

## On the properties for families of function classes over harmonic intervals and their embedding relation with Besov spaces

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The article is dedicated to the issues of studying the approximation of functions by trigonometric polynomials with a spectrum from special sets. In this paper, these special sets are harmonic intervals. To study the approximation of functions over harmonic intervals, families of function classes have been created, designed as a subsidiary tool. These families of function classes are characterized through the best approximations of functions by trigonometric polynomials over such sets and are used in the research. For these families of function classes, their properties and the connection with classical Besov spaces are shown. The results of the study are presented in the form of theorems and lemmas. In carrying out the research presented in the article, the main apparatus for proving theorems are the fundamentals of approximation theory, the method of real interpolation of spaces, and the fundamentals of the theory of embedding classes of functions and functional spaces. The article is destined for mathematicians and can be used by researchers and specialists whose interests lie in the indicated areas of mathematics.

*Keywords:* harmonic interval, spectrum, best approximation of a function by trigonometric polynomials with a spectrum from harmonic intervals, family of classes of functions, Besov spaces, embedding theorems.

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### Introduction

In approximation theory, the spectrum of approximating functions plays an important role. This spectrum can be selected from sets of different shapes, types, contours, structures, locations, etc. The spectrum can contain a wide variety of sets [1, 2].

A person's perception of information occurs through his sense organs, readings from various devices of measuring and observation, etc. And they all have a limited finite, not infinite, range. Based on this, when modeling diverse, multifaceted practical and applied problems [3–5], it is necessary to find a solution on some finite set that reflects this finite range. Special sets called harmonic intervals  $I_k^N$  [6, 7], where  $N$  defines exactly such a finite range of perception, to some extent help to conduct research and solve such problems.

The approximative properties of a function are usually characterized by the magnitude of the best approximation or the speed of approximation by a linear method. Often, during the course of research, there is a need to create auxiliary elements and tools. One of them is the families of classes of functions  $\{B_{p,q,N}^N\}_N$ , the definition of which is expressed in terms of the best approximations of functions by trigonometric polynomials with spectrum from sets  $I_k^N$ . These families of function classes are used, for example, in the theorems on the boundedness of the partial sum operator of the Fourier series, etc.

In carrying out the proofs of the statements described in the article, the fundamentals of approximation theory [8–10], the method of real interpolation of spaces [11–13] and the fundamentals of embedding theory [14] for classes of functions and function spaces are used.

In the article the properties of families of function classes  $\{B_{p,q,N}^r\}_N$  and the relationship of these families of function classes to the classical Besov spaces [15, 16] are studied.

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1 Definitions and auxiliary results

We construct families of classes of functions related to the best approximations of functions by trigonometric polynomials with a spectrum from harmonic intervals.

*Definition 1.* [17] Let  $1 \leq p, q \leq \infty, r > 0, N \in \mathbb{N}, f \in L_p [0; 2\pi)$ . The family of function classes is denoted by  $\{B_{p,q,N}^r\}_N$  and is defined in terms of the best approximations over harmonic intervals using the relation

$$B_{p,q,N}^r = \left\{ f : \|f\|_{B_{p,q,N}^r} < \infty \right\},$$

where

$$\|f\|_{B_{p,q,N}^r} = \left( \sum_{k=1}^N k^{rq-1} \left( E_{k-1}^N(f)_p \right)^q \right)^{\frac{1}{q}}.$$

*Definition 2.* [17] Let  $A^N$  and  $B^N$  be two classes of functions depending on a parameter  $N$ . The class of functions  $A^N$  is embedded in the class of functions  $B^N$  and denote it by

$$A^N \hookrightarrow B^N$$

if the following conditions are met:

- 1)  $A^N \subset B^N$ ;
- 2) there is a positive parameter  $C$  such that the inequality

$$\|f\|_{B^N} \leq C \|f\|_{A^N}$$

is correct for any  $f \in A^N$ , and the parameter  $C$  does not depend on  $f$  and  $N$ .

*Definition 3.* [17] Let two families of function classes  $\{A^N\}_N$  and  $\{B^N\}_N$  be given, where  $N \in \mathbb{N}$  and  $\{A^N\}_N \cap \{B^N\}_N = \emptyset$ . The classes of functions  $\{A^N\}_N$  and  $\{B^N\}_N$  is said to be equivalent

$$\|f\|_{A^N} \sim \|f\|_{B^N}$$

if there are positive parameters  $C_1, C_2$  such that for any  $f \in A^N$  the relation

$$C_1 \|f\|_{B^N} \leq \|f\|_{A^N} \leq C_2 \|f\|_{B^N}$$

is correct. Moreover, the parameters  $C_1, C_2$  do not depend on  $f$  and  $N$ . In this case, it is assumed that the families of function classes  $\{A^N\}_N$  and  $\{B^N\}_N$  coincide

$$\{A^N\}_N = \{B^N\}_N.$$

*Definition 4.* [11] Let  $(A_0, A_1)$  be an interpolation pair. For any  $t$  such that  $0 < t < \infty$ , the functional is defined by the equality

$$K(t, a; \bar{A}) = \inf_{a=a_0+a_1} (\|a_0\|_{A_0} + t \|a_1\|_{A_1}),$$

where  $a = A_0 + A_1$ , and  $\bar{A}=(A_0, A_1)$  is a compatible pair of spaces. This functional is called Petre's  $K$ -functional or simply Petre's functional.

*Definition 5.* [11] Let  $(A_0, A_1)$  be an interpolation pair,  $0 < \theta < 1$ . For  $1 \leq q < \infty$ , we have

$$\begin{aligned} \bar{A}_{\theta,q} &= (A_0, A_1)_{\theta,q} = \\ &= \left\{ a : a \in A_0 + A_1, \|a\|_{\bar{A}_{\theta,q}} = \left( \int_0^\infty \left[ t^{-\theta} K(t, a; \bar{A}) \right]^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty \right\}, \end{aligned}$$

and for  $q = \infty$  the equality

$$\bar{A}_{\theta, \infty} = (A_0, A_1)_{\theta, \infty} = \{a : a \in A_0 + A_1, \|a\|_{\bar{A}_{\theta, \infty}} = \sup_{0 < t < \infty} t^{-\theta} K(t, a; \bar{A}) < \infty\}$$

holds. The space  $\bar{A}_{\theta, q}$  is defined by the following relation

$$\bar{A}_{\theta, q} = \{a : \|a\|_{\bar{A}_{\theta, q}} < \infty\}.$$

*Theorem 1.* [7] Let  $m \in \mathbb{N}$ ,  $1 \leq p, q, q_0, q_1 \leq \infty$ ,  $0 < \theta < 1$ ,  $r > 0$ ,  $r_0 > r_1 > 0$ . Then there is the embedding of the form

$$B_{p, q, 2^m}^r \hookrightarrow \left( B_{p, q_0, 2^m}^{r_0}; B_{p, q_1, 2^m}^{r_1} \right)_{\theta, q},$$

where  $r = (1 - \theta)r_0 + \theta r_1$ .

*Theorem 2.* [7] Let  $N \in \mathbb{N}$ ,  $1 \leq p, q, q_1 \leq \infty$ ,  $r > 0$ , then the following embedding

$$B_{p, q, N}^r \hookrightarrow B_{p, q_1, N}^r$$

is performed for  $q < q_1$ .

*Theorem 3.* [7] Let  $f \in B_{p, q, 2^m}^r$ ,  $m \in \mathbb{N}$ ,  $\sum_{v \in \mathbb{Z}} a_v e^{ivx}$  be the trigonometric Fourier series of the function  $f$ . Then for  $1 \leq p, q \leq \infty$ ,  $r > 0$  the following relation

$$\|f\|_{B_{p, q, 2^m}^r} \sim \left( \sum_{k=1}^m 2^{rqk} \left( \delta_k(f)_p \right)^q \right)^{\frac{1}{q}}$$

is correct, where

$$\delta_k(f)_p = \left\| \sum_{s \in \mathbb{Z}} \sum_{\tau=2^{k-1}}^{2^k-1} a_{\tau+s \cdot 2^m} \cdot e^{i(\tau+s \cdot 2^m)x} \right\|_p.$$

*Lemma 1.* [6] Let  $B = [-k, k]$  be the segment in  $\mathbb{Z}$ ;  $k, d, h \in \mathbb{N}$ ,  $k < h$ ,  $\{I_B^{h, d}\}_{d=0}^{\infty}$  be the sequence of harmonic segments in  $\mathbb{Z}$ , converging to the harmonic interval  $I_B^h$ , where  $I_B^h = \bigcup_{v=-\infty}^{\infty} [B + vh]$ . If  $f \in L_p[0; 2\pi)$ ,  $1 \leq p \leq \infty$ ,  $\sum_{v \in \mathbb{Z}} a_v e^{ivx}$  is the Fourier series of the function  $f$ , then the sequence of partial sums of the Fourier series of the function  $f$  over harmonic segments

$$S_B^{h, d}(f) = \sum_{v \in I_B^{h, d}} a_v e^{ivx}$$

converges in the space  $L_p[0; 2\pi)$  as  $d \rightarrow \infty$  to the function

$$S_B^h(f) = \frac{1}{h} \sum_{r=0}^{h-1} f\left(x + \frac{2\pi r}{h}\right) D_B\left(\frac{2\pi r}{h}\right), \tag{1}$$

where  $D_B(x) = \sum_{m \in B} e^{imx}$  is the Dirichlet kernel corresponding to the segment  $B$  from  $\mathbb{Z}$ , and its Fourier series is the function  $\sum_{v \in I_B^h} a_v e^{ivx}$ .

*Theorem 4.* [7] Let  $m, d, N \in \mathbb{N}$ ,  $f \in L_p [0; 2\pi)$ ,  $1 < p < \infty$ ,  $S_m^N(f)$  and  $E_m^N(f)_p$  be the partial sum of the Fourier series and the best approximation of the function  $f$  over the harmonic interval  $I_m^N$ , respectively, then the following relation is correct

$$E_m^N(f)_p \sim \|f - S_m^N(f)\|_p.$$

*Theorem 5.* [7] Let  $f \in B_{p,q,2^m}^r$ ,  $m \in \mathbb{N}$ , then for  $1 \leq p, q \leq \infty$ ,  $r > 0$ , we have

$$\|f\|_{B_{p,q,2^m}^r} \sim \left( \sum_{k=1}^m 2^{r q k} \left( E_{2^{k-1}}^{2^m}(f)_p \right)^q \right)^{\frac{1}{q}}.$$

## 2 Properties for families of function classes $\{B_{p,q,N}^r\}_N$

Embedding Theorem 6 is the inverse of Embedding Theorem 1 for the same parameter values.

*Theorem 6.* Let  $m \in \mathbb{N}$ ,  $1 \leq p, q, q_0, q_1 \leq \infty$ ,  $0 < \theta < 1$ ,  $r > 0$ ,  $r_0 > r_1 > 0$ , then the embedding of the following type

$$\left( B_{p,q_0,2^m}^{r_0}; B_{p,q_1,2^m}^{r_1} \right)_{\theta,q} \hookrightarrow B_{p,q,2^m}^r$$

is satisfied, where  $r = (1 - \theta) r_0 + \theta r_1$ .

*Proof.* From Theorem 2 for families of function classes  $\{B_{p,q,N}^r\}_N$ , it follows that

$$B_{p,q,N}^r \hookrightarrow B_{p,\infty,N}^r.$$

Then, to prove the statement of the theorem, it is sufficient to prove the following embedding

$$\left( B_{p,\infty,2^m}^{r_0}; B_{p,\infty,2^m}^{r_1} \right)_{\theta,q} \hookrightarrow B_{p,q,2^m}^r.$$

Let  $f = f_0 + f_1$  be an arbitrary representation of a function  $f$ , where  $f_0 \in B_{p,\infty,2^m}^{r_0}$ ,  $f_1 \in B_{p,\infty,2^m}^{r_1}$ . Using Theorem 3 and Definition 1, we estimate the following expression

$$\begin{aligned} 2^{r_0 k} \delta_k(f)_p &\leq 2^{r_0 k} \delta_k(f_0)_p + 2^{(r_0-r_1)k} 2^{r_1 k} \delta_k(f_1)_p \leq \\ &\leq \|f_0\|_{B_{p,\infty,2^m}^{r_0}} + 2^{(r_0-r_1)k} \|f_1\|_{B_{p,\infty,2^m}^{r_1}}. \end{aligned}$$

Given the arbitrariness of the representation for the function  $f$  and using Definition 4, we receive the ratio

$$\begin{aligned} 2^{r_0 k} \delta_k(f)_p &\leq \inf_{f=f_0+f_1} \left( \|f_0\|_{B_{p,\infty,2^m}^{r_0}} + 2^{(r_0-r_1)k} \|f_1\|_{B_{p,\infty,2^m}^{r_1}} \right) = \\ &= K \left( 2^{(r_0-r_1)k}, f; B_{p,\infty,2^m}^{r_0}, B_{p,\infty,2^m}^{r_1} \right), \end{aligned}$$

where  $K$  is Petre's functional.

Therefore, we obtain the following inequality

$$\begin{aligned} \|f\|_{B_{p,q,2^m}^r} &\leq \left( \sum_{k=1}^m 2^{r q k} \left( \delta_k(f)_p \right)^q \right)^{\frac{1}{q}} \leq \\ &\leq \left( \sum_{k=1}^m 2^{(r-r_0)qk} \left\{ K \left( 2^{(r_0-r_1)k}, f; B_{p,\infty,2^m}^{r_0}, B_{p,\infty,2^m}^{r_1} \right) \right\}^q \right)^{\frac{1}{q}}. \end{aligned} \tag{2}$$

Taking into account that  $r - r_0 = -\theta(r_0 - r_1)$ , considering the partition of the interval  $(0; \infty)$  into half-intervals  $[2^{k(r_0-r_1)}; 2^{(k+1)(r_0-r_1)})$ ,  $k \in \mathbb{Z}$ , and using Definition 5, we transform inequality (2) in the following way

$$\begin{aligned} \|f\|_{B_{p,q,2^m}^r} &\leq \left( \sum_{k \in \mathbb{Z}} 2^{-\theta(r_0-r_1)qk} \left\{ K \left( 2^{(r_0-r_1)k}, f; B_{p,\infty,2^m}^{r_0}, B_{p,\infty,2^m}^{r_1} \right) \right\}^q \right)^{\frac{1}{q}} \leq \\ &\leq \left( \sum_{k \in \mathbb{Z}} \left\{ 2^{-\theta(r_0-r_1)k} K \left( 2^{(r_0-r_1)k}, f; B_{p,\infty,2^m}^{r_0}, B_{p,\infty,2^m}^{r_1} \right) \right\}^q \times \right. \\ &\quad \left. \times 2^{-(r_0-r_1)k} \left( 2^{(r_0-r_1)(k+1)} - 2^{(r_0-r_1)k} \right) \right)^{\frac{1}{q}} \leq \\ &\leq \left( \int_0^\infty \left\{ t^{-\theta} K \left( t, f; B_{p,\infty,2^m}^{r_0}, B_{p,\infty,2^m}^{r_1} \right) \right\}^q \frac{dt}{t} \right)^{\frac{1}{q}} = \|f\|_{(B_{p,\infty,2^m}^{r_0}; B_{p,\infty,2^m}^{r_1})_{\theta,q}}. \end{aligned}$$

As a result, we have

$$\|f\|_{B_{p,\infty,2^m}^r} \leq C \|f\|_{(B_{p,\infty,2^m}^{r_0}; B_{p,\infty,2^m}^{r_1})_{\theta,q}}.$$

This inequality determines the required statement of the theorem. The theorem is proved.

The following corollary follows directly from Theorems 1, 6 and Definitions 2, 3.

*Corollary 1.* Let  $m \in \mathbb{N}$ ,  $1 \leq p, q, q_0, q_1 \leq \infty$ ,  $0 < \theta < 1$ ,  $r > 0$ ,  $r_0 > r_1 > 0$ , then the equality

$$\left( B_{p,q_0,2^m}^{r_0}; B_{p,q_1,2^m}^{r_1} \right)_{\theta,q} = B_{p,q,2^m}^r$$

is correct, where  $r = (1 - \theta) r_0 + \theta r_1$ .

### 3 Relationship for the families of function classes with Besov spaces

The following theorems show the connection between families of function classes  $\{B_{p,q,N}^r\}_N$  and the classical Besov spaces.

*Theorem 7.* [17] If  $N \in \mathbb{N}$ ,  $1 \leq p, q, \leq \infty$ ,  $r > 0$ , then the following relation is fulfilled

$$\bigcap_{N=1}^\infty B_{p,q,N}^r = B_{p,q}^r.$$

*Theorem 8.* Let  $1 \leq p \leq q \leq \infty$ ,  $1 \leq r \leq \infty$ ,  $m \in \mathbb{N}$ ,  $\alpha > 0$ ,  $\beta > 0$ ,  $\alpha - \beta = \frac{1}{p} - \frac{1}{q}$ , then for any value of  $m$  the following embedding

$$B_{p,r}^\alpha \hookrightarrow B_{q,r,2^m}^\beta$$

holds, that is, the inequality

$$\|f\|_{B_{q,r,2^m}^\beta} \leq C \|f\|_{B_{p,r}^\alpha}$$

is satisfied, where  $C$  is a constant that does not depend on  $f$  and  $m$ .

*Proof.* First, we show that the following inequality

$$\|f\|_{B_{q,\infty,2^m}^\beta} \leq \|f\|_{B_{q,\infty}^\beta} \tag{3}$$

holds. Indeed, the above inequality is correct, since it follows from the relation

$$\|f\|_{B_{q,\infty,2^m}^\beta} = \max_{1 \leq k \leq m} 2^{\beta k} E_{2^{k-1}}^{2^m}(f)_q \leq \sup_{k \geq 1} 2^{\beta k} E_{2^k}(f)_q = \|f\|_{B_{q,\infty}^\beta}.$$

According to the Besov embedding theorem [16], for  $1 \leq p \leq q \leq \infty$ ,  $\alpha - \beta = \frac{1}{p} - \frac{1}{q}$ , we have such inequality

$$\|f\|_{B_{q,\infty}^\beta} \leq C \|f\|_{B_{p,\infty}^\alpha}, \tag{4}$$

where  $C$  is a constant that does not depend on  $f$  and  $m$ . Based on (3) and (4), we get an inequality of the form

$$\|f\|_{B_{q,\infty,2^m}^{\beta_i}} \leq C_i \|f\|_{B_{p,\infty}^{\alpha_i}}, \tag{5}$$

where  $C_i$  are constants that do not depend on  $f$  and  $m$ ,  $i = 0, 1$ .

With the parameters  $\alpha, \beta, p, q$  specified in the conditions of the theorem, we define the operator  $I$  as follows

$$If = f.$$

Let us take pairs  $(\alpha_0, \alpha_1)$  and  $(\beta_0, \beta_1)$ , where the parameters  $\alpha_0, \beta_0, \alpha_1, \beta_1$  satisfy the conditions

$$\alpha_0 < \alpha < \alpha_1, \quad \beta_0 < \beta < \beta_1, \quad \alpha_0 - \beta_0 = \alpha_1 - \beta_1 = \frac{1}{p} - \frac{1}{q}.$$

Taking into account inequality (5), we have

$$I : B_{p,\infty}^{\alpha_0} \rightarrow B_{q,\infty,2^m}^{\beta_0}$$

with the norm  $C_0$ , and

$$I : B_{p,\infty}^{\alpha_1} \rightarrow B_{q,\infty,2^m}^{\beta_1}$$

with the norm  $C_1$ . Then, by the interpolation theorem [11], we obtain

$$I : (B_{p,\infty}^{\alpha_0}; B_{p,\infty}^{\alpha_1})_{\theta,r} \rightarrow (B_{q,\infty,2^m}^{\beta_0}; B_{q,\infty,2^m}^{\beta_1})_{\theta,r}. \tag{6}$$

According to Petre's theorem [11], the equality

$$(B_{p,\infty}^{\alpha_0}; B_{p,\infty}^{\alpha_1})_{\theta,r} = B_{p,r}^{\alpha_\theta} \tag{7}$$

is valid, where

$$\alpha_\theta = (1 - \theta)\alpha_0 + \theta\alpha_1,$$

$0 < \theta < 1$ . Using a similar equality for families of classes of functions  $\{B_{p,q,N}^r\}_N$ , we have the following relation

$$(B_{q,\infty,2^m}^{\beta_0}; B_{q,\infty,2^m}^{\beta_1})_{\theta,r} = B_{q,r,2^m}^{\beta_\theta}, \tag{8}$$

where

$$\beta_\theta = (1 - \theta)\beta_0 + \theta\beta_1,$$

$0 < \theta < 1$ . Since

$$\alpha_\theta - \beta_\theta = (1 - \theta)(\alpha_0 - \beta_0) + \theta(\alpha_1 - \beta_1) = \frac{1}{p} - \frac{1}{q},$$

then there exists  $\theta \in (0; 1)$  such that

$$\beta_\theta = \beta \Rightarrow \alpha_\theta = \alpha. \tag{9}$$

As a result, taking into account (7)–(9), relation (6) determines that

$$I : B_{p,r}^\alpha \rightarrow B_{q,r,2^m}^\beta,$$

and besides

$$\|f\|_{B_{q,r,2^m}^\beta} \leq C \|f\|_{B_{p,r}^\alpha}.$$

This inequality proves the statement of the theorem. The theorem is proved.

In particular, the following lemma is correct for the conditions of Theorem 6.

*Lemma 2.* Let  $1 \leq p \leq q \leq \infty$ ,  $1 \leq r \leq \infty$ ;  $m, n \in \mathbb{N}$ ,  $\alpha > 0$ ,  $\beta > 0$ ,  $\alpha - \beta = \frac{1}{p} - \frac{1}{q}$ . For functions  $f$  that are summable to the  $p$ -th power, the inequality

$$\left( \sum_{k=1}^m 2^{\beta rk} \left( \left\| f - \frac{1}{2^{m+1}} \sum_{n=0}^{2^{m+1}-1} f \left( x + \frac{\pi n}{2^m} \right) D_{2^{k-1}} \left( \frac{\pi n}{2^m} \right) \right\|_q \right)^r \right)^{\frac{1}{r}} \leq C \|f\|_{B_{p,r}^\alpha}$$

holds, where  $D_{2^{k-1}} \left( \frac{\pi n}{2^m} \right)$  is the Dirichlet kernel.

*Proof.* By successively applying (1), Theorems 4 and 5, we obtain the relation

$$\begin{aligned} & \left( \sum_{k=1}^m 2^{\beta rk} \left( \left\| f - \frac{1}{2^{m+1}} \sum_{n=0}^{2^{m+1}-1} f \left( x + \frac{\pi n}{2^m} \right) D_{2^{k-1}} \left( \frac{\pi n}{2^m} \right) \right\|_q \right)^r \right)^{\frac{1}{r}} = \\ & = \left( \sum_{k=1}^m 2^{\beta rk} \left( \|f - S_{2^{k-1}}^{2^m}(f)\|_q \right)^r \right)^{\frac{1}{r}} \sim \\ & \sim \left( \sum_{k=1}^m 2^{\beta rk} \left( E_{2^{k-1}}^{2^m}(f)_q \right)^r \right)^{\frac{1}{r}} \sim \|f\|_{B_{q,r,2^m}^\beta}. \end{aligned}$$

From this relation, taking into account the ratio

$$\|f\|_{B_{q,r,2^m}^\beta} \leq C \|f\|_{B_{p,r}^\alpha},$$

the required inequality follows. The lemma is proved.

### Conclusion

In the article the properties of families of function classes  $\{B_{p,q,N}^r\}_N$  are presented. Embedding theorems for the specified families of function classes and the lemma on estimating the norm of the Besov space are proved. The embedding theorem showing the relationship of these families of function classes to the classical Besov spaces is proved by the method of real interpolation. The results presented in the article will be used in future directions of research on the approximation of functions over harmonic intervals.

### Author Contributions

L.A. Serikova conducted the research presented in the article under the scientific supervision of G.A. Yessenbayeva. All authors contributed equally to this work and participated in the revision of the manuscript and approved the final submission.

## Conflict of Interest

The authors declare no conflict of interest.

## References

- 1 Temlyakov, V.N. (2021). Diskretizatsiia integralnykh norm polinomov s garmonikami iz giperbolicheskogo kresta po znacheniiam v tochkakh [Discretization of integral norms for polynomials with harmonics from a hyperbolic cross by values at points]. *Funktsionalnye prostranstva, teoriia priblizhenii i smezhnye voprosy analiza: sbornik statei k 115-letiiu so dnia rozhdeniia akademika Sergeia Mikhailovicha Nikolskogo. Trudy MIAN – Functional spaces, approximation theory, and related issues of analysis. Collection of articles dedicated to the 115th anniversary of the birth of academician Sergei Mikhailovich Nikolsky. Works of MIAS, 312*, 282–293 [in Russian].
- 2 Turova, M.E. (2015). Otsenki nailuchshikh priblizhenii funktsii so spektrom iz giperbolicheskikh krestov [Estimates for the best approximations of a function with spectrum from hyperbolic crosses]. *Mezhdunarodnyi nauchno-issledovatel'skii zhurnal – International research journal*, 5–1(36), 29–31 [in Russian].
- 3 Akhmediyev, S.K., Khabidolda, O., Vatin, N.I., Yessenbayeva, G.A., & Muratkhan, R. (2023). Physical and mechanical state of cantilever triangular plates. *Journal of Mathematics, Mechanics and Computer Science*, 118(2), 64–73. <https://doi.org/10.26577/JMMCS.2023.v118.i2.07>
- 4 Nursultanov, E.D., & Tleukhanova, N.T. (2006). O nekotorykh zadachah teorii priblizhenii [On some problems of approximation theory]. *Matematicheskii zhurnal – Mathematical Journal*, 6(4(22)), 86–95 [in Russian].
- 5 Yesbayev, A.N. Yessenbayeva, G.A., Yessenbayeva G.A., & Ramazanov, M.I. (2019). On the kernel properties of the integral equation for the model with the essentially loaded heat equation. *Eurasian Physical Technical Journal*, 16(2), 105–112. <https://doi.org/10.31489/2019No2/105-112>
- 6 Nursultanov, E.D. (1999). Setevye prostranstva i ikh prilozheniia k zadacham garmonicheskogo analiza [Network spaces and their applications to problems of harmonic analysis]. *Doctor's thesis*. Moscow [in Russian].
- 7 Yessenbayeva, G.A., & Nursultanov, E.D. (2004). Interpoliatsiia nekotorykh neravenstv i apksimatsiia funktsii trigonometrskimi polinomami so spektrom iz garmonicheskikh intervalov [Interpolation of some inequalities and approximation of functions by trigonometric polynomials with spectrum from harmonic intervals]. *Evraziiskii matematicheskii zhurnal – Eurasian mathematical journal*, 1, 3–20 [in Russian].
- 8 Akishev, G. (2024). Estimates of  $M$ -term approximations of functions of several variables in the Lorentz space by a constructive method. *Eurasian mathematical journal*, 15(2), 8–32. <https://doi.org/10.32523/2077-9879-2024-15-2-08-32>
- 9 Nikolsky, S.M. (2006). *Izbrannye trudy. Teoriia priblizhenii [Selected works. Approximation theory]*. Moscow: Nauka [in Russian].
- 10 Motorny, V.P., Babenko, V.F., Dovgoshey, A.A., & Kuznetsova, O.I. (2011). *Teoria apksimatsii i garmonicheskii analiz [Approximation theory and harmonic analysis]*. Kiev: Naukova Dumka [in Russian].
- 11 Berg, J., & Lefstrom, J. (1980). *Interpoliatsionnye prostranstva. Vvedenie [Interpolation spaces. Introduction]*. Moscow: Mir [in Russian].
- 12 Bashirova, A.N., Kalidolday, A.H., & Nursultanov, E.D. (2024). Interpolation methods for anisotropic net space. *Eurasian mathematical journal*, 15(2), 33–41. <http://doi.org/10.32523/2077-9879-2024-15-2-33-41>

- 13 Krepkogorsky, V.L. (2009). Interpoliatsiia funktsionalnyukh prostranstv klassov Besova i Lizorkina-Tribelia [Interpolation of functional spaces of the Besov and Lizorkin-Triebel classes]. *Extended abstract of Doctor's thesis*. Kazan [in Russian].
- 14 Besov, O.V., Ilyin, V.P., & Nikolsky, S.M. (1996). *Integralnye predstavleniia funktsii i teoremy vlozheniia* [Integral representations of functions and embedding theorems]. Moscow: Nauka. Fizmatlit [in Russian].
- 15 Sawano, Y. (2018). *Theory of Besov Spaces*. Singapore: Springer Nature Singapore Ltd.
- 16 Nikolsky, S.M. (2003). Prostranstva Besova [Besov spaces]. *Trudy MIAN – Proceedings of MIAS*, 243, 9–12 [in Russian].
- 17 Yessenbayeva, G.A., Yesbayev, A.N., Syzdykova, N.K., & Smirnova, M.A. (2023). On the function approximation by trigonometric polynomials and the properties of families of function classes over harmonic intervals. *Bulletin of the Karaganda University. Mathematics Series*, 3(111), 181–190. <https://doi.org/10.31489/2023m3/181-190>

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