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# Liftings from Lorentzian $\alpha$-Sasakian manifolds to tangent bundles

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The goal of the current study is to investigate the complete lift of Lorentzian $\alpha$-Sasakian manifolds to the tangent bundle $\mathsf{TM}$. We also examine the complete lift of the different four types of Lorentzian $\alpha$-Sasakian manifolds and find that $(\mathsf{TM}, g^C)$ is an $\eta$-Einstein manifold in each instance. In order to show that a Lorentzian $\alpha$-Sasakian manifold exists on $\mathsf{TM}$, a non-trivial example by means of partial differential equations is built in the final section.

*Keywords:* complete lift, tangent bundle, mathematical operators, Weyl conformal curvature tensor, conharmoic curvature tensor, projective curvature tensor, concircular curvature tensor, trans-Sasakian manifolds, Lorentzian $\alpha$-Sasakian manifolds, partial differential equations.

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## *Introduction*

The geometry of trans-Sasakian manifolds received significant contributions from Blair and Oubina [1]. An almost contact metric manifold $(\mathsf{M}, \phi, \xi, \eta, g)$ with dim $= (2n+1)$ was defined due to Blair [2]. The geometry of the almost Hermitian manifold $(\bar{\mathsf{M}}, J, G)$ helps to determine the geometry of the almost contact metric manifold $(\mathsf{M}, \phi, \xi, \eta, g)$, that offers various structures on $\mathsf{M}$ (Sasakian, quasi-Sasakian, Kenmotsu, etc.) [1, 3–6], where $\bar{\mathsf{M}} = \mathsf{M} \times R$, $J$ represents the almost complex structure and $G$ stands for the Hermitian metric. A structure $(\phi, \xi, \eta, g, \alpha, \beta)$ on $\mathsf{M}$ is referred to as a trans-Sasakian structure where $\alpha, \beta$ are smooth functions using the structure in the class $W_4$ on $(\bar{\mathsf{M}}, J, G)$. On the nearly Hermitian manifold $(\bar{\mathsf{M}}, J, G)$, sixteen different types of structures are known to exist [7].

It is noted that cosymplectic [2], $\beta$-Kenmotsu [6] and $\alpha$-Sasakian [6] are trans-Sasakian structures of type $(0,0), (0,\beta)$ and $(\alpha, 0)$ respectively. De and Tripathi [8] have explored trans-Sasakian manifolds and achieved outstanding findings. Lorentzian $\alpha$-Sasakian manifolds were the subject of Yildiz and Murathan's research [9]. Yoldas developed a few classes of generalized recurrent $\alpha$-cosymplectic manifolds [10–12]. We cite [13, 14] for additional research on the aforementioned subject.

Assuming that $(\mathsf{M}, g), n = \dim \mathsf{M} > 3$ is connected semi Riemannian manifold of class $C^\infty$ and represent by $\nabla$ its Levi-Civita connection. We write the Riemannian-Christoffel curvature tensor $R$, the Weyl conformal curvature tensor $\mathcal{C}$, the conharmonic curvature tensor $K$ [13], the projective curvature tensor $P$ and the concircular curvature tensor $\tilde{\mathcal{C}}$ of $(\mathsf{M}, g)$ by

$$R(s_1,s_2)s_3 = \nabla_{s_1}\nabla_{s_2}s_3 - \nabla_{s_2}\nabla_{s_1}s_3 - \nabla_{[s_1,s_2]}s_3,$$

$$\begin{aligned} \mathcal{C}(s_1,s_2)s_3 &= R(s_1,s_2)s_3 \\ &+ \frac{1}{n-2}[S(s_1,s_3)s_2 - S(s_2,s_3)s_1 + g(s_1,s_3)Qs_2 - g(s_2,s_3)Qs_1] \\ &- \frac{\tau}{(n-1)(n-2)}[g(s_1,s_3)s_2 - g(s_2,s_3)s_1], \end{aligned}$$

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$$K(s_1,s_2)s_3 = R(s_1,s_2)s_3 - \frac{1}{n-2}[S(s_2,s_3)s_1 - S(s_1,s_3)s_2 + g(s_2,s_3)Qs_1 - g(s_1,s_3)Qs_2],$$

$$P(s_1,s_2)s_3 = R(s_1,s_2)s_3 - \frac{1}{n-2}[g(s_2,s_3)Qs_1 - g(s_1,s_3)Qs_2],$$

$$\tilde{C}(s_1,s_2)s_3 = R(s_1,s_2)s_3 - \frac{\tau}{n(n-2)}[g(s_1,s_3)s_2 - g(s_2,s_3)s_1],$$

respectively. In this scenario, $Q$ is the Ricci operator as defined by $S(s_1,s_2) = g(Qs_1,s_2)$, $s_1,s_2,s_3 \in \Im_0^1(\mathsf{M})$, S is the Ricci tensor and $\tau = tr(S)$ is the scalar curvature.

The fundamental characteristics of curvature tensors and the idea of the liftings of tensor fields and connections to their tangent bundle was developed in [15]. In their study, Dida and Hathout [16] looked into Ricci soliton structures on tangent bundles of Riemannian manifolds. Numerous scholars have examined several connections and geometric structures on the tangent bundle and established their fundamental geometric features [17–22].

These works serve as our inspiration as we investigate the complete lift of Lorentzian $\alpha$-Sasakian manifolds to tangent bundle $\mathsf{TM}$. Additionally, we investigate the complete lift of $\phi$-conformally flat, $\phi$-conharmonically flat, $\phi$-projectively flat and $\phi$-concircularly flat Lorentzian $\alpha$-Sasakian manifolds and derive $(\mathsf{TM}, g^C)$ as an $\eta^C$-Einstein manifold in each instance where $g^C$ is the Lorentzian metric.

*Notations.* The notations below appear in several places throughout the text: Both $\Im_a^b(\mathsf{M})$ and $\Im_a^b(\mathsf{TM})$ stand for the set of all tensor fields of type $(a,b)$ [23, 24].

## 1 *Preliminaries*

If there is a $(1,1)$-tensor field $\phi$, a vector field $\xi$, a 1-form $\eta$, and a Lorentzian metric $g$ that satisfy $\forall s_1,s_2 \in \Im_0^1(\mathsf{M})$ [1, 2, 8, 25]

$$\eta(\xi) = -1, \tag{1}$$
$$\phi^2 = I + \eta \otimes \xi, \tag{2}$$
$$g(\phi s_1, \phi s_2) = g(s_1,s_2) + \eta(s_1)\eta(s_2), \tag{3}$$
$$g(s_1,\xi) = \eta(s_1), \tag{4}$$
$$\phi\xi = 0, \quad \eta(\phi s_1) = 0,$$

then the differentiable manifold $\mathsf{M}$ is said to be a Lorentzian $\alpha$-Sasakian manifold.

The following relationships are also true in a Lorentzian $\alpha$-Sasakian manifold $\mathsf{M}$ [26–30]

$$\nabla_{s_1}\xi = -\alpha\phi s_1, \tag{5}$$
$$(\nabla_{s_1}\eta)s_2 = -\alpha g(\phi s_1, s_2), \tag{6}$$

and $\mathsf{M}$ becomes $\eta$-Einstein if its Ricci tensor $S$ is given by

$$S(s_1,s_2) = ag(s_1,s_2) + b\eta(s_1)\eta(s_2), \ \ \forall s_1,s_2 \in \Im_0^1(\mathsf{M}), \tag{7}$$

in above equations $\nabla$ represents the covariant differentiation operator w.r.t. $g$ and $a,b$ are functions on $\mathsf{M}$.

For curvature tensor $R$, we have [8]

$$\begin{aligned} R(\xi,s_1)s_2 &= \alpha^2(g(s_1,s_2)\xi+\eta(s_2)s_1), && (8)\\ R(s_1,s_2)\xi &= \alpha^2(\eta(s_2)s_1+\eta(s_1)s_2), && (9)\\ R(\xi,s_1)\xi &= \alpha^2(\eta(s_1)\xi+s_1), && (10)\\ S(s_1,\xi) &= (n-1)\alpha^2\eta(s_1), && (11)\\ Q\xi &= (n-1)\alpha^2\xi, && (12)\\ S(\xi,\xi) &= -(n-1)\alpha^2, && (13)\\ S(\phi s_1,\phi s_2) &= S(s_1,s_2)+(n-1)\alpha^2\eta(s_1)\eta(s_2), && (14) \end{aligned}$$

where $Q$ stands for the Ricci operator with $S(s_1,s_2)=g(Qs_1,s_2)$.

## *2 The complete lift from a Lorentzian α-Sasakian manifold to its tangent bundle*

Let us consider a local coordinate system $(x^i), i=1,...,n$ on differentiable manifold $\mathsf{M}$ and let $(x^i,y^i)$, $i=1,...,n$ be an induced local coordinate system on tangent bundle $\mathsf{TM}$. If $s_1={s_1}^i\frac{\partial}{\partial x^i}$ is a local vector field on $\mathsf{M}$, then its vertical and complete lifts in the term of partial differential equations are

$$\begin{aligned} {s_1}^V &= {s_1}^i\frac{\partial}{\partial y^i},\\ {s_1}^C &= {s_1}^i\frac{\partial}{\partial x^i}+\frac{\partial {s_1}^i}{\partial x^j}y^j\frac{\partial}{\partial y^i}. \end{aligned}$$

Let $\eta, s_1$ and $\phi$, respectively, represent 1-form, vector field, and a tensor field of type (1,1). Denote the complete and vertical lifts of $\eta, s_1$ and $\phi$ by $\eta^C, {s_1}^C, \phi^C$ and $\eta^V, {s_1}^V, \phi^V$, respectively. Then by using mathematical operators on $\eta, s_1$ and $\phi$, we have [15, 31]

$$\begin{gathered} \eta^V({s_1}^C)=\eta^C({s_1}^V)=\eta(s_1)^V,\ \eta^C({s_1}^C)=\eta(s_1)^C,\\ \phi^V{s_1}^C=(\phi s_1)^V,\ \phi^C{s_1}^C=(\phi s_1)^C,\\ [s_1,s_2]^V=[{s_1}^C,{s_2}^V]=[{s_1}^V,{s_2}^C],\ [s_1,s_2]^C=[{s_1}^C,{s_2}^C],\\ \nabla^C_{{s_1}^C}{s_2}^C=(\nabla_{s_1}s_2)^C,\qquad \nabla^C_{{s_1}^C}{s_2}^V=(\nabla_{s_1}s_2)^V, \end{gathered}$$

$\nabla^C$ is used for the complete lift of $\nabla$ on $\mathsf{TM}$.

Using the complete lift on (1)–(7), we conclude

$$\begin{aligned} \eta^C(\xi^C) &= \eta^V(\xi^V)=0,\ \ \eta^C(\xi^V)=\eta^V(\xi^C)=-1, && (15)\\ (\phi^2)^C &= I+\eta^C\otimes\xi^V+\eta^V\otimes\xi^C, && (16)\\ g^C((\phi s_1)^C,(\phi s_2)^C) &= g^C({s_1}^C,{s_2}^C)+\eta^C({s_1}^C)\eta^V({s_2}^C)\\ &+ \eta^V({s_1}^C)\eta^C({s_2}^C),\\ g^C({s_1}^C,\xi^C) &= \eta^C({s_1}^C),\\ \phi^C\xi^C &= \phi^V\xi^V=\phi^C\xi^V=\phi^V\xi^C=0, && (17)\\ \eta^C(\phi s_1)^C &= \eta^V(\phi s_1)^V=\eta^C(\phi s_1)^V=\eta^V(\phi s_1)^C=0, && (18) \end{aligned}$$

$\forall {s_1}^C, {s_2}^C\in\Im^1_0(\mathsf{TM})$, and

$$\begin{aligned} \nabla^C_{{s_1}^C}\xi^C &= -\alpha(\phi s_1)^C,\\ (\nabla^C_{{s_1}^C}\eta^C){s_2}^C &= -\alpha g^C((\phi s_1)^C,{s_2}^C). \end{aligned}$$

When the complete lift of Ricci tensor $S$ holds for

$$S^C({s_1}^C,{s_2}^C)=ag^C({s_1}^C,{s_2}^C)+b\{\eta^C({s_1}^C)\eta^V({s_2}^C)+\eta^V({s_1}^C)\eta^C({s_2}^C)\},$$

then the Lorentzian $\alpha$-Sasakian manifold $\mathsf{M}$ on $\mathsf{TM}$ is thought to be $\eta^C$-Einstein.

Taking the complete lift on (8)–(14), we infer

$$\begin{aligned}
R^C(\xi^C,{s_1}^C){s_2}^C &= \alpha^2\{g^C({s_1}^C,{s_2}^C)\xi^V+g^C({s_1}^V,{s_2}^C)\xi^V\\
&- \eta^C({s_2}^C){s_1}^V+\eta^V({s_2}^C){s_1}^C\},\\
R^C({s_1}^C,{s_2}^C)\xi^C &= \alpha^2\{\eta^C({s_2}^C){s_1}^V+\eta^V({s_2}^C){s_1}^C\\
&- \eta^C({s_1}^C){s_2}^V+\eta^V({s_1}^C){s_2}^C\},\\
R^C(\xi^C,{s_1}^C)\xi^C &= \alpha^2\{\eta^C({s_1}^C)\xi^V+\eta^V({s_1}^C)\xi^C+{s_1}^C\},\\
S^C({s_1}^C,\xi^C) &= (n-1)\alpha^2\eta^C({s_1}^C),\\
(Q\xi)^C &= (n-1)\alpha^2\xi^C,\\
S^C(\xi^C,\xi^C) &= -(n-1)\alpha^2,\\
S^C((\phi s_1)^C,(\phi s_2)^C) &= S^C({s_1}^C,{s_2}^C)+(n-1)\alpha^2\{\eta^C({s_1}^C)\eta^V({s_2}^C)\\
&+ \eta^V({s_1}^C)\eta^C({s_2}^C)\},
\end{aligned}$$

where $S^C({s_1}^C,{s_2}^C)=g^C((Qs_1)^C,{s_2}^C)$ and $S^({s_1}^V,{s_2}^C)=g^C((Qs_1)^V,{s_2}^C)$.

## 3 Main Results

*Definition 1.* Consider a differentiable manifold $(\mathsf{M}^n,g)$ with $n>3$. Then $M$ is said to be

- $\phi$-conformally flat if
$$\phi^2\mathcal{C}(\phi s_1,\phi s_2)\phi s_3=0, \tag{19}$$
- $\phi$-conharmonically flat if
$$\phi^2K(\phi s_1,\phi s_2)\phi s_3=0,$$
- $\phi$-projectively flat provided
$$\phi^2P(\phi s_1,\phi s_2)\phi s_3=0,$$
- $\phi$-concircularly flat if
$$\phi^2\tilde{\mathcal{C}}(\phi s_1,\phi s_2)\phi s_3=0.$$

*Theorem 1.* Let $\mathsf{M}^n$ be $\phi$-conformally flat Lorentzian $\alpha$-Sasakian manifold and denote by $\mathsf{TM}$ its tangent bundle. Then $(\mathsf{TM},g^C)$ is an $\eta^C$-Einstein manifold, $g^C$ being the Lorentzian metric of $\mathsf{TM}$.

*Proof.* For the given assumptions, we see that

$$\phi^2\mathcal{C}(\phi s_1,\phi s_2)\phi s_3=0\Longleftrightarrow g(\mathcal{C}(\phi s_1,\phi s_2)\phi s_3,\phi s_4)=0, \tag{20}$$

$\forall s_1,s_2,s_3,s_4\in\Im_0^1(\mathsf{M})$.

Complete lifts on (1) produce

$$\begin{aligned}
\mathcal{C}^C({s_1}^C,{s_2}^C){s_3}^C &= R^C({s_1}^C,{s_2}^C){s_3}^C\\
&+ \frac{1}{n-2}[S^C({s_1}^C,{s_3}^C){s_2}^V+S^C({s_1}^V,{s_3}^C){s_2}^C\\
&- S^C({s_2}^C,{s_3}^C){s_1}^V-S^C({s_2}^V,{s_3}^C){s_1}^C\\
&+ g^C({s_1}^C,{s_3}^C)(Qs_2)^V+g^C({s_1}^V,{s_3}^C)(Qs_2)^C\\
&- g^C({s_2}^C,{s_3}^C)(Qs_1)^V-g^C({s_2}^V,{s_3}^C)(Qs_1)^C]\\
&- \frac{\tau}{(n-1)(n-2)}[g^C({s_1}^C,{s_3}^C){s_2}^V+g^C({s_1}^V,{s_3}^C){s_2}^C\\
&- g^C({s_2}^C,{s_3}^C){s_1}^V-g^C({s_2}^V,{s_3}^C){s_1}^C].
\end{aligned}$$

In view of (20) and (21) $\phi$-conformally flat means

$$
\begin{aligned}
& (g(R(\phi s_1, \phi s_2)\phi s_3, \phi s_4))^C \\
= \ & \frac{1}{n-2}[g^C((\phi s_2)^C, (\phi s_3)^C)S^C((\phi s_1)^V, (\phi s_4)^C) \\
+ \ & g^C((\phi s_2)^V, (\phi s_3)^C)S^C((\phi s_1)^C, (\phi s_4)^C) \\
- \ & g^C((\phi s_1)^C, (\phi s_3)^C)S^C((\phi s_2)^V, (\phi s_4)^C) \\
- \ & g^C((\phi s_1)^V, (\phi s_3)^C)S^C((\phi s_2)^C, (\phi s_4)^C) \\
+ \ & S^C((\phi s_2)^C, (\phi s_3)^C)g^C((\phi s_1)^V, (\phi s_4)^C) \\
+ \ & S^C((\phi s_2)^V, (\phi s_3)^C)g^C((\phi s_1)^C, (\phi s_4)^C) \\
- \ & S^C((\phi s_1)^C, (\phi s_3)^C)g^C((\phi s_2)^V, (\phi s_4)^C) \\
- \ & S^C((\phi s_1)^V, (\phi s_3)^C)g^C((\phi s_2)^C, (\phi s_4)^C)] \\
- \ & \frac{\tau}{(n-1)(n-2)}[g^C((\phi s_2)^C, (\phi s_3)^C)g^C((\phi s_1)^V, (\phi s_4)^C) \\
+ \ & g^C((\phi s_2)^V, (\phi s_3)^C)g^C((\phi s_1)^C, (\phi s_4)^C) \\
- \ & g^C((\phi s_1)^C, (\phi s_3)^C)g^C((\phi s_2)^V, (\phi s_4)^C) \\
- \ & g^C((\phi s_1)^V, (\phi s_3)^C)g^C((\phi s_2)^C, (\phi s_4)^C)].
\end{aligned} \tag{21}
$$

Let $\{\varepsilon_i : i = 1, ..., n-1, \xi\}$ represent a local orthonormal basis. Then $\{(\phi\varepsilon_i)^C : i = 1, ..., n-1, (\phi\xi)^C\}$ is in $\mathsf{TM}$.

Setting $s_1 = s_4 = \varepsilon_i$ in (21) produces

$$
\begin{aligned}
& \sum_{i=1}^{n-1} g(R(\phi\varepsilon_i, \phi s_2)\phi s_3, \phi\varepsilon_i) \\
= \ & \frac{1}{n-2}\sum_{i=1}^{n-1}[g^C((\phi s_2)^C, (\phi s_3)^C)S^C((\phi\varepsilon_i)^V, (\phi\varepsilon_i)^C) \\
+ \ & g^C((\phi s_2)^V, (\phi s_3)^C)S^C((\phi\varepsilon_i)^C, (\phi\varepsilon_i)^C) \\
- \ & g^C((\phi\varepsilon_i)^C, (\phi s_3)^C)S^C((\phi s_2)^V, (\phi\varepsilon_i)^C) \\
- \ & g^C((\phi\varepsilon_i)^V, (\phi s_3)^C)S^C((\phi s_2)^C, (\phi\varepsilon_i)^C) \\
+ \ & S^C((\phi s_2)^C, (\phi s_3)^C)g^C((\phi\varepsilon_i)^V, (\phi\varepsilon_i)^C) \\
+ \ & S^C((\phi s_2)^V, (\phi s_3)^C)g^C((\phi\varepsilon_i)^C, (\phi\varepsilon_i)^C) \\
- \ & S^C((\phi\varepsilon_i)^C, (\phi s_3)^C)g^C((\phi s_2)^V, (\phi\varepsilon_i)^C) \\
- \ & S^C((\phi\varepsilon_i)^V, (\phi s_3)^C)g^C((\phi s_2)^C, (\phi\varepsilon_i)^C)] \\
- \ & \frac{\tau}{(n-1)(n-2)}\sum_{i=1}^{n-1}[g^C((\phi s_2)^C, (\phi s_3)^C)g^C((\phi\varepsilon_i)^V, (\phi\varepsilon_i)^C) \\
+ \ & g^C((\phi s_2)^V, (\phi s_3)^C)g^C((\phi\varepsilon_i)^C, (\phi\varepsilon_i)^C) \\
- \ & g^C((\phi\varepsilon_i)^C, (\phi s_3)^C)g^C((\phi s_2)^V, (\phi\varepsilon_i)^C) \\
- \ & g^C((\phi\varepsilon_i)^V, (\phi s_3)^C)g^C((\phi s_2)^C, (\phi\varepsilon_i)^C)].
\end{aligned} \tag{22}
$$

In view of (15), (16), (17), (18) and (19), we infer

$$\sum_{i=1}^{n-1}(g(R(\phi\varepsilon_i,\phi s_2)\phi s_3,\phi\varepsilon_i))^C = S^C((\phi s_2)^C,(\phi s_3)^C) + g^C((\phi s_2)^C,(\phi s_3)^C), \tag{23}$$

$$\sum_{i=1}^{n-1}S^C((\phi\varepsilon_i)^C,(\phi\varepsilon_i))^C = \tau-(n-1)\alpha^2, \tag{24}$$

$$\sum_{i=1}^{n-1}[g^C((\phi\varepsilon_i)^C,(\phi s_3)^C)S^C((\phi s_2)^V,(\phi\varepsilon_i)^C) + g^C((\phi\varepsilon_i)^V,(\phi s_3)^C)S^C((\phi s_2)^C,(\phi\varepsilon_i)^C) = S^C((\phi s_2)^C,(\phi s_3)^C) \tag{25}$$

$$\sum_{i=1}^{n-1}g^C((\phi\varepsilon_i)^C,(\phi\varepsilon_i))^C = n-1, \tag{26}$$

and

$$\sum_{i=1}^{n-1}[g^C((\phi\varepsilon_i)^C,(\phi s_3)^C)g^C((\phi s_2)^V,(\phi\varepsilon_i)^C) + g^C((\phi\varepsilon_i)^V,(\phi s_3)^C)g^C((\phi s_2)^C,(\phi\varepsilon_i)^C) = g^C((\phi s_2)^C,(\phi s_3)^C). \tag{27}$$

Making use of (23)–(27) the equation (22) can be expressed as

$$S^C((\phi s_2)^C,(\phi s_3)^C) = L_1 g^C((\phi s_2)^C,(\phi s_3)^C), \tag{28}$$

where $L_1 = [\frac{\tau}{n-1}-(n-1)\alpha^2-(n-2)]$. Using (17) and (19), the equation (28) becomes

$$S^C({s_2}^C,{s_3}^C) = L_1 g^C({s_2}^C,{s_3}^C) + L_1\{\eta^C({s_2}^C)\eta^V({s_3}^C)+\eta^V({s_2}^C)\eta^C({s_3}^C)\}.$$

Thus $(\mathsf{TM}, g^C)$ is an $\eta^C$-Einstein manifold.

On the similar devices of Theorem 4.1 and using definition 1, we have

*Theorem 2.* Let $\mathsf{M}^n$, $(n>3)$ be $\phi$-conharmonically flat Lorentzian $\alpha$-Sasakian manifold and $\mathsf{TM}$ be its tangent bundle. Then $(\mathsf{TM}, g^C)$ is an $\eta^C$-Einstein manifold.

*Theorem 3.* For any $\phi$-projectively flat Lorentzian $\alpha$-Sasakian manifold $\mathsf{M}^n$ $(n>3)$, $(\mathsf{TM}, g^C)$ is an $\eta^C$-Einstein manifold.

*Theorem 4.* For any $\phi$-concircularly flat Lorentzian $\alpha$-Sasakian manifold $\mathsf{M}^n$ $(n>3)$, $(\mathsf{TM}, g^C)$ will be an $\eta^C$-Einstein manifold.

## *4 Example*

Assume a differentiable manifold $\mathsf{M} = \{(u,v,w) : u,v,w \in \Re^3, w>0\}$ and denote the L.I. vector fields on $\mathsf{M}$ by $\varsigma_1,\varsigma_2,\varsigma_3$ given by [32]

$$\varsigma_1 = \varsigma^{-w}\frac{\partial}{\partial v}, \varsigma_2 = \varsigma^{-w}(\frac{\partial}{\partial u}+\frac{\partial}{\partial v}), \varsigma_3 = \alpha\frac{\partial}{\partial w} = \xi.$$

Further for 1-form $\eta$ on $\mathsf{M}$, one can write

$$g(\varsigma_1,\varsigma_2) = g(\varsigma_1,\varsigma_3) = g(\varsigma_2,\varsigma_3) = 0, \quad g(\varsigma_1,\varsigma_1) = g(\varsigma_2,\varsigma_2) = 1, \quad g(\varsigma_3,\varsigma_3) = -1$$

and

$$\eta(s_3) = g(s_3, \varsigma_3), \quad s_3 \in \Im_0^1(\mathsf{M}),$$

where $g$ is the Lorentzian metric.

Suppose $\phi$ stands for $(1,1)$-tensor field satisfying

$$\phi\varsigma_1 = \varsigma_1, \quad \phi\varsigma_2 = \varsigma_2, \quad \phi\varsigma_3 = 0.$$

With the linearity of $\phi$, one concludes $\eta(\varsigma_3) = -1$, $\phi^2\zeta_3 = s_3 + \eta(s_3)\varsigma_3$ and $g(\phi s_1, \phi s_2) = g(s_1, s_2) + + \eta(s_1)\eta(s_2)$.

Thus, (for $\varsigma_3 = \xi$) $\mathsf{M}$ becomes Lorentzian almost paracontact metric manifold with Lorentzian almost paracontact metric structure $(\phi, \xi, \eta, g)$ on $\mathsf{M}$.

Also,

$$[\varsigma_1, \varsigma_2] = 0, \quad [\varsigma_1, \varsigma_3] = \alpha\varsigma_1, \quad [\varsigma_2, \varsigma_3] = \alpha\varsigma_2.$$

The Koszul's formula is written as

$$\begin{aligned} 2g(\nabla_{\varsigma_1}\varsigma_2, s_3) &= Xg(s_2, s_3) + s_2 g(s_3, s_1) - \varsigma_3 g(s_1, s_2) \\ &- g(s_1, [s_2, s_3]) + g(s_2, [s_3, s_1]) + g(s_3, [s_1, s_2]), \end{aligned}$$

and we have [32]

$$\nabla_{\varsigma_1}\varsigma_1 = \alpha\varsigma_3, \quad \nabla_{\varsigma_1}\varsigma_3 = \alpha\varsigma_1, \quad \nabla_{\varsigma_2}\varsigma_2 = \alpha\varsigma_3, \quad \nabla_{\varsigma_2}\varsigma_3 = \alpha\varsigma_2, \tag{29}$$

$$\nabla_{\varsigma_1}\varsigma_2 = \nabla_{\varsigma_3}\varsigma_1 = \nabla_{\varsigma_3}\varsigma_2 = \nabla_{\varsigma_2}\varsigma_1 = \nabla_{\varsigma_3}\varsigma_3 = 0. \tag{30}$$

We can easily verify that

$$\begin{aligned} \nabla_{s_1}\xi &= -\alpha\phi s_1, \\ (\nabla_{s_1}\eta)s_2 &= -\alpha g(\phi s_1, s_2). \end{aligned}$$

Hence, $M$ is a "Lorentzian $\alpha$-Sasakian manifold".

Let us denote the complete and vertical lifts of $\varsigma_1, \varsigma_2, \varsigma_3$ on $T\mathsf{M}$ by $\varsigma_1^C, \varsigma_2^C, \varsigma_3^C$ and $\varsigma_1^V, \varsigma_2^V, \varsigma_3^V$. Further, suppose that $g^C$ be the complete lift of a Riemannian metric $g$ on $\mathsf{TM}$ holding

$$\begin{aligned} g^C({s_1}^V, \varsigma_3^C) &= (g^C(s_1, \varsigma_3))^V = (\eta(s_1))^V, \\ g^C({s_1}^C, \varsigma_3^C) &= (g^C(s_1, \varsigma_3))^C = (\eta(s_1))^C, \\ g^C(\varsigma_3^C, \varsigma_3^C) &= -1, \quad g^V({s_1}^V, \varsigma_3^C) = 0, \quad g^V(\varsigma_3^V, \varsigma_3^V) = 0 \end{aligned} \tag{31}$$

and so on.

Consider the $(1,1)$-tensor field $\phi$ and its complete and vertical lifts $\phi^C$ and $\phi^V$ by

$$\phi^V(\varsigma_1^V) = \varsigma_1^V, \quad \phi^C(\varsigma_1^C) = \varsigma_1^C.$$

$$\phi^V(\varsigma_2^V) = \varsigma_2^V, \quad \phi^C(\varsigma_2^C) = \varsigma_2^C,$$

$$\phi^V(\varsigma_3^V) = \phi^C(\varsigma_3^C) = 0.$$

Above equation produces

$$(\phi^2 X)^C = {s_1}^C + \eta^V(s_1)\varsigma_3^C + \eta^C(s_1)\varsigma_3^V, \tag{32}$$

$$\begin{aligned} g^C((\phi\varsigma_1)^C, (\phi\varsigma_2)^C) &= g^C(\varsigma_1^C, \varsigma_2^C) + (\eta(\varsigma_1))^C(\eta(\varsigma_2))^V \\ &+ (\eta(\varsigma_1))^V(\eta(\varsigma_2))^C. \end{aligned}$$

Thus, for $\varsigma_3 = \xi$ in (31)–(32), the structure $(\phi^C, \xi^C, \eta^C, g^C)$ is a Lorentzian almost paracontact metric structure on $\mathsf{TM}$.

The Koszul's formula for $\nabla^C$ can be viewed as

$$\begin{aligned} 2g^C(\nabla^C_{\varsigma_1^C}\varsigma_2^C, s_3^C) &= X^C g^C(s_2^C, s_3^C) + s_2^C g^C(s_3^C, s_1^C) - \varsigma_3^C g^C(s_1^C, s_2^C) \\ &- g^C(s_1^C, [s_2^C, s_3^C]) + g^C(s_2^C, [s_3^C, s_1^C]) + g^C(s_3^C, [s_1^C, s_2^C]). \end{aligned}$$

Taking the complete lift on (29) and (30), we conclude

$$\begin{aligned} \nabla^C_{\varsigma_1^C}\varsigma_1^C &= \alpha\varsigma_3^C, \;\; \nabla^C_{\varsigma_1^C}\varsigma_3^C = \alpha\varsigma_1^C, \;\; \nabla^C_{\varsigma_2^C}\varsigma_2^C = \alpha\varsigma_3^C, \;\; \nabla^C_{\varsigma_2^C}\varsigma_3^C = \alpha\varsigma_2^C, \\ \nabla^C_{\varsigma_1^C}\varsigma_2^C &= \nabla^C_{\varsigma_3^C}\varsigma_1^C = \nabla^C_{\varsigma_3^C}\varsigma_2^C = \nabla^C_{\varsigma_2^C}\varsigma_1^C = \nabla^C_{\varsigma_3^C}\varsigma_3^C = 0. \end{aligned}$$

We can easily verify that

$$\begin{aligned} \nabla^C_{s_1^C}\xi^C &= -\alpha\phi^C s_1^C, \\ (\nabla^C_{s_1^C}\eta^C)s_2^C &= -\alpha g^C((\phi s_1)^C, s_2^C). \end{aligned}$$

Hence, $(\phi^C, \xi^C, \eta^C, g^C, \mathsf{TM})$ is a Lorentzian $\alpha$-Sasakian manifold.

*Conflict of Interest*

The authors declare no conflict of interest.

*Author Contributions*

All authors contributed equally to this work.



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