Assume that φ is an admissible function, ν is a nondegenerate nonnegative Borel measure on $[0,\infty)$, h is the fundamental function of ν and $f \in \mathfrak{M}^+(0,\infty)$. If $\{x_k\}$ is a discretizing sequence for h with respect to φ, then

$$\int\limits_{[0,\infty)} \sup_{y\in(0,\infty)} \frac{|f(y)|}{\varphi(x)+\varphi(y)} d\nu(x) \approx \sum_{k\in\mathbb{Z}} \left(\sup_{y\in(0,\infty)} \frac{|f(y)|}{\varphi(x_k)+\varphi(y)} \right) \varphi(x_k)$$

$$\approx \sum_{k\in\mathbb{Z}} \left(\varphi^{-1}(x_k) \sup_{x_{k-1}\le y<x_k} |f(y)| + \sup_{x_k\le y<x_{k+1}} |f(y)|\varphi^{-1}(y) \right) h(x_k)$$

$$\approx \sum_{k\in\mathbb{Z}} \sup_{x_k\le y<x_{k+1}} |f(y)|\varphi^{-1}(y)h(y).$$

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ANALYSIS OF LIMITING VALUES OF THE FRACTIONAL DERIVATE'S ORDER FOR A CAUCHY-TYPE PROBLEM.

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In this paper we consider a Cauchy type problem for a differential equation with fractional Riemann-Liouville derivatives of different orders. Special attention is paid to the study of limiting cases of changing the order of fractional derivatives. The intervals of values for which the existence and uniqueness theorem holds are determined. The results show that when the order of the fractional derivative tends to an integer value, the solution continuously transitions to the classical case. This allows us to establish a connection between fractional and integer order equations, which confirms the correctness of the chosen approach. The methods of integral transformations and the theory of special functions is applied in the research process. This material can serve as an additional resource for studying the methods of fractional analysis and modern trends in the theory of differential equations.

In recent years, there has been a growing interest in fractional calculus within the scientific community. This trend is well justified, as many physical and chemical processes occurring in fractal and heterogeneous media are more accurately described by equations that include fractional integro-differential operators. The use of such operators not only simplifies complex models, but also enhances our understanding of various applied phenomena. These include the modeling of epidemic spread, optimization of energy production, control of ecosystems, and other real-world systems.

Fractional calculus is a branch of mathematical analysis that extends the classical concepts of differentiation and integration to non-integer (fractional) orders. Within this framework, one introduces operators of arbitrary order, which allow for more flexible modeling of processes characterized by memory and hereditary effects.

The origins of fractional calculus trace back to 1695, when G. Leibniz and G. de L'Hopital exchanged ideas on the possibility of taking derivatives of arbitrary order. Since then, numerous distinguished mathematicians have contributed to the development of the field. Among them are L. Euler, B. Riemann [1], J. Liouville, N. Abel, J. Fourier, A. Grünwald, J. Bernoulli, G. Leibniz, P.A. Nekrasov, M.M. Djrbashian, I. Podlubny [2], A.M. Nakhushev [3], J. Hadamard [4], K.A. Krug, N.Ya. Sonin, V. Miller, H. Ross, D. Caputo, and others. Today, the concepts of fractional derivatives and integrals are most commonly associated with the works of Bernhard Riemann and Joseph Liouville.

This paper addresses a Cauchy-type problem for a differential equation with Riemann–Liouville fractional derivatives of different orders. We present an explicit solution to the problem, analyze limiting cases of the derivative orders, and demonstrate that the solution is continuous with respect to changes in the order of the fractional derivative.

Solution of the Cauchy type problem.

The basic concepts used in the paper can be seen in detail in [5]-[9].

Consider an equation involving fractional derivatives of different orders:

$$ {}_0D_t^{\alpha} y(t) - \lambda_0 D_t^{\beta} y(t) = f(t) \tag{1} $$

where ${}_0D_t^{\alpha}$ is the Riemann-Liouville derivative of order $1 < \alpha < 2$, ${}_0D_t^{\alpha}$ is the Riemann-Liouville derivative of order $0 < \beta < 1$, λ is a real parameter.

For the existence of the Riemann–Liouville derivatives in equation (1), it is sufficient, by definition, that the function $f(t)$, $t \in [0, T]$, is summable. That is $f(t) \in L_1[0, T]$. Therefore, it is necessary to introduce the concept of a regular solution.

A regular solution of equation (3) is a function $y = y(t)$, which satisfies the following conditions:

1. $y(t) \in L_1([0, T])$;
2. $y(t)$ satisfies equation (3) at every point of the interval $(0, T)$.

The general solution of equation (1) is of the form [10].

$$ y(t) = f(t) * t^{\alpha-1} E_{\alpha-\beta;\alpha}\left(\lambda t^{\alpha-\beta}\right) + c_1 t^{\alpha-1} E_{\alpha-\beta;\alpha}\left(\lambda t^{\alpha-\beta}\right) + c_2 t^{\alpha-2} E_{\alpha-\beta;\alpha-1}\left(\lambda t^{\alpha-\beta}\right) \tag{2} $$

where $E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k+\beta)}$ is the Mittag-Leffler function; $\alpha > 0, \beta \in R$.

In [10] the following Cauchy-type problem with fractional derivatives of different orders was posed:

We need to find a solution to the equation

$$ {}_0D_t^{\alpha} y\,(t) - \lambda\ {}_0D_t^{\beta} y\,(t) = f\,(t)\,,\ \ t \in (0, T)\,(3) $$

where ${}_0D_t^{\alpha}$ is the Riemann-Liouville derivative of order $1 < \alpha < 2$, ${}_0D_t^{\alpha}$ is the Riemann-Liouville derivative of order $0 < \beta < 1, \lambda$ is a real parameter.

The solution is required to satisfy the following initial conditions:

$$ \begin{cases} \lim_{t\to 0}\left({}_0D_t^{\alpha-1}y(t) - \lambda_0 D_t^{\beta-1}y(t)\right) = a_0, \\ \lim_{t\to 0}\,{}_0D_t^{\alpha-2}y(t) = b_0. \end{cases} \tag{4} $$

As shown earlier, the general solution of equation (3) is given by formula (2). By applying the initial conditions (4) to this solution, we obtain

$$ y\,(t) = a_0 t^{\alpha-1} E_{\alpha-\beta,\alpha}\left(\lambda t^{\alpha-\beta}\right) + b_0 t^{\alpha-2} E_{\alpha-\beta,\alpha-1}\left(\lambda t^{\alpha-\beta}\right) + f\,(t) * \left(t^{\alpha-1} E_{\alpha-\beta,\alpha}(\lambda t^{\alpha-\beta})\right) \tag{5} $$

In [10], the existence and uniqueness theorem for the solution of the problem (3)–(4) was proved.

Theorem. The Cauchy-type problem (3) – (4) has a unique regular solution defined by formula (5), for any given values of $a_0 and b_0$.

Continuity analysis of the solution to the Cauchy-type problem with respect to the order of fractional derivatives.

Consider the problem (3) – (4) for $\alpha = 2, \beta = 1$

$$ y''\,(t) - \lambda y'\,(t) = f(t) \tag{3^1} $$

$$ \begin{cases} \lim_{t\to 0} y'(t) - \lambda y(t) = a_0 \\ \lim_{t\to 0} y(t) = b_0 \end{cases} \tag{4^1} $$

Solving the equation $(3^1), we obtain:$

$$ y\,(t) = C_1 + C_2 e^{\lambda t} + \frac{1}{\lambda}\int_0^t e^{\lambda(t-\tau)} f(\tau) d\tau - \frac{1}{\lambda}\int_0^t f\,(\varphi)\, d\tau \tag{6} $$

- the general solution of equation (3^1).

Let us apply the condition (4^1) to equation (6). To differentiate under the integral sign, we use the Leibniz rule. As a result, we obtain the following system:

$$ \begin{cases} \lim_{t\to 0}\left((C_2\lambda 2\lambda t + \frac{1}{\lambda}\left(f(t) + \int_0^t \lambda e^{\lambda(t-\tau)} f(\tau)d\tau\right) - \frac{1}{\lambda}f(t) - \lambda(C_1 + C_2 e^{\lambda t}\right. \\ \qquad \left. + \frac{1}{\lambda}\int_0^t \lambda e^{\lambda(t-\tau)} f(\tau)d\tau - \frac{1}{\lambda}\int_0^t f(\tau)d\tau\right) = a_0, \\ \lim_{t\to 0}\left(C_1 + C_2 e^{\lambda t} + \frac{1}{\lambda}\int_0^t \lambda e^{\lambda(t-\tau)} f(\tau)d\tau - \frac{1}{\lambda}\int_0^t f(\tau)d\tau\right) = b_0 \end{cases} $$

From this system, we obtain the following relations:

$$ C_1 = -\frac{a_0}{\lambda} $$

$$C_2 + C_1 = b_0 \ \rightarrow \ C_2 = b_0 + \frac{a_0}{\lambda}$$

Now let us substitute the obtained values of C_1 *and* C_2 into equation (6). This gives the following expression for the solution:

$$y\left(t\right) = -\frac{a_0}{\lambda} + \left(b_0 + \frac{a_0}{\lambda}\right) e^{\lambda t} + \frac{1}{\lambda}\int_0^t e^{\lambda(t-\tau)} f\left(\tau\right) d\tau - \frac{1}{\lambda}\int_0^t f\left(\tau\right) d\tau. \tag{7}$$

On the other hand, when $\alpha = 2, \beta = 1$ the general solution (3) of the original problem (3) – (4) has the form:

$$y\left(t\right) = a_0 t E_{1;2}\left(\lambda t\right) + b_0 E_{1;1}\left(\lambda t\right) + f\left(t\right) * \left(t E_{1;2}\left(\lambda t\right)\right).$$

From the auto-transformation formula:

$$E_{\alpha;\mu}\left(x\right) = 1 + xE; +x,$$

and setting $\alpha \ = \ 2, \ \mu \ = \ 1$ we obtain:

$$E_{1;2}\left(\lambda t\right) = \frac{1}{\lambda t}\left(e^{\lambda t} - 1\right); \ E_{1;1} = e^{\lambda t}.$$

Therefore, the solution of the Cauchy-type problem takes the form:

$$y\left(t\right) = \frac{a_0}{\lambda}\left(e^{\lambda t} - 1\right) + b_0 e^{\lambda t} + \frac{1}{\lambda}\left(\int_0^t f\left(\tau\right) e^{\lambda(t-\tau)} d\tau - \int_0^t f\left(\tau\right) d\tau. \tag{8}$$

Equation (9) coincides with equation (7). Hence, the solution of the original problem can be extended to the case where $\alpha \in (1;2]$ *and* $\beta \in (0;1]$. The results of the paper [10] can therefore be generalized to these intervals.

As a result, the following lemma is proved: For the Cauchy-type problem (3) – (4), the solution is continuous from the right with respect to the fractional orders α *and* β.

Conclusion. In this paper, the limiting cases of the fractional derivative orders in the Cauchy-type problem were investigated. It was shown that the solution obtained for these limiting values coincides with the solution of the corresponding Cauchy problem involving integer-order derivatives. This confirms the correctness of the proposed approach and its connection with the classical theory of differential equations.

The practical value of the study lies in the broad applicability of fractional calculus across various fields of science and engineering. These include physics, control theory, transport modeling, anomalous diffusion, mechanical and electrical systems with memory, and even biological and economic models. The results obtained contribute to a deeper understanding of the fundamental behavior of such systems and extend the possibilities for their modeling and application.

This article may also serve educational purposes. It can be used as supplementary material in the study of methods for solving differential equations with fractional derivatives. The findings may be useful for students studying fractional calculus and for instructors preparing specialized courses on this subject and its applications.

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NONLINEAR INTEGRAL-DIFFERENTIAL EQUATION WITH ZERO OPERATOR OF DIFFERENTIAL PART AND RAPIDLY OSCILLATING COSINE

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In this work, the regularization method of S.A. Lomov (1; 2) is generalized for problems for an integro-differential equation with a rapidly changing kernel and with a right-hand side depending on a rapidly oscillating cosine

$$\varepsilon \frac{dy}{dt} = \int\limits_0^t e^{\frac{1}{\varepsilon}\int\limits_s^t \mu(\theta)d\theta} K(t,s)y(s,\varepsilon)ds + h(t)\cos\frac{\beta(t)}{\varepsilon} + \varepsilon f(y,t), \ y(0,\varepsilon) = y^0, \ t \in [0,T]. \quad (1)$$

Problem (1) is considered under the following conditions:

1) $\mu(t), \beta'(t), h(t) \in \mathrm{C}^\infty([0,T],R), \ K(t,s) \in \mathrm{C}^\infty(0 \le s \le t \le T, R)$;

2) $\mu(t) < 0, \beta'(t) > 0 \ \forall t \in [0,T]$;

3) $f(y,t)$ is a polynomial, i.e. $f(y,t) = \sum\limits_{m=0}^{N} f_m(t)y^m$ with the coefficients $f_m(t) \in \in C^\infty([0,T],R), m = \overline{0,N}, \ N < \infty$.

We introduce a new unknown function

$$z = \int\limits_0^t e^{\frac{1}{\varepsilon}\int\limits_s^t \mu(\theta)d\theta} K(t,s)y(s,\varepsilon)ds.$$