $j = 1, \dots, n$ ,  $\gamma'_j = \gamma_j$  for  $j \in D$  and  $1 < \gamma'_j < \gamma_j$  for  $j \notin D$ . The polynomials  $t(\Omega_{N_{\mathbf{s}}}; \mathbf{x})$  are chosen for every "block"  $t_{\mathbf{s}}(\mathbf{x}) = \sum_{\mathbf{k} \in \rho(\mathbf{s})} e^{i(\mathbf{k}, \mathbf{x})}$  according to Corollary 1 and numbers

$$N_{\mathbf{s}} = \left\lfloor 2^{(\alpha_{j_1-1/p_{j_1}+1/p_0})m} 2^{-(\alpha-1/\mathbf{p}+1/\mathbf{p}_0-1, \mathbf{s})} \right\rfloor.$$

Note that according to Lemma 4

$$\begin{aligned} \sum_{m \leq (\gamma', \mathbf{s}) < \beta m} N_{\mathbf{s}} &= 2^{(\alpha_{j_1-1/p_{j_1}+1/p_0})m} \sum_{m \leq (\gamma', \mathbf{s}) < \beta m} 2^{-(\alpha-1/\mathbf{p}+1/\mathbf{p}_0-1, \mathbf{s})} \leq \\ &\leq 2^{(\alpha_{j_1-1/p_{j_1}+1/p_0})m} \left\| \left\{ 2^{-(\alpha-1/\mathbf{p}+1/\mathbf{p}_0-1, \mathbf{s})} \right\}_{\mathbf{s} \in Y^n(m, \gamma')} \right\|_{l_1} \leq \\ &\leq 2^{(\alpha_{j_1-1/p_{j_1}+1/p_0})m} 2^{-(\alpha_{j_1-1/p_{j_1}+1/p_0})m} m^{(|D|-1)} = 2^m m^{(|D|-1)} \asymp M, \end{aligned}$$

so that  $(\alpha_j - 1/p_j + 1/p_0 - 1) / (\alpha_{j_1-1/p_{j_1}+1/p_0-1}) > \gamma'_j$  at  $j \notin D$ .

Moreover according to equality (5) and Minkowski's inequality we have

$$\begin{aligned} &\|f(\cdot) - P(\Omega_M; \cdot)\|_{L_{\mathbf{q}\theta}(\mathbb{T}^n)} \leq \\ &\leq C_1 \left( \left\| \sum_{m \leq (\gamma', \mathbf{s}) < \beta m} (\Delta_{\mathbf{s}}(f, \cdot) - \Delta_{\mathbf{s}}(f, \cdot) * t(\Omega_{N_{\mathbf{s}}}; \cdot)) \right\|_{L_{\mathbf{q}\theta}(\mathbb{T}^n)} + \left\| \sum_{(\gamma', \mathbf{s}) \geq \beta m} \Delta_{\mathbf{s}}(f, \cdot) \right\|_{L_{\mathbf{q}\theta}(\mathbb{T}^n)} \right) = \\ &= C_1 (I_1(f) + I_2(f)). \end{aligned} \quad (6)$$

Firstly we estimate  $I_2(f)$ . By Lemma 3 we have

$$I_2(f) \leq C_2 \left\| \left\{ 2^{(1/\mathbf{p}-1/\mathbf{q}, \mathbf{s})} \|\Delta_{\mathbf{s}}(f, \cdot)\|_{L_{\mathbf{p}}(\mathbb{T}^n)} \right\}_{\mathbf{s} \in Y^n(\beta m, \gamma')} \right\|_{l_{\theta}} \leq I_3(f). \quad (7)$$

According to Hölder's inequality with parameters  $1/\theta = 1/\tau + 1/\varepsilon$ , where  $1/\varepsilon = (1/\theta - 1/\tau)_+$  and Lemma 4, taking into account that  $\gamma' \leq \gamma$  we find

$$\begin{aligned} I_3(f) &= \left\| \left\{ 2^{(\alpha, \mathbf{s})} \|\Delta_{\mathbf{s}}(f, \cdot)\|_{L_{\mathbf{p}}(\mathbb{T}^n)} \cdot 2^{-(\alpha_{j_1-1/p_{j_1}+1/q_{j_1})(\gamma, \mathbf{s})} \right\}_{\mathbf{s} \in Y^n(\beta m, \gamma')} \right\|_{l_{\theta}} \leq \\ &\leq \left\| \left\{ 2^{(\alpha, \mathbf{s})} \|\Delta_{\mathbf{s}}(f, \cdot)\|_{L_{\mathbf{p}}(\mathbb{T}^n)} \right\}_{\mathbf{s} \in Y^n(\beta m, \gamma')} \right\|_{l_{\tau}} \times \\ &\times \left\| \left\{ 2^{-(\alpha_{j_1-1/p_{j_1}+1/q_{j_1})(\gamma, \mathbf{s})} \right\}_{\mathbf{s} \in Y^n(\beta m, \gamma')} \right\|_{l_{\varepsilon}} \leq \end{aligned}$$

$$\begin{aligned} &\leq C_3 \|f\|_{B_{\mathbf{p}}^{\sigma}(\mathbb{T}^n)} \cdot 2^{-(\alpha_{j_1-1/p_{j_1}+1/q_{j_1}})\beta m \sum_{j \in D \setminus \{j_1\}} 1/\varepsilon_j} \leq \\ &\leq C_3 2^{-(\alpha_{j_1-1/p_{j_1}+1/q_{j_1}})\beta m \sum_{j \in D \setminus \{j_1\}} (1/\theta_j - 1/\tau_j)_+}. \end{aligned} \tag{8}$$

Inserting (8) into (7) we have

$$I_2 \leq C_4 2^{-(\alpha_{j_1-1/p_{j_1}+1/q_{j_1}})\beta m \sum_{j \in D \setminus \{j_1\}} (1/\theta_j - 1/\tau_j)_+}.$$

Next by using  $\beta$  we obtain

$$2^{-(\alpha_{j_1-1/p_{j_1}+1/q_{j_1}})\beta m} = 2^{-(\alpha_{j_1-1/p_{j_1}+1/2})\beta m \sum_{j \in D \setminus \{j_1\}} (1/2 - 1/\tau_j)_+},$$

and consequently

$$I_2(f) \leq C_4 2^{-(\alpha_{j_1-1/p_{j_1}+1/2})\beta m \sum_{j \in D \setminus \{j_1\}} (1/2 - 1/\tau_j)_+}.$$

Taking into account that  $M \asymp 2^m m^{(|D|-1)}$  we have

$$I_2(f) \leq C_5 M^{-(\alpha_{j_1-1/p_{j_1}+1/2})} (\log M)^{(|D|-1)(\alpha_{j_1-1/p_{j_1}+1/2}) + \sum_{j \in D \setminus \{j_1\}} (1/2 - 1/\tau_j)_+}. \tag{9}$$

Now, let us estimate the value  $I_1(f)$ . By using the Littlewood-Paley theorem (see [14]), we obtain

$$\begin{aligned} I_1(f) &= \left\| \sum_{m \leq (\gamma', \mathbf{s}) < \beta m} (\Delta_{\mathbf{s}}(f, \cdot) - \Delta_{\mathbf{s}}(f, \cdot) * t(\Omega_{N_{\mathbf{s}}}; \cdot)) \right\|_{L_{\mathbf{q}\theta}(\mathbb{T}^n)} \leq \\ &\leq C_6 \left( \left\| \sum_{m \leq (\gamma', \mathbf{s}) < \beta m} (\Delta_{\mathbf{s}}(f, \cdot) - \Delta_{\mathbf{s}}(f, \cdot) * t(\Omega_{N_{\mathbf{s}}}; \cdot)) \right\|_{L_{\mathbf{q}\theta}(\mathbb{T}^n)} \right)^2 \leq \\ &\leq C_6 \left\| \left\{ \left\| \Delta_{\mathbf{s}}(f, \cdot) * \left( \sum_{\mathbf{k} \in \rho(\mathbf{s})} e^{i(\mathbf{k}, \cdot)} - t(\Omega_{N_{\mathbf{s}}}; \cdot) \right) \right\|_{L_{\mathbf{q}\theta}(\mathbb{T}^n)} \right\}_{\mathbf{s} \in \aleph^n(m, \beta m, \gamma')} \right\|_{l_2} = \\ &= C_6 \left\| \left\{ \|T_{\mathbf{s}} \Delta_{\mathbf{s}}(f, \cdot)\|_{L_{\mathbf{q}\theta}(\mathbb{T}^n)} \right\}_{\mathbf{s} \in \aleph^n(m, \beta m, \gamma')} \right\|_{l_2}, \end{aligned} \tag{10}$$

where  $\aleph^n(m, \beta m, \gamma') = \{\mathbf{s} \in \mathbb{Z}_+^n : m \leq (\gamma', \mathbf{s}) < \beta m\}$ .

By using Lemma 2 and inequality of different metric for trigonometric polynomials in the Lebesgue spaces with mixed metric (see [14]) for  $1 < p_j < p_0$  ( $j = 1, \dots, n$ ), from (10) we have

$$\begin{aligned} I_1(f) &\leq C_7 \left\| \left\{ 2^{(\mathbf{1}, \mathbf{s})} N_{\mathbf{s}}^{-(1/2+1/p'_0)} \|\Delta_{\mathbf{s}}(f, \cdot)\|_{L_{\mathbf{p}_0}(\mathbb{T}^n)} \right\}_{\mathbf{s} \in \aleph^n(m, \beta m, \gamma')} \right\|_{l_2} \leq \\ &\leq C_8 \left\| \left\{ 2^{(\mathbf{1}, \mathbf{s})} N_{\mathbf{s}}^{-(1/2+1/p'_0)} 2^{(\mathbf{1}/\mathbf{p}+\mathbf{1}/\mathbf{p}_0, \mathbf{s})} \|\Delta_{\mathbf{s}}(f, \cdot)\|_{L_{\mathbf{p}}(\mathbb{T}^n)} \right\}_{\mathbf{s} \in \aleph^n(m, \beta m, \gamma')} \right\|_{l_2} = \\ &= C_8 \left\| \left\{ N_{\mathbf{s}}^{-(1/2+1/p'_0)} 2^{-(\alpha-1/\mathbf{p}+\mathbf{1}/\mathbf{p}_0-1, \mathbf{s})} \cdot 2^{(\alpha, \mathbf{s})} \|\Delta_{\mathbf{s}}(f, \cdot)\|_{L_{\mathbf{p}}(\mathbb{T}^n)} \right\}_{\mathbf{s} \in \aleph^n(m, \beta m, \gamma')} \right\|_{l_2}. \end{aligned} \tag{11}$$

According to Hölder's inequality with parameters  $\mathbf{1}/2 = \mathbf{1}/\tau + \mathbf{1}/\varepsilon$ , where  $\mathbf{1}/\varepsilon = (\mathbf{1}/2 - \mathbf{1}/\tau)_+$  and by (11) we find

$$\begin{aligned} I_1(f) &\leq C_8 \left\| \left\{ 2^{(\alpha, \mathbf{s})} \|\Delta_{\mathbf{s}}(f, \cdot)\|_{L_{\mathbf{p}}(\mathbb{T}^n)} \right\}_{\mathbf{s} \in \aleph^n(m, \beta m, \gamma')} \right\|_{l_{\tau}} \times \\ &\times \left\| \left\{ N_{\mathbf{s}}^{-(1/2+1/p'_0)} 2^{-(\alpha-1/\mathbf{p}+\mathbf{1}/\mathbf{p}_0-1, \mathbf{s})} \right\}_{\mathbf{s} \in \aleph^n(m, \beta m, \gamma')} \right\|_{l_{\varepsilon}} \leq \end{aligned}$$

$$\begin{aligned}
 &\leq C_8 \|f\|_{B_{\mathbf{p}}^{\alpha\tau}(\mathbb{T}^n)} \left\| \left\{ N_{\mathbf{s}}^{-(1/2+1/p'_0)} 2^{-(\alpha-1/\mathbf{p}+1/\mathbf{p}_0-1, \mathbf{s})} \right\}_{\mathbf{s} \in \mathbb{N}^n(m, \beta m, \gamma')} \right\|_{l_\varepsilon} \leq \\
 &\leq C_8 \left\| \left\{ N_{\mathbf{s}}^{-(1/2+1/p'_0)} 2^{-(\alpha-1/\mathbf{p}+1/\mathbf{p}_0-1, \mathbf{s})} \right\}_{\mathbf{s} \in \mathbb{N}^n(m, \beta m, \gamma')} \right\|_{l_\varepsilon} \quad (12)
 \end{aligned}$$

for any function  $f \in B_{\mathbf{p}}^{\alpha\tau}(\mathbb{T}^n)$ .

By continuing (12), according to the Lemma 4 we have

$$\begin{aligned}
 I_1(f) &\leq C_8 2^{-(1/2+1/p'_0)(\alpha_{j_1}-1/p_{j_1}+1/p_0)m} \times \\
 &\times \left\| \left\{ 2^{(1/2+1/p'_0)(\alpha-1/\mathbf{p}+1/\mathbf{p}_0-1, \mathbf{s})} \cdot 2^{-(\alpha-1/\mathbf{p}+1/\mathbf{p}_0-1, \mathbf{s})} \right\}_{\mathbf{s} \in \mathbb{N}^n(m, \beta m, \gamma')} \right\|_{l_\varepsilon} = \\
 &= C_8 2^{-(1/2+1/p'_0)(\alpha_{j_1}-1/p_{j_1}+1/p_0)m} \times \\
 &\times \left\| \left\{ 2^{-(1/2+1/p'_0)(\alpha-1/\mathbf{p}+1/\mathbf{p}_0-1, \mathbf{s})} \right\}_{\mathbf{s} \in \mathbb{N}^n(m, \beta m, \gamma')} \right\|_{l_\varepsilon} \leq \\
 &\leq C_8 2^{-(1/2+1/p'_0)(\alpha_{j_1}-1/p_{j_1}+1/p_0)m} \left\| \left\{ 2^{-(1/2+1/p'_0)(\alpha-1/\mathbf{p}+1/\mathbf{p}_0-1, \mathbf{s})} \right\}_{\mathbf{s} \in Y^n(m, \gamma')} \right\|_{l_\varepsilon} \leq \\
 &\leq C_9 2^{-(1/2+1/p'_0)(\alpha_{j_1}-1/p_{j_1}+1/p_0)m} 2^{-(1/2+1/p'_0)(\alpha_{j_1}-\frac{1}{p_{j_1}}+\frac{1}{p_0}-1)m} m^{\sum_{j \in D \setminus \{j_1\}} 1/\varepsilon_j} = \\
 &= C_9 2^{-(\alpha_{j_1}-1/p_{j_1}+1/2)m} m^{\sum_{j \in D \setminus \{j_1\}} (1/2-1/\tau_j)_+},
 \end{aligned}$$

as  $(\alpha_j - 1/p_j + 1/p_0 - 1) / (\alpha_{j_1} - 1/p_{j_1} + 1/p_0 - 1) > \gamma'_j$  at  $j \notin D$ .

Taking into account that  $M \asymp 2^m m^{(|D|-1)}$  we find

$$I_2(f) \leq C_{10} M^{-(\alpha_{j_1}-1/p_{j_1}+1/2)} (\log M)^{(|D|-1)(\alpha_{j_1}-1/p_{j_1}+1/2)+\sum_{j \in D \setminus \{j_1\}} (1/2-1/\tau_j)_+}. \quad (13)$$

Inserting (9) and (13) into (6) we obtain the inequality, which gives the upper estimate in (4).

For the proof of the lower estimate we consider the following value

$$e_M(F)_V = \sup_{f \in F} \inf_{\{b_j, \mathbf{k}_j\}_{j=1}^M} \left\| f - \sum_{j=1}^M b_j e^{i(\mathbf{k}_j, \mathbf{x})} \right\|_V,$$

which is called the best  $M$ -term approximation of the class  $F$  in metric space  $V$ .

Moreover, by the definition, the following inequality holds

$$e_M(F)_V \leq d_M^F(F, V).$$

By using the condition  $2 < q_j$  ( $j = 1, \dots, n$ ) we have

$$e_M(f)_{L_2(\mathbb{T}^n)} \leq C_{11} e_M(f)_{L_{q\theta}(\mathbb{T}^n)}.$$

For the proof of the lower estimate we will use double relation, which follows from the general results of S.M. Nikol'skii (see [15]). According to this relation for any function  $f \in L_2(\mathbb{T}^n)$  the following equality holds

$$e_M(f)_{L_2(\mathbb{T}^n)} = \inf_{\Omega_M} \sup_{P \in \mathcal{L}^\perp, \|P\|_{L_2(\mathbb{T}^n)} \leq 1} \left| \int_{\mathbb{T}^n} f(\mathbf{x}) P(\mathbf{x}) d\mathbf{x} \right|, \quad (14)$$

where  $\mathcal{L}$  is a linear span of a system of functions  $\{e^{i(\mathbf{k}, \mathbf{x})}\}_{\mathbf{k} \in \Omega_M}$ .

We consider the function

$$f(\mathbf{x}) = m^{-\sum_{j \in D' \setminus \{j_1\}} 1/\tau_j} \sum_{m \leq (\gamma', \mathbf{s}_0) < m+n} \prod_{j=1}^n 2^{-(\alpha_j+1-1/p_j)s_j^0} \sum_{\mathbf{k} \in \rho(\mathbf{s}_0)} e^{i(\mathbf{k}, \mathbf{x})},$$

where  $D' = \{j \in D : 2 < \tau_j\} \cup \{j_1\}$ ,  $\mathbf{s}_0 = (s_1^0, \dots, s_n^0)$ ,  $s_j^0 = s_j$  at  $j \in D'$  and  $s_j^0 = 0$  at  $j \notin D'$ .

In the paper [16] it was proved that the function  $C_{12}f(\mathbf{x})$  belongs to the class  $B_{\mathbf{p}}^{\alpha\tau}(\mathbb{T}^n)$ .

Let us construct the function  $P(\mathbf{x})$  satisfying the condition (14).

Let

$$u(\mathbf{x}) = \sum_{(\gamma', \mathbf{s}_0) \leq m} \sum_{\mathbf{k} \in \rho(\mathbf{s}_0)} e^{i(\mathbf{k}, \mathbf{x})}, \quad (15)$$

and  $\Omega_M$  be an arbitrary collection of integer vectors  $\mathbf{k} = (k_1, \dots, k_n)$ .

We denote by

$$v(\mathbf{x}) = \sum_{(\gamma', \mathbf{s}_0) \leq m} \sum_{\mathbf{k} \in \rho(\mathbf{s}_0) \cap \Omega_M} e^{i(\mathbf{k}, \mathbf{x})}$$

the function, containing only those terms of (15), for which  $\mathbf{k} \in \Omega_M$ . By Minkowski's inequality and Parseval's equality for function  $w(\mathbf{x}) = u(\mathbf{x}) - v(\mathbf{x})$  we have

$$\|w\|_{L_2(\mathbb{T}^n)} \leq C_{13}M^{1/2}.$$

We consider the function  $P(\mathbf{x}) = C_{13}^{-1}M^{-1/2}w(\mathbf{x})$ , then  $\|P\|_{L_2(\mathbb{T}^n)} \leq 1$ . Since the function  $w(\mathbf{x}) = u(\mathbf{x}) - v(\mathbf{x})$  does not contain the harmonics from  $\Omega_M$ , then function  $P \in \mathcal{L}^\perp$ . Thus, the function  $P(\mathbf{x})$  satisfies the conditions from (14).

According to (14) and by Lemma 5 we obtain

$$\begin{aligned} e_M(f)_{L_2(\mathbb{T}^n)} &\geq C_{14}M^{-1/2} \left| \int_{\mathbb{T}^n} f(\mathbf{x}) \omega(\mathbf{x}) d\mathbf{x} \right| \geq \\ &\geq C_{14}M^{-1/2} m^{-\sum_{j \in D' \setminus \{j_1\}} 1/\tau_j} \sum_{(\gamma', \mathbf{s}_0) = m} \prod_{j=1}^n 2^{-(\alpha_j + 1 - 1/p_j)s_j^0} \sum_{\mathbf{k} \in \rho(\mathbf{s}_0)} 1 = \\ &\geq C_{14}M^{-1/2} m^{-\sum_{j \in D' \setminus \{j_1\}} 1/\tau_j} \sum_{(\gamma', \mathbf{s}_0) = m} \prod_{j=1}^n 2^{-(\alpha_j - 1/p_j)s_j^0} = \\ &= C_{14}M^{-1/2} m^{-\sum_{j \in D' \setminus \{j_1\}} 1/\tau_j} \left\| \left\{ 2^{-(\alpha_{j_1} - 1/p_{j_1})(\mathbf{1}, \mathbf{s})} \right\}_{\mathbb{N}^{|D|}(\mathbf{1}, \mathbf{s})} \right\|_{l_1} \asymp \\ &\asymp M^{-1/2} m^{-\sum_{j \in D' \setminus \{j_1\}} 1/\tau_j} \cdot 2^{-(\alpha_{j_1} - 1/p_{j_1})m} m^{(|D|-1)}, \end{aligned} \quad (16)$$

where  $\bar{\mathbf{s}} = (s_{j_1}, \dots, s_{j_{|D|}})$ .

Taking into account that  $M \asymp 2^m m^{(|D|-1)}$  from (16) we have

$$\begin{aligned} e_M(f)_{L_2(\mathbb{T}^n)} &\geq C_{15} 2^{-(\alpha_{j_1} - 1/p_{j_1} + 1/2)m} m^{\sum_{j \in D' \setminus \{j_1\}} (1/2 - 1/\tau_j)_+} = \\ &= C_{16} M^{-(\alpha_{j_1} - 1/p_{j_1} + 1/2)} (\log M)^{(|D|-1)(\alpha_{j_1} - 1/p_{j_1} + 1/2) + \sum_{j \in D' \setminus \{j_1\}} (1/2 - 1/\tau_j)_+}. \end{aligned} \quad (17)$$

By (17) lower estimate in (4) follows.

*Remark.* Note, that for  $\mathbf{p} = (p, \dots, p)$ ,  $\tau = (\tau, \dots, \tau)$  and  $\mathbf{q} = \theta = (q, \dots, q)$  the statement of the theorem coincides with the corresponding result of A.S. Romanyuk [7].

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#### References

- 1 Ismagilov R.S. Diameter of the sets in normed linear spaces and the approximation of functions by trigonometric polynomials / R.S. Ismagilov // Russian Mathematical Surveys. — 1974. — Vol. 29, Is. 3. — P. 169–186.

- 2 Белинский Э.С. Приближение периодических функций многих переменных «плавающей» системой экспонент и тригонометрические поперечники / Э.С. Белинский // Доклады АН СССР. — 1985. — Т. 284, № 6. — С. 1294–1297.
- 3 Mayorov V.E. Trigonometric diameters of the Sobolev classes  $W_p^r$  in the space  $L_q$  / V.E. Mayorov // Mathematical notes. — 1986. — Vol. 40, Is. 2. — P. 590–597.
- 4 Makovoz Y. On trigonometric  $n$ -widths and their generalization / Y. Makovoz // Journal of approximation theory. — 1984. — Vol. 41, Is. 4. — P. 361–366.
- 5 Магарил-Ильяев Г.Г. Тригонометрические поперечники соболевских классов функций на  $R^n$  / Г.Г. Магарил-Ильяев // Труды МИАН СССР. — 1988. — Т. 181. — С. 147–155.
- 6 Temlyakov V.N. Approximations of functions with bounded mixed derivative / V.N. Temlyakov // Proc. Steklov Inst. Math. — 1989. — Vol. 178. — P. 1–121.
- 7 Romanyuk A.S. Kolmogorov and trigonometric widths of the Besov classes  $B_{p,\theta}^r$  of multivariate periodic functions / A.S. Romanyuk // Sbornik: Mathematics. — 2006. — Vol. 197, Is. 1. — P. 69–93.
- 8 Bazarkhanov D.B. Estimates for certain approximation characteristics of Nikol'skii-Besov spaces with generalized mixed smoothness / D.B. Bazarkhanov // Doklady mathematics. — 2009. — Vol. 79, Is. 3. — P. 305–308.
- 9 Bekmaganbetov K.A. On the order of the trigonometric diameter of the anisotropic Nikol'skii-Besov class in the metric anisotropic Lorentz spaces / K.A. Bekmaganbetov, Ye. Toleugazy // Analysis Mathematica. — 2019. — Vol. 45, Is. 2. — P. 237–247.
- 10 Белинский Э.С. О наименьшей величине норм смешанных производных тригонометрических полиномов с заданным числом гармоник / Э.С. Белинский, Э.М. Галеев // Вестн. МГУ. Сер. 1. Матем., мех. — 1991. — № 2. — С. 3–7.
- 11 Акишев Г. Оценки тригонометрических поперечников классов в пространстве Лоренца / Г. Акишев // Мат. журн. — 2012. — Т. 12, № 4. — С. 41–57.
- 12 Nursultanov E.D. Interpolation theorems for anisotropic function spaces and their applications / E.D. Nursultanov // Doklady mathematics. — 2004. — Vol. 69, Is. 1. — P. 16–19.
- 13 Bekmaganbetov K.A. Order of the orthoprojection widths of the anisotropic Nikol'skii – Besov classes in the anisotropic Lorentz space / K.A. Bekmaganbetov, Y. Toleugazy // Eurasian Mathematical Journal. — 2016. — Vol. 7, No. 3. — P. 8–16.
- 14 Nikol'skii S.M. Approximation of classes of functions of several variables and embedding theorems. — Springer-Verlag, Berlin, 1975. — 428 p.
- 15 Никольский С.М. Приближение функций тригонометрическими полиномами в среднем / С.М. Никольский // Изв. АН СССР. Сер. матем. — 1946. — Т. 10, № 3. — С. 207–256.
- 16 Бекмаганбетов К.А. О порядках приближения класса Бесова в метрике анизотропных пространств Лоренца / К.А. Бекмаганбетов // Уфим. матем. журн. — 2009. — Т. 1, № 2. — С. 9–16.

Қ.А. Бекмағанбетов, Қ.Е. Кервенев, Е. Төлеуғазы

## Анизотропты Лоренц кеңістігіндегі метрикасындағы аралас метрикалы Никольский-Бесов класындағы тригонометриялық көлденеңнің реті

Тригонометриялық көлденең ұғымын бірөлшемді жағдайда алғаш рет Р.С. Исмагилов енгізді және үздіксіз функциялар кеңістігінде бірқатар кластар үшін оларға бағалаулар белгіледі. Көп айнымалы функциялар үшін  $L_q$  кеңістігінде Соболевтің  $W_p^r$ , Никольскийдің  $H_p^r$  кластарындағы тригонометриялық көлденеңдер үшін дәл бағалауларды Э.С.Белинский, В.Е. Майоров, Ю. Маковоз, Г.Г. Магарил-Ильяев, В.Н. Темляков анықтады. Бұл есепті  $B_{p,q}^r$  Бесов класы үшін А.С. Романюк, Д.Б. Базарханов зерттеді. Анизотропты Никольский-Бесов  $B_{p,r}^{\alpha,\tau}(\mathbb{T}^n)$  класы үшін тригонометриялық көлденең анизотропты Лоренц кеңістіктері метрикасында К.А. Бекмаганбетов және Е. Төлеуғазымен табылды.

*Кілт сөздер:* тригонометриялық көлденең, анизотропты Лоренц кеңістіктері, аралас метрикалы Никольский-Бесов класы.

К.А. Бекмаганбетов, К.Е. Кервенов, Е. Толеугазы

## Порядок тригонометрического поперечника класса Никольского-Бесова со смешанной метрикой в метрике анизотропного пространства Лоренца

Понятие тригонометрического поперечника в одномерном случае впервые введено Р.С. Исмагиловым и им были установлены оценки для некоторых классов в пространстве непрерывных функций. Для функций многих переменных точные порядки тригонометрических поперечников класса Соболева  $W_p^r$ , Никольского  $H_p^r$  в пространстве  $L_q$  установлены Э.С.Белинским, В.Е. Майоровым, Ю. Макавозом, Г.Г. Магарил-Ильяевым, В.Н. Темляковым. Эта задача для класса Бесова  $B_{pq}^r$  исследована А.С. Романюком и Д.Б. Базархановым. Тригонометрический поперечник для анизотропного класса Никольского-Бесова  $B_{pr}^{\alpha r}(\mathbb{T}^n)$  в метрике анизотропных пространств Лоренца  $L_{q\theta}(\mathbb{T}^n)$  был найден К.А. Бекмаганбетовым и Е. Толеугазы.

*Ключевые слова:* тригонометрический поперечник, анизотропные пространства Лоренца, класс Никольского-Бесова со смешанной метрикой.

### References

- 1 Ismagilov, R.S. (1974). Diameter of the sets in normed linear spaces and the approximation of functions by trigonometric polynomials. *Russian Mathematical Surveys*, 29, 3, 169–186.
- 2 Belinsky, E.S. (1985). Priblizhenie periodicheskikh funktsii mnozhikh peremennykh «plavaiushchei» sistemoi eksponent i trihonometricheskie poperechniki [Approximation of periodic functions of several variables by «floating» system of exponentials, and trigonometric widths]. *Doklady AN SSSR – Reports of the USSR Academy of Sciences*, 284, 6, 1294–1297 [in Russian].
- 3 Mayorov, V.E. (1986). Trigonometric diameters of the Sobolev classes  $W_p^r$  in the space  $L_q$ . *Mathematical notes*, 40, 2, 590–597.
- 4 Makovoz, Y. (1984). On trigonometric  $n$ -widths and their generalization. *Journal of approximation theory*, 41, 4, 361–366.
- 5 Magaril-Il'yaev, G.G. (1988). Trihonometricheskie poperechniki sobolevskikh klassov funktsii na  $R^n$  [Trigonometric widths of Sobolev classes of function on  $R^n$ ]. *Trudy Matematicheskoho Instituta imeni V.A. Steklova – Proceedings V.A. Steklov Math. Inst.*, 181, 147–155 [in Russian].
- 6 Temlyakov, V.N. (1989). Approximations of functions with bounded mixed derivative. *Proc. Steklov Inst. Math.*, 178, 1–121.
- 7 Romanyuk, A.S. (2006). Kolmogorov and trigonometric widths of the Besov classes  $B_{p,\theta}^r$  of multivariate periodic functions. *Sbornik: Mathematics*, 197, 1, 69–93.
- 8 Bazarkhanov, D.B. (2009). Estimates for certain approximation characteristics of Nikol'skii-Besov spaces with generalized mixed smoothness. *Doklady mathematics*, 79, 3, 305–308.
- 9 Bekmaganbetov, K.A. & Toleugazy, Y. (2019). On the order of the trigonometric diameter of the anisotropic Nikol'skii-Besov class in the metric anisotropic Lorentz spaces. *Analysis Mathematica*, 45, 2, 237–247.
- 10 Belinskii, E.S. & Galeev, E.M. (1991). O naimenshei velichine norm smeshannykh proizvodnykh trihonometricheskikh polinomov s zadannym chislom harmonik [On the smallest value of mixed derivatives of trigonometric polynomials with fixed number of harmonics]. *Vestnik Moskovskogo universiteta. Seriya 1: Matematika. Mekhanika – Moscow University Bulletin. Ser. 1. Math., mech.*, 2, 3–7 [in Russian].
- 11 Akishev, G. (2012). Otsenki trihonometricheskikh poperechnikov klassov v prostranstve Lorentsa [Estimates of trigonometric width of classes in the Lorentz space]. *Matematicheskii zhurnal – Mathematical Journal*, 12, 4, 41–57 [in Russian].
- 12 Nursultanov, E.D. (2004). Interpolation theorems for anisotropic function spaces and their applications. *Doklady mathematics*, 69, 1, 16–19.
- 13 Bekmaganbetov, K.A. & Toleugazy, Y. (2016). Order of the orthoprojection widths of the anisotropic Nikol'skii-Besov classes in the anisotropic Lorentz space. *Eurasian Mathematical Journal*, 7, 3, 8–16.

- 14 Nikol'skii, S.M. (1975). *Approximation of classes of functions of several variables and embedding theorems*. Berlin: Springer-Verlag.
- 15 Nikol'skii, S.M. (1946). Priblizhenie funktsii trihonometricheskimi polinomami v srednem [Approximation of functions in the mean by trigonometrical polynomials]. *Izvestiia Akademii nauk SSSR. Serii Matematicheskaiia – Izvestiia: Mathematics*, 10, 3, 207–256 [in Russian].
- 16 Bekmaganbetov, K.A. (2009). O poriadkakh priblizheniia klassa Besova v metrike anizotropnykh prostanstv Lorentsa [About order of approximation of Besov classes in metric of anisotropic Lorentz spaces]. *Ufinskii Matematicheskii zhurnal – Ufa Mathematical Journal*, 1, 2, 9–16 [in Russian].

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