ISSN 0012-2661, Differential Equations, 2011, Vol. 47, No. 2, pp. 231–243. © Pleiades Publishing, Ltd., 2011.
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PARTIAL DIFFERENTIAL EQUATIONS

Boundary Value Problems for a Spectrally Loaded Heat Operator with Load Line Approaching the Time Axis at Zero or Infinity

M. M. Amangalieva, D. M. Akhmanova, M. T. Dzhenaliev,
and M. I. Ramazanov
Institute for Mathematics of the Ministry of Education and Science,
Almaty, Kazakhstan
Karaganda State University, Karaganda, Kazakhstan

Received April 1, 2009

Abstract—We continue the study of boundary value problems for spectrally loaded heat equations in unbounded domains for the case in which the order of the derivative in the loaded term coincides with that of the differential part of the equation and the motion of the load point with respect to the space variable is given by the law $\bar{x}(t) = t^\omega$, $-\infty < \omega < 1/2$.

DOI: 10.1134/S0012266111020091

1. INTRODUCTION

The main problems arising in the theory of boundary value problems for loaded equations in connection with numerous applications are studied by numerous authors [1–5].

As was mentioned in [6–12], there arise some new properties of a spectrally loaded differential operator. For example, the first boundary value problem for a semibounded domain becomes Fredholm; moreover, the index of the problem depends on the values of the coefficient multiplying the loaded term, which explains the introduction of the term "spectrally loaded." Various cases in which the load in the equation is specified by the trace of the solution of the boundary value problem on the line $\bar{x}(t) = t^\omega$ and is not a weak perturbation of its differential part were considered in [6, 10–14]. The following cases were considered in these papers: $\omega = 0$ [10], $\omega = 1/2$ [11], $\omega = 1$ [12, 13], and $1/2 < \omega < \infty$ [14].

In the present paper, we consider the case in which the load point moves according to the law $\bar{x}(t) = t^\omega$, where $-\infty < \omega < 1/2$, and show that the case of $\omega = 0$ is the *standard case* for it.

The considered boundary value problems can be reduced to a pair of singular Volterra integral equations of the second kind, for which the corresponding characteristic integral equations are chosen. Further, by using results on characteristic integral equations [10, 12–14], we solve the original integral equations by the Carleman–Vekua regularization method.

2. STATEMENTS OF THE PROBLEMS

In the domain $Q = \{x \in \mathbb{R}_+,\ t \in \mathbb{R}_+\}$, $\mathbb{R}_+ = (0, \infty)$, consider the following adjoint boundary value problems for the spectrally loaded heat equation:

$$\mathbb{L}_\lambda u = f \iff \begin{cases} u_t - u_{xx} + \lambda u_{xx}(x,t)|_{x=t^\omega} = f, \\ u(x,0) = 0, \quad u(0,t) = 0, \end{cases} \tag{1}$$

$$\mathbb{L}_\lambda^* v = g \iff \begin{cases} -v_t - v_{xx} + \bar{\lambda}\delta''(x - t^\omega) \otimes \int\limits_0^\infty v(\xi,t)\,d\xi = g, \\ v(x,\infty) = 0, \quad v(0,t) = v(\infty,t) = v_x(\infty,t) = 0, \end{cases} \tag{2}$$

where, unlike the earlier-studied cases, we assume that $-\infty < \omega < 1/2$. The coefficient $\lambda = \lambda_1 + i\lambda_2 \in \mathbb{C}$ multiplying the loaded term is a spectral parameter, and $\delta(x - t^\omega)$ is the delta function concentrated on the open line $x = t^\omega$ in the domain Q. In addition, we assume that

$$\begin{gathered}
e^t t^{\omega-3/2} f \in L_1(\mathbb{R}^t_+; L_\infty(\mathbb{R}^x_+)), \quad e^{-t} t^{-\omega}(x + \sqrt{t}\,)\, g \in L_\infty(\mathbb{R}^t_+; L_1(\mathbb{R}^x_+)) \quad \text{are given functions,} \\
t^{2\omega}\left(\frac{\partial^2}{\partial x^2}\int\limits_0^t\int\limits_0^\infty G(x,\xi,t-\tau) f(\xi,\tau)\,d\xi\,d\tau\right)\Bigg|_{x=t^\omega} \in L_1(\mathbb{R}_+), \\
e^t t^{\omega-3/2}\left(\frac{\partial^2}{\partial x^2}\int\limits_0^t\int\limits_0^\infty G(x,\xi,t-\tau) f(\xi,\tau)\,d\xi\,d\tau\right)\Bigg|_{x=t^\omega} \in L_1(\mathbb{R}_+), \\
\int\limits_t^\infty\int\limits_0^\infty \operatorname{erf}\left(\frac{\xi}{2\sqrt{t-\tau}}\right) g(\xi,\tau)\,d\xi\,d\tau \in L_\infty(\mathbb{R}_+), \qquad (3) \\
\operatorname{erf} a = \frac{2}{\sqrt{\pi}}\int\limits_0^a \exp(-z^2)\,dz,
\end{gathered}$$

and the Green function $G(x, \xi, t)$ is given by the formula

$$G(x,\xi,t) = \frac{1}{2\sqrt{\pi t}}\left\{\exp\left(-\frac{(x-\xi)^2}{4t}\right) - \exp\left(-\frac{(x+\xi)^2}{4t}\right)\right\}.$$

The function classes $\mathcal{U}$ and $\mathcal{V}$ for the solutions of the boundary value problems (1) and (2), respectively, and the domains $\mathcal{D}(\mathbb{L}_\lambda)$ and $\mathcal{D}(\mathbb{L}^*_\lambda)$ of the operators $\mathbb{L}_\lambda$ and $\mathbb{L}^*_\lambda$ are defined as follows:

$$\begin{aligned}
\mathcal{U} = \{u|\ & e^t t^\omega (x+\sqrt{t}\,)^{-1} u, e^t t^{\omega-3/2}(u_t - u_{xx}) \in L_1(\mathbb{R}^t_+; L_\infty(\mathbb{R}^x_+)), \\
& e^t t^{\omega-3/2} u_{xx}(x,t)|_{x=t^\omega} \in L_1(\mathbb{R}_+)\}, \qquad (4) \\
\mathcal{V} = \Bigg\{v|\ & e^{-t} t^{3/2-\omega} v, e^{-t} t^{-\omega}(x+\sqrt{t})(v_t + v_{xx}) \in L_\infty(\mathbb{R}^t_+; L_1(\mathbb{R}^x_+)), \\
& e^{-t} t^{3/2-\omega}\int\limits_0^\infty v(\xi,t)\,d\xi \in L_\infty(\mathbb{R}_+)\Bigg\}, \qquad (5) \\
\mathcal{D}(\mathbb{L}_\lambda) \equiv {} & \mathcal{D}(\mathbb{L}_0) = \{u|\ u \in \mathcal{U},\ u(x,0) = 0,\ u(0,t) = 0\}, \qquad (6) \\
\mathcal{D}(\mathbb{L}^*_\lambda) \equiv {} & \mathcal{D}(\mathbb{L}^*_0) = \{v|\ v \in \mathcal{V},\ v(x,\infty) = 0,\ v(0,t) = 0,\ v(\infty,t) = 0,\ v_x(\infty,t) = 0\}. \qquad (7)
\end{aligned}$$

The boundary value problem (2) is adjoint to problem (1). Indeed, by (1)–(7) we have

$$\langle \mathbb{L}_\lambda u, v\rangle = \langle u, \mathbb{L}^*_\lambda v\rangle \quad \forall u \in \mathcal{D}(\mathbb{L}_\lambda), \quad \forall v \in \mathcal{D}(\mathbb{L}^*_\lambda). \qquad (8)$$

Problem. Study the solvability of the boundary value problems (1) and (2) under conditions (3)–(7).

3. INTEGRAL EQUATIONS OF THE BOUNDARY VALUE PROBLEMS

By inverting the differential part in the boundary value problem (1), we obtain

$$u(x,t) = -\lambda\int\limits_0^t \operatorname{erf}\left(\frac{x}{2\sqrt{t-\tau}}\right) u_{\eta\eta}(\eta,\tau)|_{\eta=\tau^\omega}\,d\tau + \int\limits_0^t\int\limits_0^\infty G(x,\xi,t-\tau) f(\xi,\tau)\,d\xi\,d\tau, \qquad (9)$$

since $\int_0^\infty G(x,\xi,t)\,d\xi = \operatorname{erf}(x/(2\sqrt{t}\,))$.

Therefore, it follows from relation (9) that, to find the solution of problem (1), it suffices to determine the loaded term $u_{xx}(x,t)|_{x=t^\omega}$. To this end, we differentiate both sides of relation (9) twice with respect to x and set $x = t^\omega$ in the resulting relation. If we write

$$\begin{gathered}
\tilde{\mu}(t) = e^t t^{\omega-3/2} u_{xx}(x,t)|_{x=t^\omega}, \\
\tilde{\mathcal{K}}_2(t,\tau) = e^{-(\tau-t)} \left(\frac{t}{\tau}\right)^{\omega-3/2} \frac{t^\omega}{2\sqrt{\pi}(t-\tau)^{3/2}} \exp\left(-\frac{t^{2\omega}}{4(t-\tau)}\right), \\
\tilde{f}_1(t) = e^t t^{\omega-3/2} \left(\frac{\partial^2}{\partial x^2} \int\limits_0^t \int\limits_0^\infty G(x,\xi,t-\tau) f(\xi,\tau)\, d\xi\, d\tau \right)\Bigg|_{x=t^\omega},
\end{gathered} \tag{10}$$

then from (9), we obtain the integral equation

$$\tilde{\mathbf{K}}_{2\lambda}\tilde{\mu} \equiv (I - \lambda\tilde{\mathbf{K}}_2)\tilde{\mu} \equiv \tilde{\mu}(t) - \lambda \int\limits_0^t \tilde{\mathcal{K}}_2(t,\tau)\tilde{\mu}(\tau)\, d\tau = \tilde{f}_1(t), \qquad t \in \mathbb{R}_+. \tag{11}$$

Next, if we introduce the notation

$$\begin{gathered}
\mu(t) = t^{\omega-3/2} u_{xx}(x,t)|_{x=t^\omega}, \\
\mathcal{K}_2(t,\tau) = \left(\frac{t}{\tau}\right)^{\omega-3/2} \frac{t^\omega}{2\sqrt{\pi}(t-\tau)^{3/2}} \exp\left(-\frac{t^{2\omega}}{4(t-\tau)}\right), \\
f_1(t) = t^{\omega-3/2} \left(\frac{\partial^2}{\partial x^2} \int\limits_0^t \int\limits_0^\infty G(x,\xi,t-\tau) f(\xi,\tau)\, d\xi\, d\tau \right)\Bigg|_{x=t^\omega},
\end{gathered} \tag{12}$$

then from the representation (9), we obtain the integral equation

$$\mathbf{K}_{2\lambda}\mu \equiv (I - \lambda\mathbf{K}_2)\mu \equiv \mu(t) - \lambda \int\limits_0^t \mathcal{K}_2(t,\tau)\mu(\tau)\, d\tau = f_1(t), \qquad t \in \mathbb{R}_+. \tag{13}$$

We need to solve Eq. (11). However, to study Eq. (11), it suffices to solve Eq. (13).

Remark 1. The continuous nonnegative kernel $\mathcal{K}_2(t,\tau)$ has a weak singularity and satisfies the limit relation

$$\lim_{t\to+0} \int\limits_0^t \mathcal{K}_2(t,\tau)\, d\tau = 0. \tag{14}$$

Obviously, the kernel $\tilde{\mathcal{K}}_2(t,\tau)$ has the property (14) as well.

Then, by inverting the differential part in problem (2), by analogy with problem (1), we obtain the integral relation

$$\begin{aligned}
v(x,t) = -\bar{\lambda} \int\limits_t^\infty \int\limits_0^\infty G(x,\xi,\tau-t)\delta''(\xi-\tau^\omega) \otimes \int\limits_0^\infty v(\eta,\tau)\, d\eta\, d\xi\, d\tau \\
+ \int\limits_t^\infty \int\limits_0^\infty G(x,\xi,\tau-t) g(\xi,\tau)\, d\xi\, d\tau.
\end{aligned} \tag{15}$$

By setting

$$\tilde{\nu}(t) = e^{-t} t^{3/2-\omega} \int\limits_0^\infty v(\eta, t)\, d\eta,$$

from (15), we obtain the integral equation

$$\tilde{\mathbf{K}}^*_{2\lambda}\tilde{\nu} \equiv (I - \bar{\lambda}\tilde{\mathbf{K}}^*_2)\tilde{\nu} \equiv \tilde{\nu}(t) - \bar{\lambda} \int\limits_t^\infty \tilde{\mathcal{K}}_2(\tau, t)\tilde{\nu}(\tau)\, d\tau = \tilde{g}_1(t), \qquad t \in \mathbb{R}_+, \tag{16}$$

where $\tilde{\mathcal{K}}_2(\tau, t)$ is defined in (10) and we have used the notation

$$\tilde{g}_1(t) = e^{-t} t^{3/2-\omega} \int\limits_t^\infty \int\limits_0^\infty \operatorname{erf}\left(\frac{\xi}{2\sqrt{\tau - t}}\right) g(\xi, \tau)\, d\xi\, d\tau.$$

Next, by setting

$$\nu(t) = t^{3/2-\omega} \int\limits_0^\infty v(\eta, t)\, d\eta,$$

we obtain the integral equation

$$\mathbf{K}^*_{2\lambda}\nu \equiv (I - \bar{\lambda}\mathbf{K}^*_2)\nu \equiv \nu(t) - \bar{\lambda} \int\limits_t^\infty \mathcal{K}_2(\tau, t)\nu(\tau)\, d\tau = g_1(t), \qquad t \in \mathbb{R}_+, \tag{17}$$

where

$$g_1(t) = t^{3/2-\omega} \int\limits_t^\infty \int\limits_0^\infty \operatorname{erf}\left(\frac{\xi}{2\sqrt{\tau - t}}\right) g(\xi, \tau)\, d\xi\, d\tau. \tag{18}$$

Remark 2. Note that the kernel $\mathcal{K}_2(\tau, t)$ of the adjoint integral operator in (17) has the property

$$\lim_{t\to\infty} \int\limits_t^\infty \mathcal{K}_2(\tau, t)\, d\tau = 1. \tag{19}$$

Relation (19) also holds for the kernel $\tilde{\mathcal{K}}_2(\tau, t)$, and this means that the norm of the integral operator with kernel $\tilde{\mathcal{K}}_2(\tau, t)$ in the space of essentially bounded functions is equal to unity. This is where Eq. (17) substantially differs from Volterra equations of the second kind [and accordingly from Eq. (16)] for which there exists a unique solution.

Therefore, solving the adjoint boundary value problems (1) and (2) can be reduced to studying the pair of adjoint integral equations (13) and (17), which will be called the *original* equations in what follows.

Remark 3. In general, the adjointness of the boundary value problems (1) and (2) to each other is understood in the sense that the corresponding integral equations (13) and (17) are adjoint to each other.

The subsequent investigation of the solvability of the boundary value problems (1) and (2) is carried out by the following scheme: we introduce the characteristic integral equation corresponding to the leading part of the kernel of the integral operator in (17) and its adjoint, study the solvability of characteristic integral equations; analyze the solvability of the integral equations (17) and (13) by the (Carleman–Vekua) regularization method by solving the characteristic integral equations, and study the solvability of the boundary value problems (1) and (2).

4. CHARACTERISTIC INTEGRAL EQUATIONS

The characteristic integral equations for (13) and (17) are the adjoint equations

$$\mathbf{K}_\lambda\mu \equiv (I-\lambda\mathbf{K})\mu \equiv \mu(t) - \lambda\int\limits_0^t \mathcal{K}(t,\tau)\mu(\tau)\,d\tau = f_1(t), \qquad t\in\mathbb{R}_+, \tag{20}$$

$$\mathbf{K}^*_\lambda\nu \equiv (I-\bar\lambda\mathbf{K}^*)\nu \equiv \nu(t) - \bar\lambda\int\limits_t^\infty \mathcal{K}(\tau,t)\nu(\tau)\,d\tau = g_1(t), \qquad t\in\mathbb{R}_+, \tag{21}$$

respectively, where

$$\mathcal{K}(\tau,t) = \left(\frac{t}{\tau}\right)^{1+\alpha/2}\frac{\alpha^{3/2}\tau^{\alpha-1}}{2\sqrt{\pi}(\tau^\alpha-t^\alpha)^{3/2}}\exp\left(-\frac{\alpha}{4(\tau^\alpha-t^\alpha)}\right), \qquad \alpha = 1-2\omega; \tag{22}$$

throughout the following, we assume that $\alpha > 0$, because $-\infty < \omega < 1/2$.

Note that the kernel $\mathcal{K}(\tau,t)$ of the characteristic equation has the same properties as the kernel $\mathcal{K}_2(\tau,t)$; in particular, it satisfies the limit relations

$$\lim_{t\to 0}\int\limits_0^t \mathcal{K}(t,\tau)\,d\tau = 0, \qquad \lim_{t\to\infty}\int\limits_t^\infty \mathcal{K}(\tau,t)\,d\tau = 1. \tag{23}$$

Next, note that if the integral equation (22) is characteristic, then its kernel satisfies the following two conditions.

(1^0) It can be reduced to the "standard" equation (the integral equation corresponding to the case $\omega = 0$ [10]).

(2^0) The difference $\mathcal{K}_2(\tau,t) - \mathcal{K}(\tau,t) = \tilde{\mathcal{K}}(\tau,t)$ of the kernels has a weak singularity (as $t\to\infty$).

Let us verify condition (1^0). To this end, in Eq. (21), we make the change of independent variables $t = [\alpha t_1]^{1/\alpha}$ and $\tau = [\alpha\tau_1]^{1/\alpha}$ and introduce the notation

$$k(z) = \frac{1}{2\sqrt{\pi}\,z^{3/2}}\exp\left(-\frac{1}{4z}\right), \qquad z>0,$$
$$\varphi(t_1) = t_1^{2/\alpha-1/2}\mu([\alpha t_1]^{1/\alpha}), \qquad f_2(t_1) = t_1^{2/\alpha-1/2}f_1([\alpha t_1]^{1/\alpha}),$$
$$\psi(t_1) = t_1^{-1/\alpha-1/2}\nu([\alpha t_1]^{1/\alpha}), \qquad g_2(t_1) = t_1^{-1/\alpha-1/2}g_1([\alpha t_1]^{1/\alpha});$$

then Eqs. (20) and (21) can be represented in the form

$$k_\lambda\varphi \equiv (I-\bar\lambda k)\varphi \equiv \varphi(t_1) - \lambda\int\limits_0^{t_1} k(t_1-\tau_1)\varphi(\tau_1)\,d\tau_1 = f_2(t_1), \tag{24}$$

$$k^*_\lambda\psi \equiv (I-\bar\lambda k^*)\psi \equiv \psi(t_1) - \bar\lambda\int\limits_{t_1}^\infty k(\tau_1-t_1)\psi(\tau_1)\,d\tau_1 = g_2(t_1), \tag{25}$$

respectively.

Let us proceed to the proof of condition (2^0), which is a consequence of the following theorem.

Theorem 1. *If the conditions $\alpha > 0$ and $0 < t < \tau < \infty$ are satisfied, then the estimate*

$$|\mathcal{K}_2(\tau,t) - \mathcal{K}(\tau,t)| \le C(\alpha)\frac{t^{1-\alpha/2}}{\tau^{3/2}\sqrt{\tau-t}}\exp\left[-C(\alpha)\frac{\tau^{1-\alpha}}{\tau-t}\right] \tag{26}$$

and the limit relation $\lim_{t\to\infty}\int_t^\infty|\mathcal{K}_2(\tau,t)-\mathcal{K}(\tau,t)|\,d\tau = 0$ *hold.*

Proof. To prove in a simpler way that the integral equation (21) with infinite integration limit is characteristic for Eq. (17), it is more convenient to reduce Eqs. (17) and (21) to equations on the interval $(0, t)$. To this end, in these equations, we change the independent variables by the formulas $t = 1/t_1$ and $\tau = 1/\tau_1$. Further, denoting the variables t_1 and τ_1 again by t and τ, respectively, and the kernels of the resulting integral equations by $\mathcal{K}_2'(t,\tau)$ and $\mathcal{K}'(t,\tau)$, respectively, we represent them in the form

$$\mathcal{K}_2'(t,\tau) = P_2'(t,\tau)\exp\{-Q_2'(t,\tau)\}, \qquad \mathcal{K}'(t,\tau) = P'(t,\tau)\exp\{-Q'(t,\tau)\}, \tag{27}$$

where

$$P_2'(t,\tau) = \frac{t^{1/2-\alpha/2}\tau^{\alpha}}{2\sqrt{\pi}\,(t-\tau)^{3/2}}, \qquad Q_2'(t,\tau) = \frac{t\tau^{\alpha}}{4(t-\tau)};$$
$$P'(t,\tau) = \frac{\alpha^{3/2}t^{\alpha-1}\tau^{\alpha}}{2\sqrt{\pi}\,(t^{\alpha}-\tau^{\alpha})^{3/2}}, \qquad Q'(t,\tau) = \frac{\alpha t^{\alpha}\tau^{\alpha}}{4(t^{\alpha}-\tau^{\alpha})}.$$

We note that the kernels $\mathcal{K}_2'(t,\tau)$ and $\mathcal{K}'(t,\tau)$ satisfy the limit relations

$$\lim_{t\to 0}\int_0^t \mathcal{K}_2'(\tau,t)\,d\tau = 1, \qquad \lim_{t\to 0}\int_0^t \mathcal{K}'(t,\tau)\,d\tau = 1 \tag{28}$$

and write out the desired inequality (26) in the form

$$\mathcal{K}_0'(\tau,t) = |\mathcal{K}'(t,\tau) - \mathcal{K}_2'(\tau,t)| \le C(\alpha)\frac{t^{\alpha/2-1/2}}{\sqrt{t-\tau}}\exp\left[-C(\alpha)\frac{t\tau^{\alpha}}{t-\tau}\right]. \tag{29}$$

One can readily see that, under the assumptions of the theorem, we have the inequalities

$$P_0'(t,\tau) = |P'(t,\tau) - P_2'(t,\tau)| \le C(\alpha)\frac{t^{\alpha/2-1/2}}{\sqrt{t-\tau}}, \tag{30}$$
$$Q_0'(t,\tau) = |Q_2'(t,\tau) - Q'(t,\tau)| \le C_2(\alpha)t^{\alpha}. \tag{31}$$

Let us prove inequality (29) for all values of the parameter α and $0 < \tau < t < \infty$ such that $Q'(t,\tau) - Q_2'(t,\tau) \ge 0$; for the values of the parameter α and $0 < \tau < t < \infty$ for which the difference $Q'(t,\tau) - Q_2'(t,\tau)$ is negative, in the proof, it suffices to exchange the functions $Q'(t,\tau)$ with $Q_2'(t,\tau)$ and $P'(t,\tau)$ with $P_2'(t,\tau)$. We have

$$\begin{aligned}&|\mathcal{K}'(t,\tau) - \mathcal{K}_2'(t,\tau)| \\ &\qquad \le |(P'(t,\tau) - P_2'(t,\tau))|\exp\{-Q'(t,\tau)\} + P_2'(t,\tau)|(Q_2'(t,\tau) - Q'(t,\tau))|\exp\{-Q'(t,\tau)\}.\end{aligned}$$

By using relations (30) and (31) and the two-sided inequality [15, p. 55] $b_1 t^{\alpha-1}(t-\tau) \le t^{\alpha} - \tau^{\alpha} \le b_2 t^{\alpha-1}(t-\tau)$, where $b_1 = \min\{1,\alpha\}$ and $b_2 = \max\{1,\alpha\}$, we obtain

$$|(P'(t,\tau) - P_2'(t,\tau))|\exp\{-Q'(t,\tau)\} \le C_1\frac{t^{\alpha/2-1/2}}{\sqrt{t-\tau}}\exp\left\{-\frac{\alpha}{b_2}\,\frac{t\tau^{\alpha}}{4(t-\tau)}\right\},$$
$$P_2'(t,\tau)|Q_2'(t,\tau) - Q'(t,\tau)|\exp\{-Q'(t,\tau)\} \le C_3\frac{t^{\alpha/2-1/2}}{\sqrt{t-\tau}}\exp\left\{\frac{\alpha}{b_2}\,\frac{t\tau^{\alpha}}{8(t-\tau)}\right\}.$$

This completes the proof of inequality (29). Inequality (29) implies that the kernel $\mathcal{K}_0'(t,\tau)$ has a weak singularity. This completes the proof of Theorem 1.

5. SOLUTION OF CHARACTERISTIC INTEGRAL EQUATIONS

As was mentioned above, if, in the characteristic integral equations (21) and (22), we make the changes of independent variables ($\alpha = 1 - 2\omega > 0$) $t = [\alpha t_1]^{1/\alpha}$ and $\tau = [\alpha \tau_1]^{1/\alpha}$ and set

$$\tilde{\varphi}(t_1) = t_1^{1/\alpha - 1}\mu([\alpha t_1]^{1/\alpha}), \qquad \tilde{f}_2(t_1) = t_1^{1/\alpha - 1} f_1([\alpha t_1]^{1/\alpha}),$$
$$\tilde{\psi}(t_1) = \nu([\alpha t_1]^{1/\alpha}), \qquad \tilde{g}_2(t_1) = g_1([\alpha t_1]^{1/\alpha}), \qquad k(t) = \frac{1}{2\sqrt{\pi}\, t^{3/2}} \exp\left(-\frac{1}{4t}\right),$$

then we obtain the standard integral equations

$$\mathbf{k}_\lambda \tilde{\varphi} \equiv (I - \lambda \mathbf{k})\tilde{\varphi} \equiv \tilde{\varphi}(t_1) - \lambda \int\limits_0^{t_1} \left(\frac{\tau_1}{t_1}\right)^{1/\alpha + 1/2} k(t_1 - \tau_1)\tilde{\varphi}(\tau_1) d\tau_1 = \tilde{f}_2(t_1), \qquad t_1 > 0,$$

$$\mathbf{k}_\lambda^* \tilde{\psi} \equiv (I - \bar{\lambda} \mathbf{k}^*)\tilde{\psi} \equiv \tilde{\psi}(t_1) - \bar{\lambda} \int\limits_{t_1}^{\infty} \left(\frac{t_1}{\tau_1}\right)^{1/\alpha + 1/2} k(\tau_1 - t_1)\tilde{\psi}(\tau_1)\, d\tau_1 = \tilde{g}_2(t_1), \qquad t_1 > 0.$$

It follows from the results in [10, 12–14] that the solutions of the pair of adjoint characteristic integral equations (20) and (21) are defined in accordance with formulas (51) and (50) in [10] as follows ($\alpha > 0$):

$$\mu(t) \equiv t^{1-\alpha}\tilde{\varphi}([\alpha^{-1}t^\alpha]) = f_1(t) + \lambda \int\limits_0^t \left(\frac{\tau}{t}\right)^{1+\alpha/2} t^{\alpha-1}\mathbf{r}_{\lambda+}([\alpha^{-1}t^\alpha] - [\alpha^{-1}\tau^\alpha]) f_1(\tau)\, d\tau, \quad t \in \mathbb{R}_+, \tag{32}$$

$$\begin{aligned}\nu(t) \equiv \tilde{\psi}([\alpha^{-1}t^\alpha]) = g_1(t) &+ \bar{\lambda} \int\limits_t^{\infty} \left(\frac{t}{\tau}\right)^{1+\alpha/2} \tau^{\alpha-1}\mathbf{r}_{\lambda-}([\alpha^{-1}\tau^\alpha] - [\alpha^{-1}t^\alpha]) g_1(\tau)\, d\tau \\ &+ \sum_{k=-N_1}^{N_2} c_k t^{1+\alpha/2} \exp\left(-\frac{iz_k}{\alpha}\, t^\alpha\right), \qquad t \in \mathbb{R}_+,\end{aligned} \tag{33}$$

where the functions $\mathbf{r}_{\lambda-}(\theta)$ and $\mathbf{r}_{\lambda+}(\theta)$ and the numbers z_k are defined below in formulas (37), (38), (46), and (47) and satisfy the estimates (39) and (48), respectively, in [10]. (Note that analogs of those formulas in [12] are given by (50), (51), (59), (60), (42), (52), and (61).)

6. SOLUTION OF THE ORIGINAL INTEGRAL EQUATIONS BY THE CARLEMAN–VEKUA REGULARIZATION METHOD

Let us study Eq. (17) corresponding to the boundary value problem (2). We introduce the notation $\tilde{\mathcal{K}}(\tau, t) = \mathcal{K}_2(\tau, t) - \mathcal{K}(\tau, t)$ and rewrite the original integral equation (17) in the form

$$\mathbf{K}_\lambda^* \nu \equiv \nu(t) - \bar{\lambda} \int\limits_t^{\infty} \mathcal{K}(\tau, t)\nu(\tau)\, d\tau = \bar{\lambda} \int\limits_t^{\infty} \tilde{\mathcal{K}}(\tau, t)\nu(\tau)\, d\tau + g_1(t). \tag{34}$$

By inverting the characteristic part of Eq. (34) in accordance with (33), we obtain

$$\begin{aligned}\nu(t) = g_1(t) &+ \bar{\lambda} \int\limits_t^{\infty} \tilde{\mathcal{K}}(t, \tau)\nu(\tau)\, d\tau \\ &+ \bar{\lambda} \int\limits_t^{\infty} \left(\frac{t}{\tau}\right)^{1+\alpha/2} \tau^{\alpha-1}\mathbf{r}_{\lambda-}(\alpha^{-1}\tau^\alpha - \alpha^{-1}t^\alpha) \left[g_1(\tau) + \bar{\lambda} \int\limits_\tau^{\infty} \tilde{\mathcal{K}}(\zeta, \tau)\nu(\zeta)\, d\zeta\right] d\tau \\ &+ \sum_{k=-N_1}^{N_2} c_k t^{1+\alpha/2} \exp\left(-\frac{iz_k}{\alpha}\, t^\alpha\right).\end{aligned}$$

By transforming the right-hand side of the last equation, we reduce it to the form

$$\nu(t) = g_1(t) + \bar\lambda \int\limits_t^\infty \left(\frac{t}{\tau}\right)^{1+\alpha/2} \tau^{\alpha-1}\mathbf{r}_{\lambda-}(\alpha^{-1}\tau^\alpha - \alpha^{-1}t^\alpha)g_1(\tau)d\tau$$

$$+ \sum_{k=-N_1}^{N_2} c_k t^{1+\alpha/2}\exp\left(-\frac{iz_k}{\alpha}\,t^\alpha\right) + \bar\lambda\int\limits_t^\infty \tilde{\mathcal{K}}(\tau,t)\nu(\tau)\,d\tau$$

$$+ \bar\lambda^2 \int\limits_t^\infty \nu(\zeta)\,d\zeta \int\limits_t^\infty \left(\frac{t}{\tau}\right)^{1+\alpha/2} \tau^{\alpha-1}\mathbf{r}_{\lambda-}(\alpha^{-1}\tau^\alpha - \alpha^{-1}t^\alpha)\tilde{\mathcal{K}}(\zeta,\tau)d\tau. \tag{35}$$

Hence for the unknown function $\nu(t)$, we obtain the new regularized equation

$$\hat{\mathbf{K}}_\lambda^*\nu \equiv (I - \bar\lambda\hat{\mathbf{K}}^*)\nu \equiv \nu(t) - \bar\lambda\int\limits_t^\infty \hat{\mathcal{K}}(\tau,t)\nu(\tau)\,d\tau$$

$$= \hat{g}(t) + \sum_{k=-N_1}^{N_2} c_k t^{1+\alpha/2}\exp\left(-\frac{iz_k}{\alpha}\,t^\alpha\right), \tag{36}$$

where we have used the notation

$$\hat{\mathcal{K}}(\tau,t) = \tilde{\mathcal{K}}(\tau,t) + \bar\lambda\int\limits_t^\tau \left(\frac{t}{\zeta}\right)^{1+\alpha/2}\zeta^{\alpha-1}\mathbf{r}_{\lambda-}(\alpha^{-1}\zeta^\alpha - \alpha^{-1}t^\alpha)\tilde{\mathcal{K}}(\tau,\zeta)\,d\zeta = \tilde{\mathcal{K}}(\tau,t) + \bar\lambda\tilde{\tilde{\mathcal{K}}}(\tau,t), \tag{37}$$

$$\hat{g_1}(t) = g_1(t) + \bar\lambda\int\limits_t^\infty \left(\frac{t}{\tau}\right)^{1+\alpha/2}\tau^{\alpha-1}\mathbf{r}_{\lambda-}(\alpha^{-1}\tau^\alpha - \alpha^{-1}t^\alpha)g_1(\tau)\,d\tau. \tag{38}$$

Let us show that the integral equation (36) is indeed regular (has a unique solution); to this end, it suffices to prove the estimate

$$|\hat{\mathcal{K}}(\tau,t)| \le C(\alpha)\frac{t^{-\varepsilon}}{\tau^{1/2+\alpha/2-\varepsilon}(\tau-t)^{1/2}}\,\exp\left\{-C(\alpha)\,\frac{\tau^{1-\alpha}}{\tau-t}\right\},$$
$$0<\varepsilon<\alpha/2, \qquad \alpha>0, \qquad 0<t<\tau<\infty. \tag{39}$$

By changing the independent variables in (36) by the formulas $t = t_1^{-1}$ and $\tau = \tau_1^{-1}$, we obtain

$$\nu(t_1) - \bar\lambda\int\limits_0^{t_1}\mathcal{K}'(t_1,\tau_1)\nu(\tau_1)\,d\tau_1 = \bar\lambda\int\limits_0^{t_1}\tilde{\mathcal{K}}'(t_1,\tau_1)\nu(\tau_1)\,d\tau_1 + g_1(t_1), \tag{40}$$

where

$$\tilde{\mathcal{K}}'(t_1,\tau_1) = \mathcal{K}_2'(t_1,\tau_1) - \mathcal{K}'(t_1,\tau_1) \tag{41}$$

and the kernels $\mathcal{K}_2'(t_1,\tau_1)$ and $\mathcal{K}'(t_1,\tau_1)$ are given by (27). Accordingly, relations (36)–(38) acquire the form

$$\hat{\mathbf{K}}_\lambda^{*'}\nu \equiv (I - \bar\lambda\hat{\mathbf{K}}^{*'})\nu \equiv \nu(t_1) - \bar\lambda\int\limits_0^{t_1}\hat{\mathcal{K}}'(t_1,\tau_1)\nu(\tau_1)\,d\tau_1$$

$$= \hat{g}(t_1) + \sum_{k=-N_1}^{N_2} c_k t_1^{-1-\alpha/2}\exp\left(-\frac{iz_k}{\alpha}\,t_1^{-\alpha}\right), \tag{42}$$

$$\begin{aligned}\hat{\mathcal{K}}'(t_1,\tau_1) &= \tilde{\mathcal{K}}'(t_1,\tau_1) + \bar{\lambda}\int_{\tau_1}^{t_1}\left(\frac{\zeta}{t_1}\right)^{1+\alpha/2}\zeta^{-\alpha-1}\mathbf{r}_{\lambda-}([\alpha t_1^{\alpha}]^{-1} - [\alpha\zeta^{\alpha}]^{-1})\tilde{\mathcal{K}}'(\tau_1,\zeta)\,d\zeta \\ &= \tilde{\mathcal{K}}'(t_1,\tau_1) + \bar{\lambda}\tilde{\tilde{\mathcal{K}}}'(t_1,\tau_1), \qquad (43)\end{aligned}$$

$$\hat{g}_1(t_1) = g_1(t_1) + \bar{\lambda}\int_0^{t_1}\left(\frac{\tau_1}{t_1}\right)^{1+\alpha/2}\tau_1^{-\alpha-1}\mathbf{r}_{\lambda-}([\alpha t_1^{\alpha}]^{-1} - [\alpha\tau_1^{\alpha}]^{-1})g_1(\tau_1)\,d\tau_1. \qquad (44)$$

Now show that the integral equation (42) is indeed regular. To this end, it suffices to prove the following lemma.

Lemma 1. *The kernel of the integral equation* (42) *has a weak singularity*; *i.e.*, *the following estimate holds*:

$$|\hat{\mathcal{K}}'(t_1,\tau_1)| \le C\frac{t_1^{1/2+\varepsilon}}{\tau_1^{1-\alpha/2+\varepsilon}(t_1-\tau_1)^{1/2}}\exp\left\{-c(\alpha)\frac{t_1\tau_1^{\alpha}}{t_1-\tau_1}\right\}, \qquad (45)$$
$$0<\varepsilon<\alpha/2, \qquad \alpha>0, \qquad 0<\tau_1<t_1<\infty.$$

Proof. Since $\hat{\mathcal{K}}'(t_1,\tau_1)$ can be represented in the form (43), it follows that the estimate (45) is a consequence of (29), (15), and the relations presented below. By using the two-sided inequality [15, p. 55] $C_1 t_1^{\alpha-1}(t_1-\tau_1) \le t_1^{\alpha} - \tau_1^{\alpha} \le C_2 t_1^{\alpha-1}(t_1-\tau_1)$, where $C_1 = \min\{1,\alpha\}$ and $C_2 = \max\{1,\alpha\}$, and the estimate (39) in [10] for the function $\mathbf{r}_{\lambda-}(\theta)$, we obtain ($\alpha = 1-2\omega > 0$) the inequality

$$\tilde{\tilde{\mathcal{K}}}'(t_1,\tau_1) \le M_1(\alpha)\frac{\sqrt{t_1}}{\tau_1^{1-\alpha/2}}I_1(t_1,\tau_1) + M_2(\alpha)\frac{\sqrt{t_1}}{\tau_1^{1-\alpha/2}}I_2(t_1,\tau_1),$$

$$I_1 = \int_{\tau_1}^{t_1}\frac{1}{\eta^{(\alpha+1)/2}\sqrt{(\eta-\tau_1)(t_1-\eta)}}\exp\left(-\frac{C_1(\alpha)(t_1-\eta)}{t_1\eta^{\alpha}}\right)d\eta,$$

$$I_2 = \int_{\tau_1}^{t_1}\frac{t_1\eta^{(\alpha-1)/2}}{(t_1-\eta)^{3/2}(\eta-\tau_1)^{1/2}}\exp\left(-\frac{C_2(\alpha)t_1\eta^{\alpha}}{t_1-\eta}\right)d\eta.$$

Here $C_j(\alpha)$ and $M_j(\alpha)$, $j=1,2$, are constants depending only on α.

Next, we represent each of the functions $I_1(t_1,\tau_1)$ and $I_2(t_1,\tau_1)$ as the sum of two terms

$$I_1(t_1,\tau_1) = I_{11}(t_1,\tau_1) + I_{12}(t_1,\tau_1), \qquad I_2(t_1,\tau_1) = I_{21}(t_1,\tau_1) + I_{22}(t_1,\tau_1),$$

for each of which we successively have

$$\begin{aligned}I_{11}(t_1,\tau_1) &= \int_{\tau_1}^{(t_1+\tau_1)/2}\frac{1}{\eta^{(\alpha+1)/2}\sqrt{(\eta-\tau_1)(t_1-\eta)}}\exp\left(-\frac{C_1(\alpha)(t_1-\eta)}{t_1\eta^{\alpha}}\right)d\eta \\ &\le \frac{1}{\sqrt{t_1-\tau_1}}[C_1 + C_3(t_1/\tau_1)^{\varepsilon}],\end{aligned}$$

where the value of the parameter ε is chosen from the condition $0<\varepsilon<\alpha/2$:

$$I_{12}(t_1,\tau_1) = \int_{(t_1+\tau_1)/2}^{t_1}\frac{1}{\eta^{(\alpha+1)/2}\sqrt{(\eta-\tau_1)(t_1-\eta)}}\exp\left(-\frac{C_1(\alpha)(t_1-\eta)}{t_1\eta^{\alpha}}\right)d\eta \le \frac{C(\alpha)}{\sqrt{t_1-\tau_1}},$$

$$I_{21}(t_1,\tau_1) = \int_{\tau_1}^{(t_1+\tau_1)/2}\frac{\sqrt{t_1}\eta^{(\alpha-1)/2}}{(t_1-\eta)^{3/2}(\eta-\tau_1)^{1/2}}\exp\left(-\frac{C_2(\alpha)t_1\eta^{\alpha}}{t_1-\eta}\right)d\eta \le \frac{1}{\sqrt{t_1-\tau_1}}[C_1 + C_3(t_1/\tau_1)^{\varepsilon}];$$

here the last inequality is obtained just as in the estimate of the function $I_{11}(t_1,\tau_1)$, and the value of the parameter ε is chosen from the condition $0<\varepsilon<\alpha/2$:

$$I_{22}(t_1,\tau_1)=\int\limits_{(t_1+\tau_1)/2}^{t_1}\frac{t_1\eta^{(\alpha-1)/2}}{(t_1-\eta)^{3/2}(\eta-\tau_1)^{1/2}}\exp\left(-\frac{C_2(\alpha)t_1\eta^{\alpha}}{t_1-\eta}\right)d\eta\le\frac{C(\alpha)}{\sqrt{t_1-\tau_1}}.$$

In these inequalities, $C(\alpha)$ and $C_j(\alpha)$, $j=1,2,3,4$, are distinct constants and depend only on α. The resulting inequalities imply the desired estimate (45). The proof of the lemma is complete.

Thus, by virtue of the estimate (45) for a given right-hand side, Eq. (42) and, in addition, Eq. (36) have a unique solution, whose existence can be proved by the successive approximation method.

It follows from relations (37) and (34) that the homogeneous equation

$$\mathbf{K}^*_{2\lambda}\nu\equiv(I-\bar\lambda\mathbf{K}_2)\nu\equiv\nu(t)-\bar\lambda\int\limits_t^{\infty}\mathcal{K}_2(\tau,t)\nu(\tau)\,d\tau=0,\qquad t\in\mathbb{R}_+,\tag{46}$$

is equivalent to the inhomogeneous equation

$$\hat{\mathbf{K}}^*_{\lambda}\nu\equiv\nu(t)-\bar\lambda\int\limits_t^{\infty}\hat{\mathcal{K}}(\tau,t)\mu(\tau)\,d\tau=\sum_{k=-N_1}^{N_2}c_k t^{1+\alpha/2}\exp\left(-\frac{iz_k}{\alpha}\,t^{\alpha}\right),\qquad t\in\mathbb{R}_+.\tag{47}$$

Instead of (47), consider the family of integral equations

$$\begin{gathered}\hat{\mathbf{K}}^*_{\lambda}\mu\equiv\nu(t)-\bar\lambda\int\limits_t^{\infty}\hat{\mathcal{K}}(t,\tau)\nu(\tau)\,d\tau=t^{1+\alpha/2}\exp\left(-\frac{iz_k}{\alpha}\,t^{\alpha}\right),\\ k=-N_1,\ldots,0,\ldots,N_2,\qquad t\in\mathbb{R}_+.\end{gathered}\tag{48}$$

Note that the numbers N_1 and N_2 are given by (33) in [10], and the partition of the complex plane $\mathbb{C}$ into subsets D_k and Γ_{k-1}, $k=0,1,2,\ldots$, for the values of the parameter λ is defined by relations (32) in [10]. [Note that the analogs of these formulas in [12] are given by (46) and (45).]

Since each of Eqs. (48) has a unique nontrivial solution $\nu_{\lambda k}(t)$, $k=-N_1,\ldots,0,\ldots,N_2$, it follows that, for each parameter value $\lambda\in\mathbb{C}\backslash D_0$, these functions $\nu_{\lambda k}(t)$, $k=-N_1,\ldots,0,\ldots,N_2$, are eigenfunctions of the homogeneous equation (46) [and hence the homogeneous equation corresponding to (17)].

From the assertions of Lemmas 1 and 2 in [10], we obtain the following assertion. (Similar Lemmas 1 and 2 can be found in [12].)

Lemma 2. *The values* $\lambda\in D_0$ *are regular numbers of the operator* $\mathbf{K}^*_{2\lambda}$ *from* (17).

Lemma 3. *The set* $\mathbb{C}\backslash D_0$ *is formed by characteristic numbers of the operator* $\mathbf{K}^*_{2\lambda}$ *in* (17). *In addition, if* $\lambda\in D_m\cup\Gamma_{m-1}\backslash\{(-1)^m e^{m\pi}\}$, $m=1,2,\ldots$, *then* $\dim\operatorname{Ker}(\mathbf{K}^*_{\mathbf{2}\lambda})=m$, *and the corresponding eigenfunctions are given by solutions of Eq.* (48):

$$\nu_{\lambda k}(t)=[\hat{\mathbf{K}}^*_{\lambda}]^{-1}\left[t^{1+\alpha/2}\exp\left(-\frac{iz_k}{\alpha}\,t^{\alpha}\right)\right],\qquad k=1,\ldots,m=N_1+N_2+1.$$

Remark 4. The general solution of the inhomogeneous integral equation (36) as well as Eq. (17) is the function

$$\nu_\lambda(t)=[\hat{\mathbf{K}}^*_{\lambda}]^{-1}\hat g(t)+\sum_{k=1}^{m=N_1+N_2+1}c_k\nu_{\lambda k}(t),\qquad t\in\mathbb{R}_+,\tag{49}$$

where the c_k, $k=1,\ldots,m$, are arbitrary constants.

Let us proceed to the analysis of the integral equation (13), which is adjoint to Eq. (17). It follows from Remark 1 that the homogeneous integral equation corresponding to (13) has only the trivial solution for any $\lambda \in \mathbb{C}$.

Thus, by virtue of Lemmas 1–3, we have the following assertion.

Lemma 4. 1. *Each value* $\lambda \in \mathbb{C}$ *is a regular number of the operator* $\mathbf{K}_{2\lambda}$ *in* (13).

2. *The inhomogeneous integral equation* (13) *is uniquely solvable for any right-hand side* $f_1(t)$ *provided that* $\bar{\lambda} \in D_0$.

3. *If* $\lambda \in D_m \cup \Gamma_{m-1} \backslash \{(-1)^m e^{m\pi}\}$, $m = 1, 2, \ldots,$ *then, for the unique solvability of the inhomogeneous integral equation* (13), *it is necessary and sufficient that the functions* $f_1(t)$ *satisfy the orthogonality conditions*

$$\int_0^\infty \overline{\nu_{\lambda k}(t)} f_1(t)\, dt = 0, \qquad k = 1, \ldots, m = N_1 + N_2 + 1. \tag{50}$$

Remark 5. By the assertion of Lemma 4, the solution of the inhomogeneous integral equation (13) is the function

$$\mu_\lambda(t) = [\mathbf{K}_{2\lambda}]^{-1} f_1(t), \qquad t \in \mathbb{R}_+. \tag{51}$$

Remark 6. It readily follows from the above-performed considerations that

$$\tilde{\mu}_\lambda(t) = e^t \mu_\lambda(t) \in L_1(\mathbb{R}_+), \qquad \tilde{\nu}_\lambda(t) = e^{-t} \nu_\lambda(t) \in L_\infty(\mathbb{R}_+). \tag{52}$$

7. INVESTIGATION OF THE BOUNDARY VALUE PROBLEMS (1) AND (2)

By (9), we write out the solution of problem (1) in the form

$$u(x,t) = -\lambda \int_0^t e^{-\tau} \tau^{3/2-\omega} \operatorname{erf}\left(\frac{x}{2\sqrt{t-\tau}}\right) \tilde{\mu}_\lambda(\tau)\, d\tau + \int_0^t \int_0^\infty G(x,\xi,t-\tau) f(\xi,\tau)\, d\xi\, d\tau, \tag{53}$$

where the function $\tilde{\mu}_\lambda(t)$ is found from (51) and (52). Hence we find that the function (53) completely satisfies the boundary value problem (1) and belongs to the class (4).

Further, by (15), we write out a solution of problem (2) in the form

$$v(x,t) = -\bar{\lambda} \int_t^\infty e^{\tau} \tau^{\omega-3/2} G_{\xi\xi}(x,\xi,\tau-t)\Big|_{\xi=\tau^\omega} \tilde{\nu}_\lambda(\tau)\, d\tau + \int_t^\infty \int_0^\infty G(x,\xi,\tau-t) g(\xi,\tau)\, d\xi\, d\tau,$$

where $\tilde{\nu}_\lambda(t) \in L_\infty(\mathbb{R}_+)$ is the function found from (49) and (52).

The function $v(x,t)$ belongs to the class (5) if

$$e^{-t} t^{3/2-\omega} \int_t^\infty e^{\tau} \tau^{\omega-3/2} G_{\xi\xi}(x,\xi,\tau-t)\Big|_{\xi=\tau^\omega} \tilde{\nu}(\tau)\, d\tau \in L_\infty(\mathbb{R}_+^t; L_1(\mathbb{R}_+^x)), \tag{54}$$

$$e^{-t} t^{3/2-\omega} \int_t^\infty \int_0^\infty G(x,\xi,\tau-t) g(\xi,\tau)\, d\xi\, d\tau \in L_\infty(\mathbb{R}_+^t; L_1(\mathbb{R}_+^x)). \tag{55}$$

The inclusion (54) does take place by virtue of condition (3). And the inclusion (55) is equivalent to the inequality

$$e^{-t} t^{3/2-\omega} \int_0^\infty \int_t^\infty e^{\tau} \tau^{\omega-3/2} G_{\xi\xi}(x,\xi,\tau-t)\Big|_{\xi=\tau} \tilde{\nu}(\tau)\, d\tau\, dx \le \|\tilde{\nu}\|_{L_\infty(\mathbb{R}_+)} \le \infty.$$

Obviously, the derivatives $v_t(x,t)$ and $v_{xx}(x,t)$ of the function $v(x,t)$ satisfy the inclusion

$$e^{-t}t^{-\omega}(x+\sqrt{t})(v_t+v_{xx}) \in L_\infty(\mathbb{R}_+^t; L_1(\mathbb{R}_+^x)).$$

Let us state the obtained results on the solvability of the boundary value problems (1) and (2) in the following theorems.

Theorem 2. *If* $\lambda \in D_0$, *then, for arbitrary* f, *the boundary value problem* (1) *has a unique solution* $u \in \mathcal{U}$. *If* $\lambda \in \{\mathbb{C}\backslash D_0\} \cap \{D_m \cup \Gamma_{m-1}\backslash\{(-1)^m e^{m\pi}\}\}$, *then, for the unique solvability of the boundary value problem* (1) *in the class* $\mathcal{U}$, *it is necessary and sufficient that the function* f *satisfy the orthogonality condition*

$$\int\limits_0^\infty \overline{v_{\lambda k}(x,t)} f(x,t)\,dxdt = 0, \qquad k = 1,\ldots,m = N_1+N_2+1.$$

Theorem 3. *If* $\lambda \in D_0$, *then, for arbitrary* g, *the boundary value problem* (2) *has a unique solution* $v \in \mathcal{V}$. *If* $\lambda \in \{\mathbb{C}\backslash D_0\} \cap \{D_m \cup \Gamma_{m-1}\backslash\{(-1)^m e^{m\pi}\}\}$, *then, for arbitrary* g, *the boundary value problem* (2) *has a general solution* $v \in \mathcal{V}$, *which consists of the sum of the solution* $v_{\mathrm{hom}}(x,t)$ *of the homogeneous equation*

$$v_{\mathrm{hom}}(x,t) = \sum_{k=1}^m c_k v_{\lambda k}(x,t),$$

$$v_{\lambda k}(x,t) = -\bar{\lambda}\int\limits_t^\infty \tau^{\omega-3/2}\operatorname{erf}\left(\frac{x}{2\sqrt{\tau-t}}\right)\nu_{\lambda k}(\tau)d\tau, \qquad k=1,\ldots,m,$$

where

$$\nu_{\lambda k}(t) = [\hat{\mathbf{K}}_\lambda^*]^{-1}\left[\exp\left(-\frac{iz_k}{\alpha}t^\alpha\right)\right]^{-1}, \qquad k=1,\ldots,m=N_1+N_2+1,$$

c_k *are arbitrary constants*, *and the particular solution*

$$v_{\mathrm{par}}(x,t) = -\bar{\lambda}\int\limits_t^\infty \tau^{\omega-3/2}\operatorname{erf}\left(\frac{x}{2\sqrt{\tau-t}}\right)[\hat{\mathbf{K}}_\lambda^*]^{-1}\hat{g}(\tau)\,d\tau + \int\limits_t^\infty\int\limits_0^\infty G(x,\xi,t-\tau)g(\xi,\tau)\,d\xi\,d\tau.$$

We have thereby shown that the above-considered boundary value problem for the spectrally loaded heat equation with a load point with respect to the space variable [$\bar{x}(t) = t^\omega$, $-\infty < \omega < 1/2$] moving at a variable velocity, is a Fredholm problem with nonpositive index depending on values of the coefficient multiplying the loaded term.

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