Journal of Mathematical Sciences, Vol. 166, No. 5, 2010

ON JONSSON STABILITY AND SOME OF ITS GENERALIZATIONS

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Abstract. We consider Jonsson analogues of the concepts of stability and P-stability. We also consider properties and connections of a Jonsson theory and its center that concern these concepts.

Introduction

The given work is devoted to the study of some properties of Jonsson theories.

The well-known specialist in mathematical logic H. J. Keisler in the review "Fundamentals of Model Theory" in [2] has defined the basic concepts and directions of the development of model theory. He has defined two historical directions in the development of model theory. They are western and eastern model theory. Such division is connected with the fact that A. Tarski lived on the western coast of the USA, while A. Robinson lived on the eastern coast of the USA. This distinction has long ago lost its geographical value; however it is useful from the mathematical point of view. The western model theory to a greater degree was based on problems in number theory, analysis, and set theory, and in it all formulas of the first-order logic, as a rule, are used. Eastern model theory was usually based on problems in abstract universal algebra, where formulas with prefix length at most 2 were considered. In the western variant, morphisms are considered elementary morphisms, and in the eastern variant the morphisms are isomorphic embeddings and homomorphisms. On the other hand, the theory of models is uniform, i.e., theorems of "eastern" character naturally supplement the western analogues as, for example, A. Macintyre's "eastern" theorem on omitting types supplements the well-known theorem on omitting types in the "western" sense. More complete information on the connection of the two directions can be found in [17].

Jonsson theories became the object of research for the first time in [8, 13]. From the middle of 1980s, the works of T. G. Mustafin have defined a new direction at studying Jonsson theories. In particular, they have defined a natural subclass of Jonsson theories, which they call perfect. The main method of research, which has been defined by Mustafin, was the following: the study of properties of any Jonsson theory by carrying over properties of the center of a given Jonsson theory. At the beginning of the 1990s, Yeshkeyev proved the criterion of perfectness of Jonsson theories [28]. In particular, in [29, 30] there was obtained a full description of Jonsson universals of the unars and also there were found some connections between the theory of the unars and their center in the language of stability. On the other hand, one of the weak points in the research of Jonsson theories within the frames of a method offered by Mustafin was the presence of an additional axiom about the existence of strongly inaccessible cardinals in addition to the axioms of the Zermelo–Fraenkel set theory in the definition of a semantic model. During Ospanov's report at the 5th Kazakh-French colloquium on model theory, well-known experts in the field of model theory, E. A. Palyutin and B. Poizat, have specified the necessity of changing this definition. Realization of their remarks was the result of the work [18], in which Mustafin redefines notions of k-homogeneous and semantic models. An accordingly changed definition of the perfectness of Jonsson theories has appeared in [31]. In this work, the basic results obtained earlier [32] were also reproved within the framework of the new definition [31].

Thus, the study of Jonsson theories is an aspect of eastern model theory. We note that generally Jonsson theories are not complete. For today the apparatus of model theory is developed in the core for

Translated from Fundamentalnaya i Prikladnaya Matematika, Vol. 14, No. 8, pp. 117–128, 2008.

complete theories, and consequently studying model-theoretic properties of Jonsson theories is interesting in the class of incomplete inductive theories.

In this paper, stability properties of Jonsson theories are studied. Thus, Jonsson analogues of the classical definition of stability and one of its generalizations are considered.

The problem on how properties of a complete theory T are connected with properties of the theory T_P of pairs of models of the original theory started to be studied in [25]. In this work, Poizat began to study structures of a general form, in which elementary substructures are allocated. He has formulated the question on finding conditions under which the theory of elementary pairs is complete. The works [3,4,21], etc. have been devoted to studying this question. E. Bouscaren in [4] has assumed that the solution of this problem should turn out to be different in different classes of stable theories. She has given a number of necessary conditions and some sufficient conditions for the completeness of the theory of elementary pairs in stable and superstable classes. It is noted (D. Lascar) that an ω-stable theory in which elementary pairs are equivalent should be uncountably categorical. In [21], A. T. Nurtazin has given a solution to the problem of completeness of the theory of elementary pairs for a class of uncountable categorical theories. In [16], the notion of T^*-λ-stability is introduced and a number of its properties are listed, generalizing the well-known facts on λ-stability in the classical sense. In particular, the P-stability is studied, which is directly connected with the concept of elementary pairs. In [19], a description of P-stability for any superstable theory is obtained. In [23], E. A. Palyutin introduces the concept of an E^*-stable theory. The basic result of the paper [23] (Theorem 3.1) consists in the proof of the definability of types for E^*-stable theories. This notion differs from T^*-stability by the addition of a condition of continuity, which is trivially satisfied for the major special cases (stability, P-stability, etc.). In addition to the above-mentioned definability of types for stable theories, another consequence of this result is the definability of types above any P-sets in P-stable theories, which has been established by T. Nurmagambetov and B. Poizat for types above P-models [20].

One of the questions formulated by Mustafin at the first Soviet-French colloquium in 1990 was the following: If T is P-λ-stable, is T always λ-stable? To this question E. Bouscaren gives a negative answer, namely:

(1) there is a theory that is P-ω-stable and superstable, but not ω-stable;
(2) there is a P-ω-stable theory that is not superstable, but is stable.

In the given work, in some class of incomplete inductive theories we consider the above-stated question of Mustafin. With corresponding generalizations of the definitions of λ-stability and P-λ-stability in Jonsson theories from Theorems 3 and 12 of this paper, one can draw the conclusion that the question has the same sense also in the perfect Jonsson case.

1. Stable Jonsson Theories

Let us give the main definitions.

Definition 1. A theory T in a language L is called a Jonsson theory if it satisfies the following conditions:

(1) the theory T has an infinite model;
(2) the theory T is inductive;
(3) the theory T has the joint embedding property;
(4) the theory T has the amalgamation property (AP).

Definition 2. Let $k \geq \omega$. A model M of a theory T is called k-universal for T if any model of T with cardinality strictly less than k can be isomorphically embedded into M. A model M of a theory T is called k-homogeneous for T if for any two models A and A_1 of the theory T with cardinality strictly less than k that are submodels of M, for any isomorphism $f\colon A \to A_1$, and for any extension B of the model A with cardinality less than k, where B is a submodel of M and B is a model of T, there are an extension B_1 of the model A_1 that is a submodel of M and an isomorphism $g\colon B \to B_1$ that continues f.

Definition 3. A model C_T of a Jonsson theory T is called a semantic model of the theory T if it is an ω^+-homogeneous universal model of T.

Definition 4. The center of a Jonsson theory T is the elementary theory T^* of a semantic model C_T of the theory T, i.e., $T^* = \mathrm{Th}(C_T)$.

Definition 5. A Jonsson theory T is called perfect if every semantic model of the theory T is a saturated model of T^*.

It is easy to note that all semantic models are elementarily equivalent to each other.

Definition 6. The $\sharp$-companion of a Jonsson theory T is the theory $T^\sharp$ such that

(1) $(T^\sharp)_\forall = T_\forall$;
(2) for any Jonsson theory T', if $T_\forall = T'_\forall$, then $T^\sharp = (T')^\sharp$;
(3) $T \subseteq T^\sharp$.

There are the following natural examples: if $\sharp \in \{\mathrm{o}, *, \mathrm{e}, \mathrm{f}\}$, then we have, respectively, the Kaiser hull of T, the center of T, $\mathrm{Th}(E_T)$, where E_T is the class of T-existentially closed models of T, and the forcing-companion T^{f} of the theory T.

Theorem 1 ([30])**.** *Let T be a Jonsson theory. Then the following conditions are equivalent*:

(1) *T is perfect*;
(2) *T^* is the model companion of the theory T*;
(3) $\mathrm{Mod}\, T^* = E_T$;
(4) $T^* = T^{\mathrm{f}}$.

Theorem 2 ([30])**.** *Let T be a Jonsson theory. Then the following conditions are equivalent*:

(1) *T is model complete*;
(2) *T is complete.*

Let T be a Jonsson theory and $S^J(X)$ be the set of all existentially complete n-types over X that are consistent with T, for any finite n.

Definition 7. We say that a Jonsson theory T is J-λ-stable if for any T-existentially closed model A and for any subset X of A the inequality $|X| \le \lambda$ implies $|S^J(X)| \le \lambda$.

It turns out that such a generalization of the notion of stability gives the following result, which connects the notions of J-stability and classical stability.

Theorem 3. *Let T be a perfect Jonsson theory that is complete for existential sentences. Let $\lambda \ge \omega$. Then the following conditions are equivalent*:

(1) *T is J-λ-stable*;
(2) *T^* is λ-stable, where T^* is the center of the theory T.*

Proof. Let us prove the implication (1) $\Longrightarrow$ (2). If $T \subset T^*$, then $E_n(T) \subset E_n(T^*)$, where $E_n(T)$ and $E_n(T^*)$ are the respective lattices of existential formulas. The theory T is complete for existential sentences, whence $E_n(T) = E_n(T^*)$. T is perfect, whence T^* is model complete by Theorem 1 if and only if

$$\forall n < \omega\ \forall \varphi \in F_n(T^*)\ \exists \theta \in E_n(T^*)\ (T^* \vdash \varphi \leftrightarrow \theta).$$

Let T be J-λ-stable. Then by the definition for every $A \in E_T$ we have that for every $X \subset A$, if $|X| \le \lambda$, then $|S^J(X)| \le \lambda$.

Suppose that T^* is not λ-stable. In view of Theorem 1, there are $A \in E_T = \mathrm{Mod}\, T^*$ and $X \subset A$ such that if $|X| < \lambda$ and there exists $n < \omega$, then $|S^J(X)| > \lambda$. For each formula $\varphi \in p$, where $p \in S_n(X)$, we shall replace φ by θ, where θ satisfies $T^* \vdash \varphi \leftrightarrow \theta$ and $\theta \in E_n(T^*)$. Let p' be p after replacement. Then $p' \in S^J(X)$ and $|S^J(X)| > \lambda$. This contradicts the J-λ-stability of the theory T.

The implication (2) $\Longrightarrow$ (1) is trivial. $\square$

The following question of Palyutin is well known: Is there an ω-categorical universal class K that is not ω_1-categorical? We consider ω-categorical universal Jonsson theories in connection with this question.

Lemma 1. *If a Jonsson theory T is ω-categorical, then T is perfect.*

Proof. If T is ω-categorical, then by Saracino's theorem [26] there is a theory T' such that T' is the model companion of the theory T and T' is ω-categorical. Consequently, by the Eklof—Sabbagh theorem [5] E_T is an elementary class. By Theorem 1, $\mathrm{Th}(E_T) = T^*$. Therefore, T is perfect. $\square$

Lemma 2. *Let $\kappa \geq \omega$. If a Jonsson theory T is κ-categorical, then the $\sharp$-companion of the theory T is κ-categorical.*

Proof. If $T^\sharp$ is not κ-categorical, then there exist $A, B \in \mathrm{Mod}(T^\sharp)$ such that $|A| = |B| = \kappa$ and A is not isomorphic to B. Then $A, B \in \mathrm{Mod}(T)$ because $T \subseteq T^\sharp$. But T is a κ-categorical theory. We have obtained a contradiction. $\square$

A consequence of the above-stated facts is the following theorem.

Theorem 4. *In the case of the negative answer to Palyutin's question for a Jonsson theory that satisfies the conditions of Palyutin's question, the center of the Jonsson theory cannot be finitely axiomatized.*

The proof follows from the previous lemmas and Zilber's theorem about total categoricity and finite axiomatizability [33].

Definition 8. A Jonsson theory T that is complete for $\exists$-sentences is called J-non-two-cardinal if for any T-existentially closed model A and for any $\exists$-formula $\varphi(x, \bar{y})$ such that $\bar{a} \in A$ and $l(\bar{a}) = l(\bar{y})$ either the set $\varphi(A, \bar{a})$ is finite or its cardinality is equal to $|A|$.

The following results belong to E. A. Palyutin [22].

Theorem 5. *If T is an ω-categorical universal theory, then the complete theory T_∞ is non-two-cardinal.*

Theorem 6. *If T is an ω-categorical universal theory, then the following conditions are equivalent*:

(1) *T is ω_1-categorical*;
(2) *T_∞ is ω-stable*;
(3) *some inessential extension of T has a strongly minimal formula.*

We obtain the following results concerning the previous statements.

Theorem 7. *If T is an ω-categorical universal Jonsson theory, then $T^\sharp$ is a non-two-cardinal theory.*

The proof follows from Theorems 5 and 1 and Lemmas 1 and 2.

Definition 9. For any Jonsson theory T, any existentially closed model M of the theory T, and any $\bar{a} \in M$, the theory $T' = \mathrm{Th}_{\forall\exists}(M, \bar{a})$ is a J-inessential extension of T.

Definition 10. Let T be an $\exists$-complete Jonsson theory. Then an $\exists$-formula $\varphi(x, \bar{a})$ is called strongly minimal in T if it is infinite, but for any $\exists$-formula $\varphi(x, \bar{b})$ one of the formulas $\varphi(x, \bar{a}) \wedge \psi(x, \bar{a})$ and $\varphi(x, \bar{a}) \wedge \neg\psi(x, \bar{a})$ is finite in T.

Theorem 8. *If T is an ω-categorical universal Jonsson theory that is complete for $\exists$-sentences, then the following conditions are equivalent*:

(1) *T is ω_1-categorical*;
(2) *$T^\sharp_\infty$ is J-ω-stable*;
(3) *some J-inessential extension of T has a J-strongly minimal formula.*

The proof follows from Theorems 6 and 1 and Lemmas 1 and 2.

2. The Axiomatic Way of Defining the Forking in Perfect Jonsson Theories

The main aim of this section is to define the notion of forking for perfect Jonsson theories. The facts about forking for a universal class can be found in [7,24,27]. We give the definition of forking for inductive theories, which are Jonsson theories.

One can observe different ways of axiomatization of forking in [1, 10, 15]. In this paper, we follow basically [15] in "Jonsson" generalizations of the basic notions in [15, Theorem 19.1].

Definition 11. Let M be an $\exists$-saturated, existentially closed model of cardinality κ of a Jonsson theory T (κ is an arbitrarily large cardinal and $\exists$-saturation means a saturation with respect to existential types). Let $\mathbf{A}$ be the class of all subsets of M and $\mathbf{P}$ be the class of all $\exists$-types (not necessarily complete). Let $\mathrm{JNF} \subseteq \mathbf{P} \times \mathbf{A}$ be some binary relation. We demand for JNF the following axioms.

Axiom 1. If $(p, A) \in \mathrm{JNF}$ and $f\colon A \to B$ is an automorphism of M, then $\big(f(p), f(A)\big) \in \mathrm{JNF}$.

Axiom 2. If $(p, A) \in \mathrm{JNF}$ and $q \subseteq p$, then $(q, A) \in \mathrm{JNF}$.

Axiom 3. If $A \subseteq B \subseteq C$ and $p \in S^J(C)$, then $(p, A) \in \mathrm{JNF}$ if and only if $(p, B) \in \mathrm{JNF}$ and $(p\lceil B, A) \in \mathrm{JNF}$.

Axiom 4. If $A \subseteq B$, $\mathrm{dom}(p) \subseteq B$, and $(p, A) \in \mathrm{JNF}$, then there exists $q \in S^J(B)$ such that $p \subseteq q$ and $(q, A) \in \mathrm{JNF}$.

Axiom 5. There is a cardinal μ such that if $A \subseteq B \subseteq C$, $p \in S^J(B)$, and $(p, A) \in \mathrm{JNF}$, then

$$|\{q \in S^J(C)\colon p \subseteq q,\ (q, A) \in \mathrm{JNF}\}| < \mu.$$

Axiom 6. There is a cardinal ρ such that for all $\rho \in \mathbf{P}$ and for all $A \in \mathbf{A}$, if $(p, A) \in \mathrm{JNF}$, then there exists $A_1 \subseteq A$ such that $|A_1| < \rho$ and $(p, A_1) \in \mathrm{JNF}$.

Axiom 7. If $p \in S^J(A)$, then $(p, A) \in \mathrm{JNF}$.

The classical notion of forking belongs to Shelah [27].

Definition 12. A set of formulas $\{\varphi(\bar{x}, \bar{a}_i)\colon i < k\} = p$ is called k-inconsistent for some positive integer k if each finite subset p of size k is inconsistent, i.e.,

$$\models \neg\bar{x}\big(\varphi(\bar{x}, \bar{a_{i_1}}) \wedge \dots \wedge \varphi(\bar{x}, \bar{a_{i_k}})\big)$$

for every $i_1 < \dots < i_k < k$.

The partial type p divides over a set with respect to $k \in \omega$ if there exist a formula $\varphi(\bar{x}, \bar{a})$ and a sequence $\langle \bar{a}_i\colon i \in \omega\rangle$ such that

(1) $p \vdash \varphi(\bar{x}, \bar{a})$;
(2) $\mathrm{tp}(\bar{a}/A) = \mathrm{tp}(\bar{a}_i/A)$ for all i;
(3) $\{\varphi(\bar{x}, \bar{a}_i)\colon i \in \omega\}$ is k-inconsistent.

Also, p divides over A if p divides over A with respect to some k. Further, p forks over A in T if there are formulas $\{\varphi(\bar{x}, \bar{a}_i)\colon i \in \omega\}$ k such that

(1) $p \models \vee_{0 \le i \le n} \phi_i(\bar{x}, \bar{a}_i)$;
(2) $\phi_i(\bar{x}, \bar{a_i})$ divides over A for each i.

In the proof of Theorem 10, we shall use the following results.

Theorem 9 (F. P. Ramsey [9, p. 173])**.** *Let I be an infinite set, let $n < \omega$, and let $[I]^n$ be the family of all subsets of the set I that consist of exactly n elements. If $[I]^n = A_0 \cup \dots \cup A_{k-1}$, $k < \omega$, and $A_i \cap A_j = \varnothing$ for $i < j < k$, then there is an infinite set $J \subset I$ such that $[J]^n \subset A_i$ for some $i < k$.*

Lemma 3 ([14, Lemma 14.9])**.** *Let T be a stable theory, M be a saturated model of cardinality μ^+, and types $p_1, p_2 \in S(M)$ do not fork over A. Then if $p_1\lceil A = p_2\lceil A$, then there is an elementary monomorphism f identical on A such that $f(d_1) \sim d_2$, where d_1 and d_2 are the schemes defining p_1 and p_2, respectively.*

Theorem 10. *Let T be a perfect Jonsson theory complete for $\exists$-sentences. Then the following conditions are equivalent*:

(1) *the relation* JNF *satisfies Axioms* 1–7 *with respect to theory T*;
(2) *T^* is stable and for any $p \in \mathbf{P}$ and any $A \in \mathbf{A}$ we have $(p, A) \in$ JNF if and only if p does not fork over A.*

Proof. (1) $\Longrightarrow$ (2) Let $\lambda = 2^{\rho\cdot|T|\cdot\mu}$, where λ, ρ, and μ are cardinals corresponding to Axioms 1–7. Now we shall show that T is J-λ-stable. Then by Theorem 3 we shall obtain that T^* is λ-stable. It is obvious that $\lambda^\rho = \lambda$. Let $|A| = \lambda$. If $p \in S^J(A)$, then by Axiom 7 $(p, A) \in$ JNF and by Axiom 6 there is $A_p \subseteq A$ such that $|A_p| < \rho$ and $(p, A_p) \in$ JNF. Then by Axiom 3 $(p\lceil A_p, A_p) \in$ JNF. We denote $p\lceil A_p$ by $g(p)$. Then by Axiom 5 $|\{q \in S^J(A) \colon g(q) = g(p)\}| < \mu$. Hence

$$|S^J(A)| \le |\{g(p) \colon p \in S^J(A)\}| \cdot \mu \le |A^\rho| \cdot 2^{\rho\cdot|T|} \cdot \mu \le \lambda^\rho \cdot \lambda \cdot \lambda = \lambda^\rho = \lambda.$$

Hence T is J-λ-stable. We conclude that T^* is λ-stable by Theorem 3.

Now let $(p, A) \in$ JNF. We show that p does not fork over A. Let $B = \operatorname{dom}(p)$. Then by Axiom 4 there is $q \in S^J(B)$ such that $p \subseteq q$ and $(q, A) \in$ JNF. We have proved that q does not fork over A (then p does not fork over A by Axiom 2). We assume the converse. Then by Definition 12 and by the perfectness of the theory T, there is a finite set of existential formulas Σ such that $q \vdash \bigvee\{\varphi \colon \varphi \in \Sigma\}$ and each formula $\varphi \in \Sigma$ divides over A. Let $C = B \cup D$, D be the set of constants occurring in at least one of the formulas from Σ. By Axiom 4 there is $q_0 \in S^J(C)$ such that $q \subseteq q_0$ and $(q_0, A) \in$ JNF. It is obvious that $q_0 \vdash \bigvee\{\varphi \colon \varphi \in \Sigma\}$, whence there is $\varphi(\bar{x}, \bar{a}) \in q_0 \cap \Sigma$. Using Theorem 9, the theorem of compactness, and the divisibility of $\varphi(\bar{x}, \bar{a})$ over A, we can show the existence of a sequence $\langle \bar{a_\alpha} \colon \alpha < \mu^+\rangle$ and elementary monomorphisms f_α, where $\alpha < \mu^+$, identical on A such that $\bar{a_0} = \bar{a}$, $\bar{a}_\alpha = f_\alpha(\bar{a})$, where $\alpha < \mu^+$, and $\{\varphi(\bar{x}, \bar{a_\alpha}) \colon \alpha < \mu^+\}$ is k-inconsistent for some $k < \omega$.

Let $E = C \cup \{\bar{a_\alpha} \colon \alpha < \mu^+\}$ and $q_\alpha = f_\alpha(q_0)$, where $0 < \alpha < \mu^+$. By Axiom 1, $(q_\alpha, A) \in$ JNF, where $\alpha < \mu^+$. By Axiom 4 there are $q'_\alpha \in S^J(E)$ such that $q_\alpha \subseteq q'_\alpha$ and $(q'_\alpha, A) \in$ JNF. Clearly, $\varphi(\bar{x}, \bar{a_\alpha}) \in q'_\alpha$ and $q \subseteq q'_\alpha$, where $\alpha < \mu^+$. We have $|\{q'_\alpha \colon \alpha < \mu^+\}| = \mu^+$ because $\{\varphi(\bar{x}, \bar{a_\alpha}) \colon \alpha < \mu^+\}$ is k-inconsistent. We have obtained a contradiction with Axiom 5. Hence q does not fork over A. Thus, we see that if $(p, A) \in$ JNF, p does not fork over A.

Let us prove the converse. Let p not fork over A. Since the theory T is perfect, the theory T^* is model complete (Theorem 1) and it suffices to work only with existential types and to consider $\exists$-saturated, existentially closed models of the theory T. We must prove that $(p, A) \in$ JNF. Let $M \supseteq A$, $M \supseteq \operatorname{dom}(p)$, $|M| > 2^{\rho\cdot|T|\cdot\mu}$, and M be an $\exists$-saturated model of the theory T^*, $t \in S^J(M)$, $p \subseteq t$, t not fork over A. By Axiom 7, $(t\lceil A, A) \in$ JNF and by Axiom 5 there is $q \in S^J(M)$ such that $q \supseteq t\lceil A$ and $(q, A) \in$ JNF. As was shown above, $(q, A) \in$ JNF implies that q does not fork over A. By Lemma 3, there is an automorphism f of model M such that f is identical on A and $t = f(q)$. Then by Axiom 1 $(t, A) \in$ JNF and by Axiom 2 $(p, A) \in$ JNF. Thus, the implication (1) $\Longrightarrow$ (2) is proved.

(2) $\Longrightarrow$ (1) It is easy to see that this follows from the proof of Theorem 19.1 in [15] with the generalization of the corresponding notions to Jonsson analogues. □

3. P-Stability in Jonsson Theories

Let T be any Jonsson theory of the signature σ, C be its semantic model, A be a subset of the model C, and P be a new unary predicate symbol. Let us consider in the signature $\sigma_P(A) = \sigma_A \cup \{P\}$ the following (in general, incomplete) theory:

$$T_P^J(A) = \mathrm{Th}_{\forall\exists}(C, a)_{a\in A} \cup \{P(c_a) \mid a \in A\} \cup \{\text{“}P \subseteq\text{”}\},$$

where $\{\text{“}P \subseteq\text{”}\}$ is an infinite set of sentences expressing the fact that the interpretation of the symbol P is an existentially closed submodel in the signature σ. The requirement of existential closeness for the submodel is essential in the sense that it should not be finite and it is also essential in view of Theorem 10. By S_p^J we shall denote the set of all $\exists$-completions of the theory $T_P^J(A)$. Let λ be any cardinal.

Definition 13. The Jonsson theory T is called Jonsson P-λ-stable (further J-P-λ-stable) if $|S^J_P| \leq \lambda$ for any set A with cardinality $\leq \lambda$.

The Jonsson theory T is called J-P-stable if T is J-P-λ-stable for some λ.

Let A and B be existentially closed models of a Jonsson theory T and let the inclusion $A \subsetneq B$ hold. Let $\sigma_P = \sigma \cup \{P\}$ and let the interpretation of the unary predicate symbol P in B be A.

Definition 14. The model (A, B) is called a Jonsson elementary pair of the theory T.

The theory of Jonsson elementary pairs is the theory T^J_P of the class K, where K is the set of all Jonsson elementary pairs of the theory T.

Theorem 11 ([6, Theorem 8.1.2])**.** *Let L be a first-order language and T be a theory in the language L. Let us assume that T has the joint embedding property, and let A and B be existentially closed models of the theory T. Then every $\forall_2$-sentence of the language L that is true in A is also true in B.*

Lemma 4. *If T is a perfect Jonsson theory, then $T^J_P(A)$ is a perfect Jonsson theory.*

Proof. By the definition of $T^J_P(A)$, it suffices to show that $T^J_P(A)$ has a semantic model saturated in its cardinality. As the given model we shall take a semantic model C of the theory T, and, depending on the subset A and the interpretation of the unary predicate P in C, the model $(C, M, a)_{a \in A}$ is saturated in its cardinality. ☐

Theorem 12. *Let T be a perfect Jonsson $\exists$-complete theory. Then the following conditions are equivalent*:

(1) *the center of the theory T is P-λ-stable* (*in the sense of* [19]);

(2) *the theory T is J-P-λ-stable.*

Proof. The proof of the implication from (1) to (2) is trivial, since if the number of all completions is less than or equal to λ, then there are at most λ $\exists$-completions.

Let us prove the implication from (2) to (1). Let the theory T be J-P-λ-stable. This is equivalent to that $T^J_P(A)$ in the signature $\sigma_P(A) = \sigma_A \cup \{P\}$ is the Kaiser hull T^0. By the perfectness of the theory T, we see that $T^0 = T^*$ and $E_T = \operatorname{Mod} T^*$ (Theorem 1) and by Lemma 4 $T^J_P(A) = T^0$ is a perfect Jonsson theory. Let the theory T^0 have no more than λ $\exists$-completions. The center of the theory T in the new signature $\sigma_P(A) = \sigma_A \cup \{P\}$ is equal to

$$\operatorname{Th}(C, a)_{a \in A} \cup \{P(c_a) \mid a \in A\} \cup \{\text{“}P \preceq\text{”}\}.$$

We must show that the number of all completions of T^* is not greater than λ. Then T^* will be P-λ-stable (in the sense of [19]).

Let us understand why T^* is not complete in the new signature. The addition of constants gives only inessential expansions, which will not change the number of types of existentially closed submodels C. The essential role is played by realizations of P. In this case, the realization of a predicate P will be some elementary submodel M of the model C. Since the semantic model C of the Jonsson theory T is existentially closed [17], from the meaning of P in C ($M \preceq C$) it follows that $M \in E_T$. Let us consider any completion T' of the theory T^* in the new signature. By the definition of T^*, there exists a model M from E_T such that $T' = \operatorname{Th}(C, M, a)_{a \in A}$, where M is the interpretation of P in the semantic model C. By Lemma 4, $T' = \operatorname{Th}(C, M, a)_{a \in A}$ is a Jonsson theory. In this case, T' is a model complete theory by Theorem 3. Thus, in this case by the model completeness of T' any formula in T' is equivalent to some existential formula in T'. Then by the $\exists$-completeness of the theory T there are at most λ such completions under condition (2). The statement is proved. ☐

As a consequence of this result and Lemma 2, one can obtain the following statement.

Corrolary. *Let T be an uncountably categorical Jonsson $\exists$-complete theory. Then the following conditions are equivalent*:

(1) *the theory of Jonsson elementary pairs T^J_P is an $\exists$-complete theory*;

(2) *the theory of elementary pairs of the center of the theory T is complete* (*in the sense of* [21]).

Conclusion

Let us note that since a theory that is complete for existential sentences satisfies the joint embedding property, but the converse is not true [11, p. 157], we see that the condition of ∃-completeness in Theorem 12 and in Corollary 3 cannot be eliminated. In connection with the fact that there exists a continuum of existentially closed groups that are not elementarily equivalent [12] and the theory of groups is a Jonsson theory, one can draw the conclusion that one cannot omit the requirement of perfectness in the condition of Theorem 12.

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