МАТЕМАТИКА

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On a singular integral equation of Volterra and its adjoint one

The paper deals solutions of mutually conjugate Volterra integral equations of the second kind with a variable limit of integration. Feature of equations consists of incompressibility of the corresponding kernels and, consequently, of the non-applicability of the successive approximation method. The non-zero solutions of singular homogeneous Volterra integral equations of the second kind were obtained. By direct verification it is established that the resulting functions are non-trivial solutions of the homogeneous Volterra integral equation of the second kind with a singular kernel and its adjoint equation.

Key words: volterra integral equation of the second kind, a special kernel, non-trivial solution, adjoint equation.

In the paper [1] solutions mutually conjugate of Volterra integral equations of the second kind for $a=1$ are obtained

$$\varphi(t)-\frac{1}{2a\sqrt{\pi}}\int_0^t\left[\frac{t+\tau}{(t-\tau)^{3/2}}\exp\left\{-\frac{(t+\tau)^2}{4a^2(t-\tau)}\right\}+\frac{1}{(t-\tau)^{1/2}}\exp\left\{-\frac{t-\tau}{4a^2}\right\}\right]\varphi(\tau)\,d\tau=0, \qquad (1)$$

$(t>0)$

and

$$\psi(t)-\frac{1}{2a\sqrt{\pi}}\int_t^\infty\left[\frac{\tau+t}{(\tau-t)^{3/2}}\exp\left\{-\frac{(\tau+t)^2}{4a^2(\tau-t)}\right\}+\frac{1}{(\tau-t)^{1/2}}\exp\left\{-\frac{\tau-t}{4a^2}\right\}\right]\psi(\tau)\,d\tau=0, \qquad (2)$$

$(t>0)$.

As it was noted, a feature of the investigated equations consists in incompressibility of kernels and is expressed in the fact that the corresponding inhomogeneous equations can not be solved by successive approximation method.

Equations of this type arise in the study of some non-local inner-boundary value problems for a parabolic equation, spectrally loaded parabolic equations, problems with a moving boundary and inverse problems for parabolic equations, etc.

The aforementioned article it was shown that the solution of the integral equation (1) is the function

$$\varphi(t)=C\cdot\left\{\frac{1}{\sqrt{t}}e^{-\frac{t}{4a^2}}+\frac{\sqrt{\pi}}{2a}erf\left(\frac{\sqrt{t}}{2a}\right)+\frac{\sqrt{\pi}}{2a}\right\}, \qquad (3)$$

and a solution of the adjoint equation (2) is the function

$$\psi(t)=\frac{C}{a\sqrt{\pi}}\sum_{n=0}^{\infty}(2n+1)\exp\left(-\frac{n^2+n}{a^2}t\right). \qquad (4)$$

The reason for writing this article was a doubt some readers that the functions (3) and (4) are indeed solution of homogeneous Volterra integral equations of the second kind (1) and (2) accordingly. By direct verification we will show that these functions are solutions of the corresponding equations. In addition, the calculations themselves are of independent interest in terms of finding unusual ways of solving non-standard integral equations.

1. Nontrivial solution of the direct homogeneous Volterra integral equation of the second kind with a singular kernel

In the course of solving the integral equation (1) was established [2; 183] that it is sufficient for its solution to find a solution of «simplified» equation

$$\varphi(t)-\int_0^t k(t,\tau)\varphi(\tau)\,d\tau=0, \tag{5}$$

where

$$k(t,\tau)=\frac{1}{2a\sqrt{\pi}}\left\{\frac{2t}{(t-\tau)^{3/2}}\exp\left\{-\frac{t\tau}{a^2(t-\tau)}\right\}+\frac{1}{(t-\tau)^{1/2}}\left(1-\exp\left\{-\frac{t\tau}{a^2(t-\tau)}\right\}\right)\right\}.$$

In [1] «simplified» integral equation (3) is reduced to the Abel integral equation of the second kind

$$\varphi(t)-\frac{1}{2a\sqrt{\pi}}\int_0^t\frac{\varphi(\tau)}{\sqrt{t-\tau}}\,d\tau=\frac{C}{\sqrt{t}}. \tag{6}$$

The solution of the Abel equation (6), ie the solution of the «simplified» homogeneous equation (5) is a function

$$\varphi_0(t)=\frac{1}{\sqrt{t}}+\frac{\sqrt{\pi}}{2a}e^{\frac{t}{4a^2}}erf\left(\frac{\sqrt{t}}{2a}\right)+\frac{\sqrt{\pi}}{2a}e^{\frac{t}{4a^2}}. \tag{7}$$

By [2], after multiplication of equality (7) to $e^{-\frac{t}{4a^2}}$ we obtain a solution of the homogeneous equation (1), that is, the function (3) (For simplicity, we adopted $C=1$).

We show that the function (7) satisfies the equation (5). After substituting (7) into (5), considering that the function (7) is the solution of equation (6), we need to prove that the function (7) satisfies equation

$$\frac{1}{2a\sqrt{\pi}}\int_0^t\frac{t+\tau}{(t-\tau)^{3/2}}\exp\left\{-\frac{t\tau}{a^2(t-\tau)}\right\}\varphi(\tau)d\tau=\frac{1}{\sqrt{t}}. \tag{8}$$

We substitute (7) in the left side of (8):

$$\frac{1}{2a\sqrt{\pi}}\int_0^t\frac{t+\tau}{(t-\tau)^{3/2}}\exp\left\{-\frac{t\tau}{a^2(t-\tau)}\right\}\left(\frac{1}{\sqrt{\tau}}+\frac{\sqrt{\pi}}{2a}e^{\frac{\tau}{4a^2}}erf\left(\frac{\sqrt{\tau}}{2a}\right)+\frac{\sqrt{\pi}}{2a}e^{\frac{\tau}{4a^2}}\right)d\tau=\frac{1}{\sqrt{t}},$$

or

$$\frac{1}{2a\sqrt{\pi}}\left[\int_0^t\frac{t+\tau}{(t-\tau)^{3/2}}\exp\left\{-\frac{t\tau}{a^2(t-\tau)}\right\}\frac{1}{\sqrt{\tau}}d\tau+\frac{\sqrt{\pi}}{2a}\int_0^t\frac{t+\tau}{(t-\tau)^{3/2}}\exp\left\{-\frac{t\tau}{a^2(t-\tau)}\right\}e^{\frac{\tau}{4a^2}}erf\left(\frac{\sqrt{\tau}}{2a}\right)d\tau+\right.$$

$$\left.+\frac{\sqrt{\pi}}{2a}\int_0^t\frac{t+\tau}{(t-\tau)^{3/2}}\exp\left\{-\frac{t\tau}{a^2(t-\tau)}\right\}e^{\frac{\tau}{4a^2}}d\tau\right]=\frac{1}{\sqrt{t}}.$$

Thus we need to show validity of the equality

$$\frac{1}{2a\sqrt{\pi}}\left(I_1(t)+I_2(t)+I_3(t)\right)=\frac{1}{\sqrt{t}}, \tag{9}$$

where

$$I_1(t)=\int_0^t\frac{t+\tau}{(t-\tau)^{3/2}}\exp\left\{-\frac{t\tau}{a^2(t-\tau)}\right\}\frac{1}{\sqrt{\tau}}d\tau;$$

$$I_2(t)=\frac{\sqrt{\pi}}{2a}\int_0^t\frac{t+\tau}{(t-\tau)^{3/2}}\exp\left\{-\frac{t\tau}{a^2(t-\tau)}\right\}e^{\frac{\tau}{4a^2}}erf\left(\frac{\sqrt{\tau}}{2a}\right)d\tau;$$

$$I_3(t)=\frac{\sqrt{\pi}}{2a}\int_0^t\frac{t+\tau}{(t-\tau)^{3/2}}\exp\left\{-\frac{t\tau}{a^2(t-\tau)}\right\}e^{\frac{\tau}{4a^2}}d\tau.$$

We calculate the first integral

$$I_1(t)=\int_0^t \frac{t+\tau}{(t-\tau)^{3/2}\sqrt{\tau}}\exp\left\{-\frac{t\tau}{a^2(t-\tau)}\right\}d\tau.$$

The change is $z=\sqrt{\frac{\tau}{t-\tau}}$.

Then

$$I_1(t)=2\left[\int_0^\infty \exp\left\{-\frac{t}{a^2}z^2\right\}dz+\int_0^\infty \frac{z^2}{z^2+1}\exp\left\{-\frac{t}{a^2}z^2\right\}dz\right].$$

The first integral is Euler-Poisson integral, for the second integral we use the formula (3.466 (2)) of [3]. As a result, we obtain

$$I_1(t)=\frac{2a\sqrt{\pi}}{\sqrt{t}}-\pi\exp\left\{\frac{t}{a^2}\right\}\cdot erfc\left\{\frac{\sqrt{t}}{a}\right\}. \tag{10}$$

We calculate the integral

$$I_3(t)=\frac{\sqrt{\pi}}{2a}\exp\left\{\frac{t}{a^2}\right\}\int_0^t \frac{t+\tau}{(t-\tau)^{3/2}}\exp\left\{-\frac{t^2}{a^2(t-\tau)}\right\}\exp\left\{\frac{\tau}{4a^2}\right\}d\tau.$$

The change is $z=\sqrt{t-\tau}$. Then

$$I_3(t)=\frac{\sqrt{\pi}}{a}\exp\left\{\frac{5t}{4a^2}\right\}\int_0^{\sqrt{t}} \frac{2t-z^2}{z^2}\exp\left\{-\frac{t^2}{a^2z^2}-\frac{z^2}{4a^2}\right\}dz.$$

As

$$\frac{t^2}{a^2z^2}+\frac{z^2}{4a^2}=\left(\frac{t}{az}+\frac{z}{2a}\right)^2-\frac{t}{a^2}$$

and

$$\left(\frac{2t}{z^2}-1\right)dz=d\left(-\frac{2t}{z}-z\right)=-2a\,d\left(\frac{t}{az}+\frac{z}{2a}\right),$$

than

$$I_3(t)=-2\sqrt{\pi}\exp\left\{\frac{9t}{4a^2}\right\}\int_0^{\sqrt{t}} \exp\left\{-\left(\frac{t}{az}+\frac{z}{2a}\right)^2\right\}d\left(\frac{t}{az}+\frac{z}{2a}\right).$$

The change $\xi=\frac{t}{az}+\frac{z}{2a}$ leads to the result

$$I_3(t)=\pi\exp\left\{\frac{9t}{4a^2}\right\}\cdot erfc\left(\frac{3\sqrt{t}}{2a}\right). \tag{11}$$

We consider the integral

$$I_2(t)=\frac{\sqrt{\pi}}{2a}\int_0^t \frac{t+\tau}{(t-\tau)^{3/2}}\exp\left\{-\frac{t\tau}{a^2(t-\tau)}\right\}\exp\left\{\frac{\tau}{4a^2}\right\}erf\left(\frac{\sqrt{\tau}}{2a}\right)d\tau.$$

As the $erf(t)=\frac{2}{\sqrt{\pi}}\int_0^t \exp\{-x^2\}dx$, then

$$I_2(t)=\frac{1}{a}\int_0^t \frac{t+\tau}{(t-\tau)^{3/2}}\exp\left\{-\frac{t\tau}{a^2(t-\tau)}\right\}\exp\left\{\frac{\tau}{4a^2}\right\}\int_0^{\frac{\sqrt{\tau}}{2a}} \exp\{-x^2\}dx\,d\tau.$$

We interchange order of integration

$$I_2(t)=\frac{1}{a}\int_0^{\frac{\sqrt{t}}{2a}} \exp\{-x^2\}\int_{4a^2x^2}^t \frac{t+\tau}{(t-\tau)^{3/2}}\exp\left\{-\frac{t\tau}{a^2(t-\tau)}\right\}\exp\left\{\frac{\tau}{4a^2}\right\}d\tau\,dx.$$

To calculate the inner integral we introduce the change $z=\sqrt{t-\tau}$. Then (analogously as for calculating integral $I_3(t)$)

$$I_2(t)=\frac{2}{a}\exp\left\{\frac{5t}{4a^2}\right\}\int\limits_0^{\frac{\sqrt{t}}{2a}}\exp\left\{-x^2\right\}\int\limits_0^{\sqrt{t-4a^2x^2}}\frac{2t-z^2}{z^2}\exp\left\{-\frac{t^2}{a^2z^2}-\frac{z^2}{4a^2}\right\}dz\,dx=$$

$$=-4\exp\left\{\frac{9t}{4a^2}\right\}\int\limits_0^{\frac{\sqrt{t}}{2a}}\exp\left\{-x^2\right\}\cdot\left\{\int\limits_0^{\sqrt{t-4a^2x^2}}\exp\left\{-\left(\frac{t}{az}+\frac{z}{2a}\right)^2\right\}d\left(\frac{t}{az}+\frac{z}{2a}\right)\right\}dx.$$

The change $\xi=\dfrac{t}{az}+\dfrac{z}{2a}$ leads to the result

$$I_2(t)=4\exp\left\{\frac{9t}{4a^2}\right\}\int\limits_0^{\frac{\sqrt{t}}{2a}}\exp\left\{-x^2\right\}\cdot\left\{\int\limits_{\frac{t}{a\sqrt{t-4a^2x^2}}+\frac{\sqrt{t-4a^2x^2}}{2a}}^{\infty}\exp\left\{-\xi^2\right\}d\xi\right\}dx=$$

$$=2\sqrt{\pi}\exp\left\{\frac{9t}{4a^2}\right\}\int\limits_0^{\frac{\sqrt{t}}{2a}}\exp\left\{-x^2\right\}\cdot erfc\left\{\frac{t}{a\sqrt{t-4a^2x^2}}+\frac{\sqrt{t-4a^2x^2}}{2a}\right\}dx. \qquad (12)$$

The results (10)–(12) we substitute in (9):

$$\frac{1}{2a\sqrt{\pi}}\left(\frac{2a\sqrt{\pi}}{\sqrt{t}}-\pi\exp\left\{\frac{t}{a^2}\right\}\cdot erfc\left\{\frac{\sqrt{t}}{a}\right\}+\pi\exp\left\{\frac{9t}{4a^2}\right\}\cdot erfc\left(\frac{3\sqrt{t}}{2a}\right)+\right.$$

$$\left.+2\sqrt{\pi}\exp\left\{\frac{9t}{4a^2}\right\}\int\limits_0^{\frac{\sqrt{t}}{2a}}\exp\left\{-x^2\right\}\cdot erfc\left\{\frac{t}{a\sqrt{t-4a^2x^2}}+\frac{\sqrt{t-4a^2x^2}}{2a}\right\}dx\right)=\frac{1}{\sqrt{t}}.$$

After simplification we obtain the equality

$$\int\limits_0^{\frac{\sqrt{t}}{2a}}\exp\left\{-x^2\right\}\cdot erfc\left\{\frac{t}{a\sqrt{t-4a^2x^2}}+\frac{\sqrt{t-4a^2x^2}}{2a}\right\}dx=\frac{\sqrt{\pi}}{2}\left(\exp\left\{-\frac{5t}{4a^2}\right\}\cdot erfc\left\{\frac{\sqrt{t}}{a}\right\}-erfc\left(\frac{3\sqrt{t}}{2a}\right)\right).$$

We differentiate this equality on t on both sides

$$\frac{1}{4a\sqrt{t}}\exp\left\{-\frac{t}{4a^2}\right\}\lim_{x\to\frac{\sqrt{t}}{2a}}\left(erfc\left\{\frac{t}{a\sqrt{t-4a^2x^2}}+\frac{\sqrt{t-4a^2x^2}}{2a}\right\}\right)-$$

$$-\frac{2}{\sqrt{\pi}}\int\limits_0^{\frac{\sqrt{t}}{2a}}\exp\left\{-x^2\right\}\cdot\frac{d}{dt}\left\{\frac{t}{a\sqrt{t-4a^2x^2}}+\frac{\sqrt{t-4a^2x^2}}{2a}\right\}\cdot\exp\left\{-\left(\frac{t}{a\sqrt{t-4a^2x^2}}+\frac{\sqrt{t-4a^2x^2}}{2a}\right)^2\right\}dx=$$

$$=\frac{\sqrt{\pi}}{2}\left(\exp\left\{-\frac{5t}{4a^2}\right\}\cdot\left[-\frac{5}{4a^2}erfc\left\{\frac{\sqrt{t}}{a}\right\}-\frac{1}{a\sqrt{\pi t}}\exp\left\{-\frac{t}{a^2}\right\}\right]+\frac{3}{2a\sqrt{\pi t}}\exp\left(-\frac{9t}{4a^2}\right)\right). \qquad (13)$$

The limit in the first summand of the left-hand side of the equality (13) is zero.
As

$$\frac{d}{dt}\left\{\frac{t}{a\sqrt{t-4a^2x^2}}+\frac{\sqrt{t-4a^2x^2}}{2a}\right\}=\frac{3t-20a^2x^2}{4a\left(t-4a^2x^2\right)^{3/2}}$$

and

$$\left(\frac{t}{a\sqrt{t-4a^2x^2}}+\frac{\sqrt{t-4a^2x^2}}{2a}\right)^2=\frac{t^2}{a^2\left(t-4a^2x^2\right)}+\frac{5t}{4a^2}-x^2,$$

then (13) takes the form

$$\int_0^{\frac{\sqrt{t}}{2a}} \frac{3t-20a^2x^2}{\left(t-4a^2x^2\right)^{3/2}} \cdot \exp\left\{-\frac{t^2}{a^2\left(t-4a^2x^2\right)}\right\} dx = \frac{5\pi}{4a} erfc\left\{\frac{\sqrt{t}}{a}\right\} - \frac{\sqrt{\pi}}{2\sqrt{t}} \exp\left(-\frac{t}{a^2}\right). \tag{14}$$

Now we calculate the integral on the left-hand side of (14), which we denote $J(t)$.

We introduce the change $z=\sqrt{t-4a^2x^2}$. The left-hand side of equation (14) takes the form

$$J(t) = \int_0^{\frac{\sqrt{t}}{2a}} \frac{5\left(t-4a^2x^2\right)-2t}{\left(t-4a^2x^2\right)^{3/2}} \cdot \exp\left\{-\frac{t^2}{a^2\left(t-4a^2x^2\right)}\right\} dx = \int_0^{\sqrt{t}} \frac{5z^2-2t}{z^2} \cdot \exp\left\{-\frac{t^2}{a^2z^2}\right\} \frac{1}{2a\sqrt{t-z^2}} dz =$$

$$= \frac{5}{2a}\int_0^{\sqrt{t}} \cdot \exp\left\{-\frac{t^2}{a^2z^2}\right\} \frac{1}{\sqrt{t-z^2}} dz - \frac{t}{a}\int_0^{\sqrt{t}} \frac{1}{z^2} \cdot \exp\left\{-\frac{t^2}{a^2z^2}\right\} \frac{1}{\sqrt{t-z^2}} dz.$$

After the change $\upsilon = z^2$ we get:

$$J(t) = \frac{5}{4a}\int_0^t \upsilon^{-1/2} \cdot (t-\upsilon)^{-1/2} \exp\left\{-\frac{t^2}{a^2\upsilon}\right\} d\upsilon - \frac{t}{2a}\int_0^t \upsilon^{-3/2} \cdot (t-\upsilon)^{-1/2} \exp\left\{-\frac{t^2}{a^2\upsilon}\right\} d\upsilon.$$

For the first integral we apply (3.471 (2)) of [3] with

$$\mu = \frac{1}{2}; \quad \nu = \frac{1}{2}; \quad \beta = \frac{t^2}{a^2}; \quad u = t.$$

For the second integral we apply (3.471 (3)) with $\mu = \frac{1}{2}; \quad \beta = \frac{t^2}{a^2}; \quad u = t.$

Then

$$J(t) = \frac{5\sqrt{\pi}}{4a}\left(\frac{a}{t}\right)^{1/2} t^{1/4} \exp\left\{-\frac{t}{2a^2}\right\} \cdot W_{-\frac{1}{4};\frac{1}{4}}\left(\frac{t}{a^2}\right) - \frac{t}{2a}\frac{a\sqrt{\pi}}{t} t^{-1/2} \exp\left\{-\frac{t}{a^2}\right\}.$$

By virtue of the formula (9.224) of [3] with $\mu = -\frac{1}{4}$ we have

$$W_{-\frac{1}{4};\frac{1}{4}}\left(\frac{t}{a^2}\right) = \left(\frac{t}{a^2}\right)^{1/4} \exp\left\{\frac{t}{2a^2}\right\} \cdot \int_{\frac{t}{a^2}}^{\infty} u^{-1/2} e^{-u}\, du.$$

Then the integral $J(t)$ takes the form

$$J(t) = \frac{5\sqrt{\pi}}{4a}\int_{\frac{t}{a^2}}^{\infty} u^{-1/2} e^{-u}\, du - \frac{\sqrt{\pi}}{2\sqrt{t}} \exp\left\{-\frac{t}{a^2}\right\}.$$

Using the integral (3.381 (3)), we obtain

$$J(t) = \frac{5\sqrt{\pi}}{4a}\Gamma\left(\frac{1}{2}; \frac{t}{a^2}\right) - \frac{\sqrt{\pi}}{2\sqrt{t}} \exp\left\{-\frac{t}{a^2}\right\}.$$

By virtue of the formula (8.359 (3)) we finally obtain

$$J(t) = \frac{5\pi}{4a} erfc\left(\frac{\sqrt{t}}{a}\right) - \frac{\sqrt{\pi}}{2\sqrt{t}} \exp\left\{-\frac{t}{a^2}\right\}.$$

(cf. on the right side of (1)).

So, from the proved identity (14) it follows that function (7) satisfies the equation (5) and, as a consequence, the function (3) is indeed a solution of the homogeneous Volterra integral equation of the second kind (1).

2. Nontrivial solution of the adjoint homogeneous Volterra integral equation of the second kind with a singular kernel

In the solving integral equation (2) it was noted [2; 183] that it is sufficient for its solution to find a solution of the «simplified» equation

$$\psi(t)-\frac{1}{2a\sqrt{\pi}}\int_t^{\infty}\left\{\frac{2\tau}{\left(\tau-t\right)^{3/2}}\exp\left\{-\frac{\tau t}{a^2(\tau-t)}\right\}+\frac{1}{\sqrt{\tau-t}}\left(1-\exp\left\{-\frac{\tau t}{a^2(\tau-t)}\right\}\right)\right\}\psi(\tau)\,d\tau=0. \tag{15}$$

With the help of the Laplace transform it was obtained the solution of the equation (15), which has the form

$$\psi_0\left(t\right)=\frac{C}{a\sqrt{\pi}}\sum_{n=0}^{\infty}(2n+1)\exp\left(-\frac{(2n+1)^2}{4a^2}t\right). \tag{16}$$

We show that the function (16) is the solution of the «simplified» homogeneous equation (15).

We substitute function (16) into equation (15):

$$\psi_0(t)-\frac{1}{2a\sqrt{\pi}}\int_t^{\infty}\left\{\frac{2\tau}{\left(\tau-t\right)^{3/2}}\exp\left\{-\frac{\tau t}{a^2(\tau-t)}\right\}+\right. \tag{17}$$

$$\left.+\frac{1}{\sqrt{\tau-t}}\left(1-\exp\left\{-\frac{\tau t}{a^2(\tau-t)}\right\}\right)\right\}\frac{C}{a\sqrt{\pi}}\sum_{n=0}^{\infty}(2n+1)\exp\left(-\frac{(2n+1)^2}{4a^2}\tau\right)d\tau=0.$$

We rewrite (17) as

$$\psi_0(t)=\frac{C}{2a^2\pi}\sum_{n=0}^{\infty}(2n+1)\left[\int_t^{\infty}\frac{2\tau}{\left(\tau-t\right)^{3/2}}\exp\left\{-\frac{(\tau-t+t)t}{a^2(\tau-t)}\right\}\exp\left(-\frac{(2n+1)^2}{4a^2}\tau\right)d\tau+\right.$$

$$\left.+\int_t^{\infty}\frac{1}{\sqrt{\tau-t}}\exp\left(-\frac{(2n+1)^2}{4a^2}\tau\right)d\tau-\int_t^{\infty}\frac{1}{\sqrt{\tau-t}}\exp\left\{-\frac{(\tau-t+t)t}{a^2(\tau-t)}\right\}\exp\left(-\frac{(2n+1)^2}{4a^2}\tau\right)d\tau\right]= \tag{18}$$

$$=\frac{C}{2a^2\pi}\sum_{n=0}^{\infty}(2n+1)\left(I_1\left(n,t\right)+I_2\left(n,t\right)-I_3\left(n,t\right)\right),$$

where

$$I_1\left(n,t\right)=2\exp\left\{-\frac{t}{a^2}\right\}\int_t^{\infty}\frac{\tau}{\left(\tau-t\right)^{3/2}}\exp\left\{-\frac{t^2}{a^2(\tau-t)}\right\}\exp\left(-\frac{(2n+1)^2}{4a^2}\tau\right)d\tau;$$

$$I_2\left(n,t\right)=\int_t^{\infty}\frac{1}{\sqrt{\tau-t}}\exp\left(-\frac{(2n+1)^2}{4a^2}\tau\right)d\tau;$$

$$I_3\left(n,t\right)=\exp\left\{-\frac{t}{a^2}\right\}\int_t^{\infty}\frac{1}{\sqrt{\tau-t}}\exp\left\{-\frac{t^2}{a^2(\tau-t)}\right\}\exp\left(-\frac{(2n+1)^2}{4a^2}\tau\right)d\tau.$$

After changing $z=\sqrt{\tau-t}$ the integral $I_2\left(n,t\right)$ takes the form

$$I_2\left(n,t\right)=\frac{2a\sqrt{\pi}}{2n+1}\exp\left(-\frac{(2n+1)^2}{4a^2}t\right). \tag{19}$$

We calculate the first integral

$$I_1\left(n,t\right)=2\exp\left\{-\frac{t}{a^2}\right\}\int_t^{\infty}\frac{\tau-t+t}{\left(\tau-t\right)^{3/2}}\exp\left\{-\frac{t^2}{a^2(\tau-t)}\right\}\exp\left(-\frac{(2n+1)^2}{4a^2}(\tau-t+t)\right)d\tau=$$

$$=2\exp\left\{-\frac{t}{a^2}\right\}\exp\left(-\frac{(2n+1)^2}{4a^2}t\right)\left[\int_t^{\infty}\frac{1}{\sqrt{\tau-t}}\exp\left\{-\frac{t^2}{a^2(\tau-t)}\right\}\exp\left(-\frac{(2n+1)^2}{4a^2}(\tau-t)\right)d\tau+\right.$$

$$\left.+t\int_t^{\infty}\frac{1}{\left(\tau-t\right)^{3/2}}\exp\left\{-\frac{t^2}{a^2(\tau-t)}\right\}\exp\left(-\frac{(2n+1)^2}{4a^2}(\tau-t)\right)d\tau\right].$$

After changing $z=\sqrt{\tau-t}$ the integral $I_1\left(n,t\right)$ takes the form

$$I_1\left(n,t\right)=4\exp\left\{-\frac{t}{a^2}\right\}\exp\left(-\frac{(2n+1)^2}{4a^2}t\right)\left[\int_0^{\infty}\exp\left\{-\frac{t^2}{a^2z^2}\right\}\exp\left(-\frac{(2n+1)^2z^2}{4a^2}\right)dz+\right.$$

$$+t\int_0^\infty \frac{1}{z^2}\exp\left\{-\frac{t^2}{a^2z^2}\right\}\exp\left(-\frac{(2n+1)^2z^2}{4a^2}\right)dz\Bigg].$$

By virtue of the known relations

$$\int_0^\infty \exp\left\{-\mu x^2-\frac{\eta}{x^2}\right\}dx=\frac{1}{2}\frac{\sqrt{\pi}}{\sqrt{\mu}}\exp\left\{-2\sqrt{\mu\eta}\right\};$$

$$\int_0^\infty \exp\left\{-\mu x^2-\frac{\eta}{x^2}\right\}\frac{dx}{x^2}=\frac{1}{2}\frac{\sqrt{\pi}}{\sqrt{\eta}}\exp\left\{-2\sqrt{\mu\eta}\right\}$$

we have

$$I_1(n,t)=4\exp\left\{-\frac{t}{a^2}\right\}\exp\left(-\frac{(2n+1)^2}{4a^2}t\right)\left[\frac{a\sqrt{\pi}}{2n+1}\exp\left(-\frac{(2n+1)t}{a^2}\right)+t\cdot\frac{a\sqrt{\pi}}{2t}\exp\left(-\frac{(2n+1)t}{a^2}\right)\right].$$

We finally get

$$I_1(n,t)=\frac{2a\sqrt{\pi}(2n+3)}{2n+1}\exp\left(-\frac{(2n+3)^2}{4a^2}t\right). \tag{20}$$

We calculate the third integral under the sum in the right-hand side of equality (18)

$$I_3(n,t)=\exp\left\{-\frac{t}{a^2}\right\}\int_t^\infty\frac{1}{\sqrt{\tau-t}}\exp\left\{-\frac{t^2}{a^2(\tau-t)}\right\}\exp\left(-\frac{(2n+1)^2}{4a^2}(\tau-t+t)\right)d\tau=$$

$$=\exp\left\{-\frac{t}{a^2}\right\}\exp\left(-\frac{(2n+1)^2}{4a^2}t\right)\int_t^\infty\frac{1}{\sqrt{\tau-t}}\exp\left\{-\frac{t^2}{a^2(\tau-t)}\right\}\exp\left(-\frac{(2n+1)^2}{4a^2}(\tau-t)\right)d\tau.$$

After changing $z=\sqrt{\tau-t}$ the integral $I_3(n,t)$ takes the form

$$I_3(n,t)=2\exp\left\{-\frac{t}{a^2}\right\}\exp\left(-\frac{(2n+1)^2}{4a^2}t\right)\int_0^\infty\exp\left\{-\frac{t^2}{a^2z^2}\right\}\exp\left(-\frac{(2n+1)^2z^2}{4a^2}\right)dz.$$

Applying again the ratio

$$\int_0^\infty \exp\left\{-\mu x^2-\frac{\eta}{x^2}\right\}dx=\frac{1}{2}\frac{\sqrt{\pi}}{\sqrt{\mu}}\exp\left\{-2\sqrt{\mu\eta}\right\},$$

we get

$$I_3(n,t)=2\exp\left\{-\frac{t}{a^2}\right\}\exp\left(-\frac{(2n+1)^2}{4a^2}t\right)\cdot\frac{1}{2}\cdot\frac{2a\sqrt{\pi}}{2n+1}\exp\left(-\frac{(2n+1)}{a^2}t\right).$$

We finally have

$$I_3(n,t)=\frac{2a\sqrt{\pi}}{2n+1}\exp\left(-\frac{(2n+3)^2}{4a^2}t\right). \tag{21}$$

The results (19)–(21) we substitute in (18)

$$\psi_0(t)=\frac{C}{2a^2\pi}\sum_{n=0}^\infty(2n+1)\left\{\frac{2a\sqrt{\pi}(2n+3)}{2n+1}\exp\left(-\frac{(2n+3)^2}{4a^2}t\right)+\right.$$

$$\left.+\frac{2a\sqrt{\pi}}{2n+1}\exp\left(-\frac{(2n+1)^2}{4a^2}t\right)-\frac{2a\sqrt{\pi}}{2n+1}\exp\left(-\frac{(2n+3)^2}{4a^2}t\right)\right\}=$$

$$=\frac{C}{a\sqrt{\pi}}\sum_{n=0}^\infty\left\{(2n+2)\exp\left(-\frac{(2n+3)^2}{4a^2}t\right)+\exp\left(-\frac{(2n+1)^2}{4a^2}t\right)\right\}=$$

$$=\frac{C}{a\sqrt{\pi}}\left[\exp\left(-\frac{t}{4a^2}\right)+\sum_{n=1}^\infty\exp\left(-\frac{(2n+1)^2}{4a^2}t\right)+\sum_{n=0}^\infty(2n+2)\exp\left(-\frac{(2n+3)^2}{4a^2}t\right)\right]=$$

$$=\frac{C}{a\sqrt{\pi}}\left[\exp\left(-\frac{t}{4a^2}\right)+\sum_{n=0}^\infty\exp\left(-\frac{(2n+3)^2}{4a^2}t\right)+\sum_{n=0}^\infty(2n+2)\exp\left(-\frac{(2n+3)^2}{4a^2}t\right)\right]=$$

$$=\frac{C}{a\sqrt{\pi}}\left[\exp\left(-\frac{t}{4a^2}\right)+\sum_{n=0}^{\infty}(2n+3)\exp\left(-\frac{(2n+3)^2}{4a^2}t\right)\right].$$

As the with $n=-1 \;\Rightarrow\; (2n+3)\exp\left(-\frac{(2n+3)^2}{4a^2}t\right)=\exp\left(-\frac{t}{4a^2}\right)$, then the last equality takes the form

$$\psi_0(t)=\frac{C}{a\sqrt{\pi}}\sum_{n=-1}^{\infty}(2n+3)\exp\left(-\frac{(2n+3)^2}{4a^2}t\right)=\frac{C}{a\sqrt{\pi}}\sum_{n=0}^{\infty}(2n+1)\exp\left(-\frac{(2n+1)^2}{4a^2}t\right).$$

(cf. (16)).

Thus, the function (16) satisfies the «simplified» equation (15).

Then the solution of the full equation (2) by [2; 183] has the form

$$\psi(t)=\psi_0(t)\exp\left\{\frac{t}{4a^2}\right\}=\frac{C}{a\sqrt{\pi}}\sum_{n=0}^{\infty}(2n+1)\exp\left(-\frac{n^2+n}{a^2}t\right).$$

Thus, homogeneous integral equation (2) has a nontrivial solution represented by the formula (4).

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Д.М.Ахманова, М.Т.Жиеналиев, М.Т.Космакова, М.Ы.Рамазанов

Вольтерра ерекше интегралды теңдеуі мен оның түйіндесі туралы

Мақалада интегралдаудың айнымалы шегімен екінші ретті өзара-түйіндес интегралды Вольтерра теңдеуінің шешімдері қарастырылды. Теңдеудің ерекшелігі — сәйкес ядроның сығылмауында және, сәйкесінше, оған біртіндеп жуықтау әдісін қолдана алмауда. Екінші ретті ерекше біртекті интегралды Вольтерра теңдеуінің нөлдік емес шешімдері алынды. Тікелей тексеру әдісін қолдану арқылы алынған функциялар ерекше ядролы біртекті екінші ретті интегралды Вольтерра теңдеуі мен оның түйіндесінің тривиалды емес шешімі екені анықталды.

Д.М.Ахманова, М.Т.Дженалиев, М.Т.Космакова, М.И.Рамазанов

Об особом интегральном уравнении Вольтерра и его сопряженном

В статье рассмотрены решения взаимно-сопряженных интегральных уравнений типа Вольтерра второго рода с переменным пределом интегрирования. Особенность уравнений заключается в несжимаемости соответствующих ядер и, соответственно, в неприменимости метода последовательных приближений. Получены ненулевые решения особых однородных интегральных уравнений Вольтерра второго рода. Путем непосредственной проверки установлено, что полученные функции являются нетривиальными решениями однородного с особым ядром интегрального уравнения Вольтерра второго рода и его сопряженного.

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