

Ö. Özer¹, D. Bellaouar²

¹Kırklareli University, Turkey;

²University 08 Mai 1945 Guelma, Guelma, Algeria
(E-mail: ozenozer39@gmail.com)

Some results on a special type of real quadratic fields

In this paper, we determine the real quadratic fields $Q(\sqrt{d})$ coincide with positive square-free integers d including the continued fraction expansion form of $w_d = \left[a_0; \underbrace{\overline{t_1, t_2, \dots, t_{\ell-1}}}_{\ell-1}, a_\ell \right]$. Furthermore, we deal with determining fundamental units and Yokoi's d -invariants n_d and m_d in the relation to continued fraction expansion of w_d where $\ell(d)$ is a period length of w_d for the such type of real quadratic number fields $Q(\sqrt{d})$. The present paper improve the theory of fundamental unit which generates the unit group of real quadratic fields and also determine the special form of continued fraction expansion of integral basis element in real quadratic fields.

Keywords: continued fraction, real quadratic fields, fundamental unit, Yokoi's invariants, integer sequences, integer basic element.

1 Introduction

Some relations among i -th approachment of quadratic irrationals were proved by Elezović in [1]. Jeongho obtained lower bound for regulator of real quadratic fields by considering quadratic integers with fixed norm in [2]. In [3], Benamar et al. described polynomials and also gave lower bound of the number of some types of polynomials. Badziahin and Shallit confirmed the conjecture of Hanna and Wilson by considering specific type of continued fraction of real numbers and got some results on transcendental numbers in [4]. Zhang and Yue [5] described some congruences relations between the coefficients of fundamental unit of real quadratic fields and odd class number. Also, Tomita [6] gave some results on fundamental unit by use of the continued fraction expansion of integral basis element where period length is equal to 3. Clemens and his co-authors [7] explored some relationship between continued fraction expansion and infinite series representation for real numbers. Louboutin [8] obtained significant results on principal or non principal real quadratic fields as well as significant conditions for principality of continued fraction expansion. Tomita and Kawamoto [9] showed a relation between real quadratic fields of class number one and minimal type of the simple continued fraction expansion of certain quadratic irrationals. Both Sasaki [10] and Mollin [11] achieved many useful results on lower bound of fundamental unit for real quadratic number fields. Williams and Buck [12] got comparison between period length of \sqrt{d} and $\frac{1+\sqrt{d}}{2}$. Besides, first author in this paper obtained some special results for different forms of continued fraction expansion of w_d in [13] and [14] where $d \equiv 2, 3 \pmod{4}$ is square-free positive integer. Also, in [15] she got significant results for varied types of continued fraction expansion of w_d where $d \equiv 1 \pmod{4}$ or $d \equiv 2, 3 \pmod{4}$. Yokoi defined n_d and m_d invariants important for class number problem [16–19]. Readers unfamiliar fundamental unit and continued fraction expansions are referred to books [11, 20–23].

Throughout this paper, $I(d)$ is the set of all quadratic irrational numbers in $Q(\sqrt{d})$, we say that α in $I(d)$ is reduced if $\alpha > 1$ and $-1 < \alpha' < 1$ where α' is the conjugate of α and denoted by $R(d)$ is the set includes of all reduced quadratic irrational numbers in $I(d)$. Then, it is well known that any number α in $R(d)$ is purely periodic in the continued fraction expansion and the denominator of its modular automorphism is equal to fundamental unit ϵ_d of $Q(\sqrt{d})$. Yokoi's invariants are expressed by $m_d = \left[\left[\begin{array}{c} u_d^2 \\ t_d \end{array} \right] \right]$ and $n_d = \left[\left[\begin{array}{c} t_d \\ u_d^2 \end{array} \right] \right]$. it is also well known $[[x]]$ represents the floor of x for any number x .

In this paper, we deal with the problem for demonstrating the continued fraction expansions which have got partial constant elements equal each others and written as $7s$ (except the last digit of the period) according to period length for d square-free integer (for $d \equiv 1 \pmod{4}$ or $d \equiv 2, 3 \pmod{4}$).

Moreover, we demonstrate the general parametrization of d square-free positive integer and fundamental unit ϵ_d as well as t_d, u_d the coefficients of fundamental unit. Additionally, we get fix on Yokoi's invariants for such types of real quadratic fields. Then, we give some results on fundamental units, continued fraction expansion and Yokoi's invariants with numerical Tables.

2 Preliminaries and Basic Results

In this section, readers can find some basic and important definitions and theorems for using our new results.

Definition 2.1. $\{\tau_i\}$ is a sequence defined by the following recurrence relation

$$\tau_i = 7\tau_{i-1} + \tau_{i-2}$$

for $i \geq 2$ where $\tau_0 = 0$ and $\tau_1 = 1$.

Lemma 2.2. [6]. For a square-free positive integer d congruent to 1 modulo 4, let $\omega_d = \frac{1 + \sqrt{d}}{2}$, $a_0 = [[\omega_d]]$, $\omega_R = a_0 - 1 + \omega_d$. Then $\omega_d \notin R(d)$, but $\omega_R \in R(d)$ holds. Moreover for the period $l = l(d)$ of ω_R , we get $\omega_R = [2a_0 - 1, a_1, \dots, a_{l-1}]$ and $\omega_d = [a_0, a_1, \dots, a_{l-1}, 2a_0 - 1]$. Furthermore, let $\omega_R = \frac{(P_l \omega_R + P_{l-1})}{(Q_l \omega_R + Q_{l-1})} = [2a_0 - 1, a_1, \dots, a_{l-1}, \omega_R]$ be a modular automorphism of ω_R , then the fundamental unit ϵ_d of $Q(\sqrt{d})$ is given by the following formula

$$\epsilon_d = \frac{t_d + u_d \sqrt{d}}{2}$$

and

$$t_d = (2a_0 - 1) \cdot Q_{l(d)} + 2Q_{l(d)-1}, \quad u_d = Q_{l(d)},$$

where Q_i is determined by $Q_0 = 0, Q_1 = 1$ and $Q_{i+1} = a_i Q_i + Q_{i-1}$ ($i \geq 1$).

Lemma 2.3. For a square-free positive integer d congruent to 2, 3 modulo 4, let $\omega_d = \sqrt{d}$, $a_0 = [[\omega_d]]$, $\omega_R = a_0 + \omega_d$. Then $\omega_d \notin R(d)$, but $\omega_R \in R(d)$ holds. Moreover for the period $l = l(d)$ of ω_R , we get $\omega_R = [2a_0, a_1, \dots, a_{l-1}]$ and $\omega_d = [a_0, a_1, \dots, a_{l-1}, 2a_0]$. Furthermore, let $\omega_R = \frac{(P_l \omega_R + P_{l-1})}{(Q_l \omega_R + Q_{l-1})} = [2a_0, a_1, \dots, a_{l-1}, \omega_R]$ be a modular automorphism of ω_R , then the fundamental unit ϵ_d of $Q(\sqrt{d})$ is given by the following formula:

$$\epsilon_d = \frac{t_d + u_d \sqrt{d}}{2} = (a_0 + \sqrt{d})Q_{l(d)} + Q_{l(d)-1}$$

and

$$t_d = 2a_0 \cdot Q_{l(d)} + 2Q_{l(d)-1}, \quad u_d = 2Q_{l(d)},$$

where Q_i is determined by $Q_0 = 0, Q_1 = 1$ and $Q_{i+1} = a_i Q_i + Q_{i-1}$, ($i \geq 1$).

Remark 2.4. Let $\{\tau_n\}$ be a sequence defined as in Definition 2.1. Then, we state that:

$$\tau_n \equiv \begin{cases} 0 \pmod{4}, & n \equiv 0 \pmod{6}; \\ 1 \pmod{4}, & n \equiv 1, 4, 5 \pmod{6}; \\ 3 \pmod{4}, & n \equiv 2 \pmod{6}; \\ 2 \pmod{4}, & n \equiv 3 \pmod{6}, \end{cases}$$

where $n \geq 0$.

3 Main Theorems and Results

In this section, we present our results as follows:

Theorem 3.1. Let d be the square-free positive integer and ℓ be a positive integer holding that ℓ is different from $0 \pmod{3}$ and $\ell > 1$. We assume that parametrization of d is

$$d = \frac{(7 + (2t + 1)\tau_\ell)^2}{4} + ((2t + 1)\tau_{\ell-1}) + 1$$

for $t \geq 0$ positive integer. In this case, we get following:

- (1) If $\ell \equiv 1(mod6)$ and $t \equiv 1(mod2)$ positive integer then $d \equiv 2(mod4)$ holds.
- (2) If $\ell \equiv 2(mod6)$ and $t \equiv 0(mod2)$ positive integer then $d \equiv 3(mod4)$ holds.
- (3) If $\ell \equiv 4(mod6)$ and $t \equiv 0(mod2)$ positive integer then $d \equiv 3(mod4)$ holds.
- (4) If $\ell \equiv 5(mod6)$ and $t \equiv 0(mod2)$ positive integer then $d \equiv 2(mod4)$ holds.

In this case, we obtain

$$w_d = \left[\frac{(2t+1)\tau_\ell + 7}{2}; \underbrace{7, 7, \dots, 7}_{\ell-1}, (2t+1)\tau_\ell + 7 \right]$$

and $\ell = \ell(d)$. Moreover, we have following equations:

$$\begin{aligned} \epsilon_d &= \left(\frac{(2t+1)\tau_\ell^2}{2} + \frac{7\tau_\ell}{2} + \tau_{\ell-1} \right) + \tau_\ell \sqrt{d}; \\ t_d &= (2t+1)\tau_\ell^2 + 7\tau_\ell + 2\tau_{\ell-1} \quad \text{and} \quad u_d = 2\tau_\ell \end{aligned}$$

for ϵ_d, t_d and u_d .

Remark 3.2. Note that d is not integer for $\ell \equiv 0(mod3)$. That's why we assume that ℓ different from $0(mod3)$.

Proof. We assume that $\ell \equiv 1(mod6)$ positive odd integer, $\ell > 1$ and $t \equiv 1(mod2)$ positive integer. So, we can get $d \equiv 2(mod4)$ by substituting the equivalents into the parametrization of d . We can easily obtain the other cases in a similar way. By using Lemma 2.3 , we put

$$w_R = \frac{(2t+1)\tau_\ell + 7}{2} + \left[\frac{(2t+1)\tau_\ell + 7}{2}; \underbrace{7, 7, \dots, 7}_{\ell-1}, (2t+1)\tau_\ell + 7 \right],$$

so we get

$$\begin{aligned} w_R &= ((2t+1)\tau_\ell + 7) + \frac{1}{7 + \frac{1}{7 + \frac{1}{\dots}}} = ((2t+1)\tau_\ell + 7) + \frac{1}{7} + \dots + \frac{1}{w_R} + \\ &+ \frac{1}{7 + \frac{1}{w_R}} \end{aligned}$$

By induction, we get

$$w_R = ((2t+1)\tau_\ell + 7) + \frac{\tau_{\ell-1}w_R + \tau_{\ell-2}}{\tau_\ell w_R + \tau_{\ell-1}}.$$

If we rearrange and use the Definition 2.1 into the above equality, we have

$$w_R^2 - ((2t+1)\tau_\ell + 7)w_R - (1 + (2t+1)\tau_{\ell-1}) = 0.$$

This requires that $w_R = \frac{(2t+1)\tau_\ell + 7}{2} + \sqrt{d}$ since $w_R > 0$. If we consider Lemma 2.3, we get

$$w_d = \sqrt{d} = \left[\frac{(2t+1)\tau_\ell + 7}{2}; \underbrace{7, 7, \dots, 7}_{\ell-1}, (2t+1)\tau_\ell + 7 \right]$$

and $\ell = \ell(d)$. This shows that the first part of proof is completed.

Now, we should determine ϵ_d, t_d and u_d using Lemma 2.3, we get

$$Q_1 = 1 = \tau_1, \quad Q_2 = a_1.Q_1 + Q_0 \Rightarrow Q_2 = 7 = \tau_2;$$

$$Q_3 = a_2.Q_2 + Q_1 = 7\tau_2 + \tau_1 = 7^2 + 1 = 50 = \tau_3, \quad Q_4 = \tau_4, \dots$$

So, this implies that $Q_i = \tau_i$ by using mathematical induction for every $i \geq 0$. If we substitute these values of sequence into the $\epsilon_d = \frac{t_d + u_d \sqrt{d}}{2} = (a_0 + \sqrt{d})Q_{l(d)} + Q_{l(d)-1}$ and rearrange, we have

$$\epsilon_d = \left(\frac{(2t+1)\tau_\ell^2}{2} + \frac{7\tau_\ell}{2} + \tau_{\ell-1} \right) + \tau_\ell \sqrt{d};$$

$$t_d = (2t + 1)\tau_\ell^2 + 7\tau_\ell + 2\tau_{\ell-1} \quad \text{and} \quad u_d = 2\tau_\ell$$

for ϵ_d, t_d and u_d . So, we complete the proof of Theorem 3.1.

Corollary 3.3. Let d be a square-free positive integer and ℓ be a positive integer holding that $\ell > 1$ different from $0 \pmod{3}$. We assume that parametrization of d is

$$d = \frac{(7 + \tau_\ell)^2}{4} + \tau_{\ell-1} + 1$$

then we obtain $d \equiv 2, 3 \pmod{4}$ and

$$w_d = \left[\frac{\tau_\ell + 7}{2}; \underbrace{7, 7, \dots, 7}_{\ell-1}, \tau_\ell + 7 \right]$$

and $\ell = \ell(d)$. Besides, we get following equalities

$$\begin{aligned} \epsilon_d &= \left(\frac{\tau_\ell^2}{2} + \frac{7\tau_\ell}{2} + \tau_{\ell-1} \right) + \tau_\ell \sqrt{d}; \\ t_d &= \tau_\ell^2 + 7\tau_\ell + 2\tau_{\ell-1}; \quad \text{and} \quad u_d = 2\tau_\ell; \\ m_d &= \begin{cases} 1, & \text{if } \ell = 2; \\ 3, & \text{if } \ell > 4 \end{cases} \end{aligned}$$

for ϵ_d, t_d, u_d and Yokoi's invariant m_d .

Furthermore, we prepare Table 1 where fundamental unit is ϵ_d , integral basis element is w_d and Yokoi's invariant is m_d for $2 < \ell(d) \leq 11$. (In this Table, we will rule out $\ell(d) = 4, 8, 10$ since d is not a square-free positive integer with these periods. Besides, d is not congruent to 2 or 3 (modulo 4) for $\ell(d) = 7$).

Table 1

Square-free positive integers d with $2 \leq \ell(d) \leq 11$

d	$\ell(d)$	m_d	w_d	ϵ_d
51	2	1	$[7; 7, 14]$	$50 + 7\sqrt{51}$
1633642	5	3	$[1278; 7, 7, 7, 7, 2556]$	$3257979 + 2549\sqrt{1633642}$
28516639237941410	11	3	$[168868704; 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 337737408]$	$57033277246500097 + 337737401\sqrt{28516639237941410}$

Proof. This corollary is gotten if we substitute $t = 0$ in Theorem 3.1. Now, we have to prove that

$$m_d = \begin{cases} 1, & \text{if } \ell = 2; \\ 3, & \text{if } \ell > 4. \end{cases}$$

If we put t_d and u_d into the m_d and rearrange, then we obtain

$$m_d = \left[\left[\frac{u_d^2}{t_d} \right] \right] = \left[\left[\frac{4\tau_\ell^2}{\tau_\ell^2 + 7\tau_\ell + 2\tau_{\ell-1}} \right] \right].$$

By using the above equality, we have $m_d = 1$ for $\ell = 2$. From the assumption since τ_ℓ is increasing we get,

$$4 > 4 \cdot \left(1 + \frac{7}{\tau_\ell} + \frac{2\tau_{\ell-1}}{\tau_\ell^2} \right)^{-1} > 3,988$$

for $\ell > 4$. Therefore, we obtain $m_d = 3$ for $\ell > 4$ and we have $m_d = \begin{cases} 1, & \text{if } \ell = 2; \\ 3, & \text{if } \ell > 4. \end{cases}$ which completes the proof of Corollary 3.3.

Furthermore, Table 1 is gotten as a numerical results of the corollary.

Corollary 3.4. Let d be the square-free positive integer and ℓ be a positive integer such that ℓ is not congruent to 0 as for $(\text{mod}3)$, $\ell > 1$. We suppose that parametrization of d is

$$d = \frac{(7 + 3\tau_\ell)^2}{4} + 3\tau_{\ell-1} + 1$$

then, we obtain $d \equiv 2(\text{mod}4)$ and

$$w_d = \left[\frac{3\tau_\ell + 7}{2}; \underbrace{\overline{7, 7, \dots, 7}}_{\ell-1}, 3\tau_\ell + 7 \right]$$

and $\ell = \ell(d)$. Moreover, we have following equalities:

$$\epsilon_d = \left(\frac{3\tau_\ell^2}{2} + \frac{7\tau_\ell}{2} + \tau_{\ell-1} \right) + \tau_\ell \sqrt{d};$$

$$t_d = 3\tau_\ell^2 + 7\tau_\ell + 2\tau_{\ell-1} \quad \text{and} \quad u_d = 2\tau_\ell$$

$m_d = 1$ for $\ell \geq 7$ for ϵ_d, t_d, u_d and Yokoi's invariant m_d . Besides, we state the following Table 2, where fundamental unit is ϵ_d , integral basis element is w_d and Yokoi's invariant is m_d for $2 < \ell(d) \leq 13$.

Table 2

Square-free positive integers d with $\ell(d) = 7$ or $\ell(d) = 13$

d	$\ell(d)$	m_d	w_d	ϵ_d
37996589930	7	1	[194927; $\overline{7, 7, 7, 7, 7, 7, 7}, 389854]$	$25330586923 + 129949\sqrt{37996589930}$
667032581096785339826	13	1	[25826973905; $\overline{7, 7, \dots, 7, 51653947810}$]	$444688387335182490505 + 17217982601\sqrt{667032581096785339826}$

Proof. Corollary is obtained if we substitute $t = 1$ in Theorem 3.1. We should prove that $m_d = 1$ for $\ell \geq 7$. If we put t_d and u_d into the m_d and rearrange, then we obtain

$$2 > 4 \cdot \left(3 + \frac{7}{\tau_\ell} + \frac{2\tau_{\ell-1}}{\tau_\ell^2} \right)^{-1} > 1,333$$

for $\ell \geq 7$ since τ_ℓ is increasing sequence. By using the above equality, we have $m_d = 1$ for $\ell \geq 7$. Also, Table 2 is given as an illustration of this corollary.

Corollary 3.5. We assume that d and ℓ are defined as in Theorem 3.1. If we choose the parametrization of d as

$$d = \frac{(7 + 5\tau_\ell)^2}{4} + 5\tau_{\ell-1} + 1$$

then $d \equiv 2, 3(\text{mod}4)$ and

$$w_d = \left[\frac{5\tau_\ell + 7}{2}; \underbrace{\overline{7, 7, \dots, 7}}_{\ell-1}, 5\tau_\ell + 7 \right]$$

with $\ell = \ell(d)$. Also, we have the following equalities:

$$\epsilon_d = \left(\frac{5\tau_\ell^2}{2} + \frac{7\tau_\ell}{2} + \tau_{\ell-1} \right) + \tau_\ell \sqrt{d};$$

$$t_d = 5\tau_\ell^2 + 7\tau_\ell + 2\tau_{\ell-1} \quad \text{and} \quad u_d = 2\tau_\ell,$$

$$n_d = 1$$

for $\ell \geq 2$.

Additionally, we prepare the Table 3 where fundamental unit is ϵ_d , integral basis element is w_d and and Yokoi's invariant is n_d for $2 \leq \ell(d) \leq 11$. (In this table, we rule out both $\ell(d) = 10$ since d is not a square-free positive integer in this period and also d is not congruent to 2 or 3 (modulo 4) for $\ell(d) = 7$).

Table 3

Square-free positive integers d with $2 \leq \ell(d) \leq 11$

d	$\ell(d)$	n_d	w_d	ϵ_d
447	2	1	[21; $\overline{7, 42}$]	$148 + 7\sqrt{447}$
803067	4	1	[896; $\overline{7, 7, 7, 1792}$]	$319922 + 357\sqrt{803067}$
40655162	5	1	[6376; $\overline{7, 7, 7, 7, 12752}$]	$16252781 + 2549\sqrt{40655162}$
5380595841067	8	1	[2319611; $\overline{7, 7, 7, 7, 7, 7, 4639222}$]	$2152234959022 + 927843\sqrt{5380595841067}$
712915956360881002	11	1	[844343506; $\overline{7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 1688687012}$]	$285166381314969699 + 337737401\sqrt{712915956360881002}$

Proof. It is gotten if we substitute $t = 2$ in Theorem 3.1. Let's prove that Yokoi's d- invariant is $n_d = 1$ for $\ell \geq 2$.

We know from H. Yokoi's references [16–19] that $n_d = \left[\left[\frac{t_d}{u_d^2} \right] \right]$. If we substitute t_d and u_d into n_d , then we get

$$n_d = \left[\left[\frac{t_d}{u_d^2} \right] \right] = \left[\left[\frac{5\tau_\ell^2 + 7\tau_\ell + 2\tau_{\ell-1}}{4\tau_\ell^2} \right] \right] = 1,$$

since τ_ℓ is increasing and $1 < \frac{5}{4} + \frac{7}{4\tau_\ell} + \frac{\tau_{\ell-1}}{2\tau_\ell^2} < 1,510$ for $\ell \geq 2$. Therefore, we obtain $n_d = 1$ for $\ell \geq 2$. As illustration, we give Table 3.

Theorem 3.6. Let d be a square-free positive integer and $\ell > 1$ be a positive integer.

(i) We suppose

$$d = (2t\tau_\ell + 7)^2 + 8t\tau_{\ell-1} + 4$$

for $t > 0$ positive integer. In this case, we obtain that $d \equiv 1 \pmod{4}$ and

$$w_d = \left[t\tau_\ell + 4; \underbrace{\overline{7, 7, \dots, 7}}_{\ell-1}, 2t\tau_\ell + 7 \right]$$

and $\ell = \ell(d)$. Moreover, in this case it holds

$$t_d = 2t\tau_\ell^2 + 7\tau_\ell + 2\tau_{\ell-1} \quad \text{and} \quad u_d = \tau_\ell$$

for $\epsilon_d = \frac{t_d + u_d\sqrt{d}}{2}$.

(ii) If $\ell \equiv 0 \pmod{3}$ and

$$d = (t\tau_\ell + 7)^2 + 4t\tau_{\ell-1} + 4$$

for $t > 0$ positive odd integer then $d \equiv 1 \pmod{4}$ and

$$w_d = \left[\frac{t}{2}\tau_\ell + 4; \underbrace{\overline{7, 7, \dots, 7}}_{\ell-1}, t\tau_\ell + 7 \right]$$

and $\ell = \ell(d)$. Furthermore, in this case

$$t_d = t\tau_\ell^2 + 7\tau_\ell + 2\tau_{\ell-1} \quad \text{and} \quad u_d = \tau_\ell$$

hold for $\epsilon_d = \frac{t_d + u_d\sqrt{d}}{2}$.

Remark 3.7. For the case (ii) in Theorem 3.6, it is clear that τ_ℓ is odd number if ℓ is not divided by 3. For the case (ii), assume that ℓ is not divided by 3. Then we get d is not integer if we put t positive odd integer into the parametrization of d . Besides, if we consider t is positive even integer, then the parametrization of d will be the case (i). So, we approve that $\ell \equiv 0(\text{mod}3)$ and t is positive odd integer in the case (ii).

Proof. (1) It is clear that $d \equiv 1(\text{mod}4)$ holds since $(2t\tau_\ell + 7)^2$ is odd integer for any $t > 0$ and $\ell > 1$ positive integers. We prove the theorem in a similar to Theorem 3.1. From Lemma 2.2, we know that $\omega_d = \frac{1 + \sqrt{d}}{2}$, $a_0 = [[\omega_d]]$, $\omega_R = a_0 - 1 + \omega_d$.

Considering above equations, we have

$$w_R = t\tau_\ell + 3 + \left[t\tau_\ell + 4; \underbrace{7, 7, \dots, 7}_{\ell-1}, 2t\tau_\ell + 7 \right]$$

so we get

$$w_R = (2t\tau_\ell + 7) + \frac{1}{7 + \frac{1}{7 + \frac{1}{\ddots + \frac{1}{7 + \frac{1}{w_R}}}}}$$

Rearranging and using Lemma 2.2 with Definition 2.1 into the above equality, we obtain

$$w_R^2 - (2t\tau_\ell + 7)w_R - (1 + 2t\tau_{\ell-1}) = 0.$$

This requires that $w_R = (t\tau_\ell + 4) - 1 + \frac{1 + \sqrt{d}}{2}$ since $w_R > 0$. If we consider Lemma 2.1, we get

$$w_d = \left[t\tau_\ell + 4; \underbrace{7, 7, \dots, 7}_{\ell-1}, 2t\tau_\ell + 7 \right]$$

and $\ell = \ell(d)$.

We obtained that $Q_i = \tau_i$ by using mathematical induction for all $i \geq 0$ in Theorem 3.1. Now, we get t_d and u_d using Lemma 2.2 as follows

$$t_d = 2t\tau_\ell^2 + 7\tau_\ell + 2\tau_{\ell-1} \quad \text{and} \quad u_d = \tau_\ell$$

for $\epsilon_d = \frac{t_d + u_d\sqrt{d}}{2}$. This shows that the first part of proof is completed.

(2) If we assume that $\ell \equiv 0(\text{mod}3)$ and the parametrization of

$$d = (t\tau_\ell + 7)^2 + 4t\tau_{\ell-1} + 4$$

for $t > 0$, then we have $d \equiv 1(\text{mod}4)$ since τ_ℓ is even integer. By taking $\frac{t}{2}$ instead of t into the case (1), we get

$$w_d = \left[\frac{t}{2}\tau_\ell + 4; \underbrace{7, 7, \dots, 7}_{\ell-1}, t\tau_\ell + 7 \right]$$

and $\ell = \ell(d)$ for $\ell \equiv 0(\text{mod}3)$. Furthermore,

$$t_d = t\tau_\ell^2 + 7\tau_\ell + 2\tau_{\ell-1} \quad \text{and} \quad u_d = \tau_\ell$$

hold for $\epsilon_d = \frac{t_d + u_d\sqrt{d}}{2}$ which completes the proof.

Corollary 3.8. Let d be the square-free positive integer and $\ell > 1$ is a positive integer. We assume that parametrization of d is

$$d = (2\tau_\ell + 7)^2 + 8\tau_{\ell-1} + 4$$

then we obtain $d \equiv 1 \pmod{4}$ and

$$w_d = \left[\tau_\ell + 4; \underbrace{\overline{7, 7, \dots, 7}}_{\ell-1}, 2\tau_\ell + 7 \right]$$

and $\ell = \ell(d)$. Moreover, we have

$$t_d = 2\tau_\ell^2 + 7\tau_\ell + 2\tau_{\ell-1} \quad \text{and} \quad u_d = \tau_\ell$$

for $\epsilon_d = \frac{t_d + u_d\sqrt{d}}{2}$ and, Yokoi's invariant n_d is determined as follows:

$$n_d = \begin{cases} 3, & \text{if } \ell = 2; \\ 2, & \text{if } \ell > 2. \end{cases}$$

Also, we give Table 4 where fundamental unit is ϵ_d , integral basis element is w_d and Yokoi's invariant is n_d for $2 \leq \ell(d) \leq 11$. (In this table, we rule out $\ell(d) = 4, 5, 10$ since d is not a square-free positive integer with these periods).

Table 4

Square-free positive integers d with $2 \leq \ell(d) \leq 11$

d	$\ell(d)$	n_d	w_d	ϵ_d
453	2	3	$[11; \overline{7, 21}]$	$(149 + 7\sqrt{453})/2$
11509	3	2	$[54; \overline{7, 7, 107}]$	$(5364 + 50\sqrt{11509})/2$
1325490045	6	2	$[18204; \overline{7, 7, \dots, 7, 36407}]$	$(662612498 + 18200\sqrt{1325490045})/2$
67550754629	7	2	$[129953; \overline{7, 7, \dots, 7, 259905}]$	$(33774431245 + 129949\sqrt{67550754629})/2$
3443597549845	8	2	$[927847; \overline{7, 7, \dots, 7, 1855693}]$	$(1721792020097 + 927843\sqrt{3443597549845})/2$
175554743008597	9	2	$[6624854; \overline{7, 7, \dots, 7, 13249707}]$	$(87777323274636 + 6624850\sqrt{175554743008597})/2$
456266217972000829	11	2	$[337737405; \overline{7, 7, \dots, 7, 675474809}]$	$(228133106527234995 + 337737401\sqrt{456266217972000829})/2$

Proof. The corollary is had if we substitute $t = 1$ into the case (1) in Theorem 3.6. Let's show that

$$n_d = \begin{cases} 3, & \text{if } \ell = 2; \\ 2, & \text{if } \ell > 2. \end{cases}$$

If we put t_d and u_d into the n_d and rearrange, then we obtain

$$n_d = \left[\left[\begin{array}{c} t_d \\ u_d^2 \end{array} \right] \right] = \left[\left[\begin{array}{c} \frac{2\tau_\ell^2 + 7\tau_\ell + 2\tau_{\ell-1}}{\tau_\ell^2} \end{array} \right] \right].$$

By using the above equality, we have $n_d = 3$ for $\ell = 2$. From the assumption since τ_ℓ is increasing sequence, we get,

$$2, 1456 \geq \left(2 + \frac{7}{\tau_\ell} + \frac{2\tau_{\ell-1}}{\tau_\ell^2} \right) > 2$$

for $\ell > 2$. Therefore, we obtain $n_d = \begin{cases} 3, & \text{if } \ell = 2; \\ 2, & \text{if } \ell > 2. \end{cases}$ Then Corollary 3.8 is proved. To give numerical examples for Corollary 3.8, we prepare Table 4.

Corollary 3.9. Let d be the square-free positive integer and $\ell > 1$ is a positive integer holding that $\ell \equiv 0 \pmod{3}$. We assume that parametrization of d is

$$d = (\tau_\ell + 7)^2 + 4\tau_{\ell-1} + 4$$

then we have $d \equiv 1 \pmod{4}$ and

$$w_d = \left[\frac{\tau_\ell}{2} + 4; \underbrace{\overline{7, 7, \dots, 7}}_{\ell-1}, \tau_\ell + 7 \right]$$

and $\ell = \ell(d)$. Moreover, we have

$$t_d = \tau_\ell^2 + 7\tau_\ell + 2\tau_{\ell-1} \quad \text{and} \quad u_d = \tau_\ell$$

for $\epsilon_d = \frac{t_d + u_d\sqrt{d}}{2}$ and, Yokoi's invariant $n_d = 1$.

Besides, we prepare Table 5 where fundamental unit is ϵ_d , integral basis element is w_d and Yokoi's invariant is n_d for $3 \leq \ell(d) \leq 12$ (in this Table, we rule out $\ell(d) = 9$ since d is not a square-free positive integer with these periods).

Table 5

Square-free positive integers d with $3 \leq \ell(d) \leq 12$

d	$\ell(d)$	n_d	w_d	ϵ_d
3281	3	1	[29; $\overline{7, 7, 57}$]	$(2864 + 50\sqrt{3281})/2$
331505049	6	1	[9104; $\overline{7, 7, \dots, 7, 18207}$]	$(331372498 + 18200\sqrt{331505049})/2$
58151567292136400057	12	1	[1205731804; $\overline{7, 7, \dots, 7, 2411463607}$]	$(5815156711680680002 + 2411463600\sqrt{58151567292136400057})/2$

Proof. If we substitute $t = 1$ into the case (2) in Theorem 3.6, we have the Corollary 3.9. Let's prove that $n_d = 1$. If we put t_d and u_d into the n_d and rearrange, then we obtain

$$n_d = \left[\left[\begin{array}{c} t_d \\ u_d^2 \end{array} \right] \right] = \left[\left[\begin{array}{c} \tau_\ell^2 + 7\tau_\ell + 2\tau_{\ell-1} \\ \tau_\ell^2 \end{array} \right] \right].$$

From the assumption since τ_ℓ is increasing sequence, we get

$$1, 1416 \geq \left(1 + \frac{7}{\tau_\ell} + \frac{2\tau_{\ell-1}}{\tau_\ell^2} \right) > 1$$

for $\ell \geq 3$ which completes the proof of the Corollary 3.9. We prepare Table 5 as an numerical illustrations of Corollary 3.9.

Corollary 3.10. Let d be a square-free positive integer congruent to modulo 4. If we suppose that d is holding the conditions of Theorem 3.6, then always satisfy that Yokoi's invariant n_d is different from zero. It means that $m_d = 0$.

Proof. It can be proven as similar of Corollary 3.2 in [15].

Remark 3.11. We should say that the present paper has got the most general theorems for given type real quadratic fields. Also, we can obtain infinitely many values of d which corresponds to $Q(\sqrt{d})$ and determine the structures of such fields by using our results.

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Ö. Öзер, Д. Беллауар

Нақты квадраттық өрістің арнаулы түрі бойынша кейбір нәтижелер

Мақалада нақты квадраттық өрістер бүтін еркін оң $w_d = \left[a_0; \underbrace{7, 7, \dots, 7}_{\ell-1}, a_\ell \right]$ тізбекті бөлшектің жік-

телу түрімен қоса, d квадраттарымен сәйкес келетіні анықталды. Сондай-ақ негізгі бірлік және Йоккой бірліктері анықталып, n_d және m_d d -инварианттары w_d бөлшегін үздіксіз жіктеуге қолданылды, мұндағы $\ell(d)$ w_d — кезең ұзындығы $Q(\sqrt{d})$ нақты квадраттық сандар өрісі түрі үшін. Авторлар нақты квадраттық өрістің бірлік группасы туындайтын іргелі бірлік теориясын жаңартып, нақты квадраттық өрісте бүтінсанды базистік элементтің үздіксіз бөлшекті жіктелуінің ерекше түрін анықтаған.

Кілт сөздер: тізбекті бөлшек, квадраттық өріс, негізгі бірлік, Йоккой инварианттары, бүтінсанды базистік элемент.

Ö. Öзер, Д. Беллауар

Некоторые результаты по специальному типу вещественных квадратичных полей

В статье определено, что действительные квадратичные поля совпадают с целыми свободными положительными квадратами d , включая форму разложения цепной дроби $w_d = \left[a_0; \underbrace{\overline{1, 1, \dots, 1}}_{\ell-1}, a_\ell \right]$.

Кроме того, определены основные единицы и единицы Йокой, d -инварианты n_d и m_d применительно к непрерывному разложению дроби w_d , где $\ell(d)$ — длина периода w_d для такого типа поля действительных квадратичных чисел $Q(\sqrt{d})$. Авторами статьи улучшена теория фундаментальной единицы, которая порождает единичную группу вещественных квадратичных полей, а также определена особая форма непрерывного дробного разложения целочисленного базисного элемента в вещественных квадратичных полях.

Ключевые слова: цепная дробь, квадратичное поле, основная единица, инварианты Йокой, целочисленный базисный элемент.