

## $q$ -Analogues of Lyapunov-type inequalities involving Riemann–Liouville fractional derivatives

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In this article, new  $q$ -analogues of Lyapunov-type inequalities are presented for two-point fractional boundary value problems involving the Riemann–Liouville fractional  $q$ -derivative with well-posed  $q$ -boundary conditions. The study relies on the properties of the  $q$ -Green’s function, which is constructed to solve such problems and allows for the analytical derivation of the inequalities. These inequalities find application in two directions: establishing precise lower bounds for the eigenvalues of corresponding  $q$ -fractional spectral problems and formulating criteria for the absence of real zeros in  $q$ -analogues of Mittag-Leffler functions. The obtained results generalize classical and fractional Lyapunov inequalities, offering new perspectives for the analysis of stability and spectral properties of  $q$ -fractional differential systems. The relevance of the work is driven by the growing interest in  $q$ -calculus in discrete models, such as viscoelastic systems or quantum circuits, where discrete dynamics play a key role. The convenience of closed-form analytical expressions makes the results practically applicable. The research lays the foundation for further generalizations, including Caputo derivatives or multidimensional  $q$ -systems, which may stimulate new discoveries in discrete fractional analysis.

**Keywords:**  $q$ -calculus, fractional  $q$ -derivative, Lyapunov-type inequality, Riemann–Liouville fractional derivative, Green’s function, Mittag-Leffler function, eigenvalue problems, fractional integral.

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### Introduction

Fractional calculus investigates integrals and derivatives of arbitrary (non-integer) order, has become an indispensable framework for modelling complex phenomena in physics, biology, engineering, and economics [1, 2]. Fractional differential equations (FDEs) naturally describe memory effects, non-local interactions, and anomalous diffusion; a representative example is C.F. Li et al.’s proof of positive solutions for nonlinear FDEs with boundary constraints [3].

A central analytical tool for boundary-value problems (BVPs) in the fractional setting is the Lyapunov-type inequality. R.A.C. Ferreira obtained the first variant for a Riemann–Liouville derivative with Dirichlet conditions [4]; M. Jleli and B. Samet extended the result to mixed boundary conditions [5]; and D. Basu et al. treated fractional boundary conditions, applying the inequality to spectral questions [6]. Subsequent refinements yielded sharper eigenvalue bounds and zero-free intervals for Mittag-Leffler functions [7].

Parallel to the continuous theory,  $q$ -fractional calculus blends quantum calculus with fractional analysis. Its origins trace back to Jackson’s introduction of  $q$ -difference operators and integrals [8, 9] and R.D. Carmichael’s work on  $q$ -difference equations [10]. Modern expositions by V. Kac and P. Cheung [11], T. Ernst [12, 13], and M.H. Annaby, Z.S. Mansour [14] have systematised the subject.

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Foundational notions of *q*-fractional integrals and derivatives, proposed by W.A. Al-Salam [15] and R.P. Agarwal [16], were rigorously formalised by P.M. Rajkovic et al. [17, 18].

Applications of *q*-fractional differential equations range from quantum mechanics to discrete dynamical systems. R.A.C. Ferreira analysed non-trivial and positive solutions for several classes of *q*-fractional BVPs [19, 20]; S. Shaimardan and collaborators established existence and uniqueness results for Cauchy-type problems with Riemann–Liouville derivatives [21]. The *q*-fractional framework has been connected with time–scale calculus through the work of F.M. Atici and P.W. Elloe [22]; with three-point and other non-local boundary conditions in the papers of S. Liang, J. Zhang, C. Yu, J. Wang, S. Wang et al. [23–25]; and further refined for related non-local problems by C. Zhai, J. Ren [26] and Y. Zhao, H. Chen, Q. Zhang [27]. Lyapunov-type inequalities for *q*-fractional equations were first obtained by M. Jleli and B. Samet [28].

In this work we derive two new Lyapunov-type inequalities for the *q*-fractional boundary-value problem

$$\begin{cases} D_{q,a}^\alpha u(t) + q(t)u(t) = 0, & a \leq t \leq b, \ 1 < \alpha \leq 2, \ 0 \leq \beta \leq 1, \\ u(a) = 0, \quad D_{q,a}^\beta u(b) = 0, & 0 < q < 1, \end{cases}$$

by exploiting properties of the associated *q*-Green function. The analysis combines topological fixed-point techniques [29], and existence principles in the Caratheodory framework [30]. Our results sharpen eigenvalue estimates, offer criteria for the real zeros of *q*-Mittag-Leffler functions, and advance the spectral theory of discrete fractional models.

## 1 Preliminaries

In this section, we introduce essential definitions and foundational concepts, including key aspects of *q*-calculus, which underpin the present study. For a comprehensive exploration of these topics, readers are referred to the monographs [11, 14].

For  $\alpha \in \mathbb{R}$ , the *q*-real number  $[\alpha]_q$  is given by

$$[\alpha]_q = \frac{1 - q^\alpha}{1 - q}, \quad q \neq 1,$$

where  $\lim_{q \rightarrow 1} \frac{1 - q^\alpha}{1 - q} = \alpha$ .

We introduce for  $k \in \mathbb{N}$ :

$$(a; q)_0 = 1, \ (a; q)_n = \prod_{k=0}^{n-1} (1 - q^k a), \ (a; q)_\infty = \lim_{n \rightarrow \infty} (a; q)_n, \ (a; q)_\alpha = \frac{(a; q)_\infty}{(q^\alpha a; q)_\infty}.$$

The *q*-factorial  $[n]_q!$ , serving as the *q*-analogue of the binomial coefficient factorial, is defined as

$$[n]_q! = \begin{cases} 1, & \text{if } n = 0, \\ [1]_q \times [2]_q \times \cdots \times [n]_q, & \text{if } n \in \mathbb{N}. \end{cases}$$

The *q*-gamma function  $\Gamma_q(x)$  is given by

$$\Gamma_q(x) = \frac{(q; q)_\infty}{(q^x; q)_\infty} (1 - q)^{1-x}, \quad x \in \mathbb{R} \setminus \{0, -1, -2, \dots\}$$

and satisfies the functional relation  $\Gamma_q(x + 1) = [x]_q \Gamma_q(x)$ .

*Definition 1.* [11] The  $q$ -analogue differential operator  $D_q f(x)$  is

$$D_q f(x) = \frac{f(x) - f(qx)}{x(1-q)},$$

and the  $q$ -derivatives  $D_q^n(f(x))$  of higher order are defined inductively as follows:

$$D_q^0(f(x)) = f(x), \quad D_q^n(f(x)) = D_q(D_q^{n-1}f(x)) \quad (n = 1, 2, 3, \dots),$$

where  $0 < q < 1$ . Be aware that  $\lim_{q \rightarrow 1} D_q f(x) = f'(x)$ .

$$\begin{aligned} D_{q,x}(x-s)_q^{(\gamma)} &= [\gamma]_q (x-s)_q^{(\gamma-1)}, \\ D_{q,s}(x-s)_q^{(\gamma)} &= -[\gamma]_q (x-qs)_q^{(\gamma-1)}. \end{aligned} \quad (1)$$

The  $q$ -integral (or Jackson integral)  $\int_a^b f(x) d_q x$  is defined by

$$\int_0^a f(x) d_q x := (1-q)a \sum_{m=0}^{\infty} q^m f(aq^m),$$

for  $a = 0$  and

$$\int_a^b f(x) d_q x = \int_0^b f(x) d_q x - \int_0^a f(x) d_q x,$$

for  $0 < a < b$ . For further details, see [8, 9].

*Definition 2.* [21] For  $\alpha > 0$ , and a function  $f$  defined on  $[a, b]$ , the fractional  $q$ -integral of Riemann–Liouville type is characterized by  $(I_{q,a}^0 f)(x) = f(x)$  and

$$(I_{q,a}^\alpha f)(x) = \frac{1}{\Gamma_q(\alpha)} \int_a^x (x-qt)_q^{(\alpha-1)} f(t) d_q t, \quad x \in [a, b].$$

*Definition 3.* [16]. Given  $\alpha, \beta > 0$ , the Riemann–Liouville fractional  $q$ -derivative is defined by setting  $(D_{q,a}^0 f)(x) = f(x)$  and

$$(D_{q,a}^\alpha f)(x) = \left( D_{q,a}^{[\alpha]} I_{q,a}^{[\alpha]-\alpha} f \right)(x),$$

where  $[\alpha]$  is the smallest integer greater than or equal to  $\alpha$ .

For  $\lambda \in (-1, \infty)$ , the following is valid [9]:

$$\left( D_{q,a}^\alpha (x-a)^\lambda \right)(x) = \frac{\Gamma_q(\lambda+1)}{\Gamma_q(\lambda-\alpha+1)} (x-a)^{\lambda-\alpha}. \quad (2)$$

The space  $L_q^p = L_q^p[a, b]$  corresponding to  $1 \leq p < \infty$  is defined by

$$L_q^p[a, b] := \left\{ f : \left( \int_a^b |f(x)|^p d_q x \right)^{\frac{1}{p}} < \infty \right\}.$$

Let  $0 < a < b < \infty$  and  $0 \leq \lambda \leq 1$ . Then we introduce the space  $C_{q,\lambda}[a, b]$  of functions  $f$  given on  $[a, b]$ , such that the functions with the norm

$$\|f\|_{C_{q,\lambda}[a,b]} := \max_{x \in [a,b]} \left| (x - qa)_q^{(\lambda)} f(x) \right| < \infty.$$

The collection of all  $q$ -absolutely continuous functions on  $[a, b]$  is denoted  $AC_q[a, b]$ . For  $n \in \mathbb{N} := 1, 2, 3, \dots$  we denote by  $AC_q^n[a, b]$  the space of real-valued functions  $f(x)$  which have  $q$ -derivatives up to order  $n - 1$  on  $[a, b]$  such that  $D_q^{n-1}f(x) \in AC_q[a, b]$ :

$$AC_q^n[a, b] := \{f : [a, b] \rightarrow \mathbb{R}; D_q^{n-1}f(x) \in AC_q[a, b]\}.$$

*Lemma 1.* [18] Assume  $\alpha > 0$ ,  $\beta > 0$ , and  $1 \leq p < \infty$ . The semigroup property for the  $q$ -fractional integral holds as follows:

1.  $(I_{q,a}^\beta I_{q,a}^\alpha f)(x) = (I_{q,a}^{\alpha+\beta} f)(x),$
2.  $(D_{q,a}^\alpha I_{q,a}^\alpha f)(x) = f(x),$
3.  $(D_{q,a}^\beta I_{q,a}^\alpha f)(x) = (I_{q,a}^{\alpha-\beta} f)(x),$

where  $f(x) \in L_q^p[a, b]$  for all  $x \in [a, b]$ .

*Lemma 2.* Suppose  $\alpha > 0$ ,  $p \in \mathbb{N}$ ,  $q \in (0, 1)$ , and let  $f \in AC_q^p[a, b]$  be a function with  $q$ -derivatives  $D_{q,a}^k f$  defined at  $x = a$  for  $k = 0, 1, \dots, p - 1$ . Following [19], the Riemann–Liouville  $q$ -fractional integral  $I_{q,a}^\alpha$  and derivative  $D_{q,a}^\alpha$  satisfy

$$(I_{q,a}^\alpha D_{q,a}^\alpha f)(x) = (D_{q,a}^\alpha I_{q,a}^\alpha f)(x) - \sum_{k=0}^{p-1} \frac{(x-a)^{\alpha-p+k}}{\Gamma_q(\alpha+k-p+1)} (D_{q,a}^k f)(a), \quad x \in [a, b].$$

*Lemma 3.* For  $\gamma > -1$ ,  $q \in (0, 1)$ ,  $a < b$ , and  $x \geq b$ , the  $q$ -integral of the  $q$ -power function is given by

$$\int_a^b (x - qs)_q^{(\gamma)} d_q s = \frac{(x-a)^{\gamma+1}}{[\gamma+1]_q}, \quad (3)$$

where  $(x - qs)_q^{(\gamma)} = (x - qs)^\gamma$  and  $[\gamma+1]_q = \frac{1-q^{\gamma+1}}{1-q}$ . See [9] for details.

## 2 Main Results

*Theorem 1.* Let  $1 < \alpha \leq 2$ ,  $0 \leq \beta \leq 1$ ,  $0 < \alpha - \beta < 1$ ,  $q \in (0, 1)$ , and  $h \in L_q^1[a, b]$ . The  $q$ -fractional boundary value problem

$$D_{q,a}^\alpha u(t) + h(t) = 0, \quad t \in [a, b], \quad (4)$$

with boundary conditions

$$u(a) = 0, \quad D_{q,a}^\beta u(b) = 0, \quad (5)$$

has a unique solution given by

$$u(t) = \int_a^b G_q(t, s) h(s) d_q s,$$

where the  $q$ -Green's function  $G_q(t, s)$  is defined as

$$G_q(t, s) = \frac{1}{\Gamma_q(\alpha)} \begin{cases} \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-\beta-1}} (b - qs)_q^{(\alpha-\beta-1)}, & a \leq t \leq s \leq b, \\ \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-\beta-1}} (b - qs)_q^{(\alpha-\beta-1)} - (t - qs)_q^{(\alpha-1)}, & a \leq s \leq t \leq b. \end{cases} \quad (6)$$

*Proof.* By applying the operator  $I_{q,a}^\alpha$  from definition 2 to both sides of (4) and employing Lemma 2 with  $p = 2$ , we obtain

$$u(t) = -I_{q,a}^\alpha h(t) + C_1(t-a)^{\alpha-1} + C_2(t-a)^{\alpha-2}, \quad (7)$$

for some  $C_1, C_2 \in \mathbb{R}$ . Applying the operator  $D_{q,a}^\beta$  in condition (5) to both parts of the equation (7) and using the Lemma 1, we obtain

$$\begin{aligned} D_{q,a}^\beta u(t) &= -D_{q,a}^\beta I_{q,a}^\alpha h(t) + C_1 D_{q,a}^\beta (t-a)^{\alpha-1} \\ &\quad + C_2 D_{q,a}^\beta (t-a)^{\alpha-2}, \end{aligned}$$

proceeding further, and using formula (2), we arrive at

$$\begin{aligned} D_{q,a}^\beta u(t) &= -I_{q,a}^{\alpha-\beta} h(t) + C_1 \frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha-\beta)} (t-a)^{\alpha-\beta-1} \\ &\quad + C_2 \frac{\Gamma_q(\alpha-1)}{\Gamma_q(\alpha-\beta-1)} (t-a)^{\alpha-\beta-2}. \end{aligned} \quad (8)$$

Using the boundary condition  $u(a) = 0$  in equation (7) gives  $C_2 = 0$ . Applying the condition  $D_{q,a}^\beta u(b) = 0$  to equation (8) then leads to

$$C_1 = \frac{1}{\Gamma_q(\alpha)(b-a)^{\alpha-\beta-1}} \int_a^b (b-qs)_q^{(\alpha-\beta-1)} h(s) d_qs.$$

Substituting the explicit expressions for  $C_1$  and  $C_2$  into equation (7), we obtain the unique solution of (4) as

$$\begin{aligned} u(t) &= -\frac{1}{\Gamma_q(\alpha)} \int_a^t (t-qs)_q^{(\alpha-1)} h(s) d_qs \\ &\quad + \frac{1}{\Gamma_q(\alpha)} \int_a^b \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-\beta-1}} (b-qs)_q^{(\alpha-\beta-1)} h(s) d_qs \\ &= \frac{1}{\Gamma_q(\alpha)} \int_a^t \left[ \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-\beta-1}} (b-qs)_q^{(\alpha-\beta-1)} - (t-qs)_q^{(\alpha-1)} \right] h(s) d_qs \\ &\quad + \frac{1}{\Gamma_q(\alpha)} \int_t^b \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-\beta-1}} (b-qs)_q^{(\alpha-\beta-1)} h(s) d_qs \\ &= \int_a^b G_q(t,s) h(s) d_qs. \end{aligned}$$

Hence, the result follows.

*Corollary 1.* Let  $1 < \alpha \leq 2$ ,  $0 \leq \beta \leq 1$ ,  $1 \leq \alpha - \beta < 2$ ,  $q \in (0, 1)$ , and  $h \in L_q^1[a, b]$ . The  $q$ -fractional boundary value problem

$$D_{q,a}^\alpha u(t) + h(t) = 0, \quad t \in [a, b],$$

with boundary conditions

$$u(a) = 0, \quad D_{q,a}^\beta u(b) = 0,$$

has a unique solution  $u \in AC_q^\alpha[a, b]$  given by

$$u(t) = \int_a^b G_q(t, s) h(s) d_qs,$$

where the  $q$ -Green's function  $G_q(t, s)$  is defined as

$$G_q(t, s) = \frac{1}{\Gamma_q(\alpha)} \begin{cases} \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-\beta-1}} (b-qs)_q^{(\alpha-\beta-1)}, & a \leq t \leq s \leq b, \\ \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-\beta-1}} (b-qs)_q^{(\alpha-\beta-1)} - (t-qs)_q^{(\alpha-1)}, & a \leq s \leq t \leq b. \end{cases} \quad (9)$$

*Proof.* The result follows from Theorem 1 by identical arguments for the case  $1 \leq \alpha - \beta < 2$ ; the details are omitted.

We proceed to demonstrate the nonnegativity of the  $q$ -Green's functions and establish upper bounds for both the functions and their  $q$ -integrals.

*Theorem 2.* Let  $1 < \alpha \leq 2$ ,  $0 \leq \beta \leq 1$ ,  $0 < \alpha - \beta < 1$ ,  $q \in (0, 1)$ , and let the  $q$ -Green's function  $G_q(t, s)$  be defined as in Theorem 1. Then,

$$G_q(t, s) \geq 0 \quad \text{for all } (t, s) \in [a, b] \times [a, b].$$

*Proof.* We analyze the  $q$ -Green's function  $G_q(t, s)$  defined in Theorem 1, considering its piecewise structure.

*Case 1:*  $a \leq t \leq s \leq b$ . Here,

$$G_q(t, s) = \frac{1}{\Gamma_q(\alpha)} \cdot \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-\beta-1}} \cdot (b-qs)_q^{(\alpha-\beta-1)}.$$

Since  $\Gamma_q(\alpha) > 0$ ,  $(b-a)^{\alpha-\beta-1} > 0$ ,  $(t-a)^{\alpha-1} \geq 0$  for  $t \geq a$ , and  $(b-qs)_q^{(\alpha-\beta-1)} \geq 0$  for  $s \leq b$  (as  $qs \leq s$ ,  $q \in (0, 1)$ , and  $0 < \alpha - \beta < 1$ ), it follows that  $G_q(t, s) \geq 0$ .

*Case 2:*  $a \leq s \leq t \leq b$ . In this case,

$$G_q(t, s) = \frac{1}{\Gamma_q(\alpha)} \left[ \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-\beta-1}} (b-qs)_q^{(\alpha-\beta-1)} - (t-qs)_q^{(\alpha-1)} \right].$$

Since  $s \leq t$ , the  $q$ -power function is monotonic, so  $t-qs \geq t-a$ , and thus  $(t-qs)_q^{(\alpha-1)} \leq (t-a)^{\alpha-1}$ . Additionally, as  $qs \leq s \leq t \leq b$ , we have  $b-qs \geq b-a$ , implying  $(b-qs)_q^{(\alpha-\beta-1)} \geq (b-a)^{\alpha-\beta-1}$ . Therefore,

$$\frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-\beta-1}} (b-qs)_q^{(\alpha-\beta-1)} \geq (t-a)^{\alpha-1} \geq (t-qs)_q^{(\alpha-1)}.$$

Hence,

$$G_q(t, s) \geq \frac{1}{\Gamma_q(\alpha)} \left[ (t-a)^{\alpha-1} - (t-qs)_q^{(\alpha-1)} \right] \geq 0.$$

Combining both cases, we conclude that  $G_q(t, s) \geq 0$  for all  $(t, s) \in [a, b] \times [a, b]$ .

*Remark 1.* The nonnegativity of the  $q$ -Green's function  $G_q(t, s)$ , established in Theorem 2, is crucial for the qualitative analysis of the  $q$ -fractional boundary value problem in Theorem 1. Specifically, it ensures that the solution

$$u(t) = \int_a^b G_q(t, s) h(s) d_qs, \quad h \in L_q^1[a, b],$$

preserves the sign of the source term  $h(s)$ . For instance, if  $h(s) \geq 0$  on  $[a, b]$ , then  $u(t) \geq 0$ ; similarly, if  $h(s) \leq 0$ , then  $u(t) \leq 0$ , for all  $t \in [a, b]$ .

*Corollary 2.* Let  $1 < \alpha \leq 2$ ,  $0 \leq \beta \leq 1$ ,  $1 \leq \alpha - \beta < 2$ ,  $q \in (0, 1)$ , and let the  $q$ -Green's function  $G_q(t, s)$  be defined as in Corollary 1 for  $a < b$ . Then,

$$G_q(t, s) \geq 0 \quad \text{for all } (t, s) \in [a, b] \times [a, b].$$

*Proof.* We analyze the piecewise definition of  $G_q(t, s)$  from Corollary 1.

*Case 1:*  $a \leq t \leq s \leq b$ . Here,

$$G_q(t, s) = \frac{1}{\Gamma_q(\alpha)} \cdot \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-\beta-1}} \cdot (b-qs)_q^{(\alpha-\beta-1)}.$$

Since  $\Gamma_q(\alpha) > 0$ ,  $(t-a)^{\alpha-1} \geq 0$ ,  $(b-a)^{\alpha-\beta-1} \geq 0$  (as  $\alpha - \beta - 1 \geq 0$ ), and  $(b-qs)_q^{(\alpha-\beta-1)} \geq 0$  (as  $qs \leq s \leq b$ ,  $q \in (0, 1)$ ), it follows that  $G_q(t, s) \geq 0$ .

*Case 2:*  $a \leq s \leq t \leq b$ . In this case,

$$G_q(t, s) = \frac{1}{\Gamma_q(\alpha)} \left[ \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-\beta-1}} (b-qs)_q^{(\alpha-\beta-1)} - (t-qs)_q^{(\alpha-1)} \right].$$

Since  $a \leq qs \leq s \leq t \leq b$ , we have  $b-qs \geq b-a$ , so  $(b-qs)_q^{(\alpha-\beta-1)} \geq (b-a)^{\alpha-\beta-1}$ . Also,  $qs \geq a$ , so  $t-qs \leq t-a$ , and the monotonicity of the  $q$ -power function [14] implies  $(t-qs)_q^{(\alpha-1)} \leq (t-a)^{\alpha-1}$ . Thus,

$$\frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-\beta-1}} (b-qs)_q^{(\alpha-\beta-1)} \geq (t-a)^{\alpha-1} \geq (t-qs)_q^{(\alpha-1)}.$$

Hence,

$$G_q(t, s) \geq \frac{1}{\Gamma_q(\alpha)} \left[ (t-a)^{\alpha-1} - (t-qs)_q^{(\alpha-1)} \right] \geq 0.$$

Thus,  $G_q(t, s) \geq 0$  for all  $(t, s) \in [a, b] \times [a, b]$ .

*Theorem 3.* Let  $1 < \alpha \leq 2$ ,  $0 \leq \beta \leq 1$ ,  $0 < \alpha - \beta < 1$ ,  $q \in (0, 1)$ ,  $a < b$ , and let the  $q$ -Green's function  $G_q(t, s)$  be defined as in (6). Then, for  $s \in [a, b]$ ,

$$\max_{t \in [a, b]} \frac{G_q(t, s)}{(b-qs)_q^{(\alpha-\beta-1)}} = \frac{G_q(s, s)}{(b-qs)_q^{(\alpha-\beta-1)}},$$

and

$$\max_{s \in [a, b]} \frac{G_q(s, s)}{(b-qs)_q^{(\alpha-\beta-1)}} = \frac{(b-a)^\beta}{\Gamma_q(\alpha)}.$$

*Proof.* We analyze the ratio  $\frac{G_q(t, s)}{(b-qs)_q^{(\alpha-\beta-1)}}$  for fixed  $s \in [a, b]$ . Since  $qs \leq s \leq b$ ,  $q \in (0, 1)$ , and  $0 < \alpha - \beta < 1$ , we have  $\alpha - \beta - 1 \in (-1, 0)$ , but  $(b-qs)_q^{(\alpha-\beta-1)} \geq 0$  as per [14].

*Case 1:*  $a \leq t \leq s \leq b$ . From (6),

$$\frac{G_q(t, s)}{(b-qs)_q^{(\alpha-\beta-1)}} = \frac{1}{\Gamma_q(\alpha)} \cdot \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-\beta-1}}.$$

Using the  $q$ -derivative (1),

$$D_{q,t}[(t-a)^{\alpha-1}] = [\alpha-1]_q (t-a)^{\alpha-2},$$

we obtain

$$D_{q,t} \left[ \frac{G_q(t, s)}{(b-qs)_q^{(\alpha-\beta-1)}} \right] = \frac{(t-a)^{\alpha-2}}{(b-a)^{\alpha-\beta-1} \Gamma_q(\alpha-1)} \geq 0,$$

since  $\alpha - 2 > -1$ . At  $t = a$ ,  $(t - a)^{\alpha-2}$  may be singular ( $\alpha - 2 \in (-1, 0]$ ), but the  $q$ -derivative is defined for  $t \in (a, s]$ . Thus, the ratio is non-decreasing on  $[a, s]$ .

Case 2:  $a \leq s \leq t \leq b$ . Here,

$$\frac{G_q(t, s)}{(b - qs)_q^{(\alpha-\beta-1)}} = \frac{1}{\Gamma_q(\alpha)} \left[ \frac{(t - a)^{\alpha-1}}{(b - a)^{\alpha-\beta-1}} - \frac{(t - qs)_q^{(\alpha-1)}}{(b - qs)_q^{(\alpha-\beta-1)}} \right].$$

Computing the  $q$ -derivative,

$$D_{q,t} \left[ \frac{G_q(t, s)}{(b - qs)_q^{(\alpha-\beta-1)}} \right] = \frac{1}{\Gamma_q(\alpha - 1)} \left[ \frac{(t - a)^{\alpha-2}}{(b - a)^{\alpha-\beta-1}} - \frac{(t - qs)_q^{(\alpha-2)}}{(b - qs)_q^{(\alpha-\beta-1)}} \right].$$

Since  $qs \leq s \leq b$ , we have  $b - qs \geq b - a$ , so  $(b - qs)_q^{(\alpha-\beta-1)} \geq (b - a)^{\alpha-\beta-1}$ . Also,  $qs \geq a$ , so  $t - qs \leq t - a$ , and the monotonicity of the  $q$ -power function [14] implies  $(t - qs)_q^{(\alpha-2)} \leq (t - a)^{\alpha-2}$ . Thus,

$$\frac{(t - a)^{\alpha-2}}{(b - a)^{\alpha-\beta-1}} \geq \frac{(t - qs)_q^{(\alpha-2)}}{(b - qs)_q^{(\alpha-\beta-1)}},$$

so

$$D_{q,t} \left[ \frac{G_q(t, s)}{(b - qs)_q^{(\alpha-\beta-1)}} \right] \leq 0.$$

Hence, the ratio is non-increasing on  $[s, b]$ . Combining both cases, the maximum occurs at  $t = s$ , where

$$\frac{G_q(s, s)}{(b - qs)_q^{(\alpha-\beta-1)}} = \frac{1}{\Gamma_q(\alpha)} \cdot \frac{(s - a)^{\alpha-1}}{(b - a)^{\alpha-\beta-1}}.$$

For the second part, consider

$$\frac{G_q(s, s)}{(b - qs)_q^{(\alpha-\beta-1)}} = \frac{(s - a)^{\alpha-1}}{(b - a)^{\alpha-\beta-1} \Gamma_q(\alpha)}.$$

Since  $(s - a)^{\alpha-1}$  is increasing on  $[a, b]$  ( $\alpha - 1 > 0$ ), the maximum occurs at  $s = b$ , yielding

$$\frac{(b - a)^{\alpha-1}}{(b - a)^{\alpha-\beta-1} \Gamma_q(\alpha)} = \frac{(b - a)^\beta}{\Gamma_q(\alpha)}.$$

This completes the proof.

*Corollary 3.* Let  $1 < \alpha \leq 2$ ,  $0 \leq \beta \leq 1$ ,  $1 \leq \alpha - \beta < 2$ ,  $q \in (0, 1)$ ,  $a < b$ , and let the  $q$ -Green's function  $G_q(t, s)$  be defined as in (9). Then, for  $s \in [a, b]$ ,

$$\max_{t \in [a, b]} G_q(t, s) = G_q(s, s),$$

and

$$\max_{s \in [a, b]} G_q(s, s) = \frac{(b - a)^\beta b^{\alpha-\beta-1} (1 - q)^{\alpha-\beta-1}}{\Gamma_q(\alpha)}.$$

*Proof.* The statement follows from Theorem 3 by identical arguments applied to the range  $1 \leq \alpha - \beta < 2$ ; the details are omitted.



*Corollary 4.* Let  $1 < \alpha \leq 2$ ,  $0 \leq \beta \leq 1$ ,  $1 \leq \alpha - \beta < 2$ ,  $q \in (0, 1)$ ,  $a < b$ , and let the  $q$ -Green's function  $G_q(t, s)$  be defined as in (6) and (9). Then:

$$\max_{t \in [a, b]} \int_a^b G_q(t, s) d_qs = \frac{[\alpha - 1]_q^{\alpha-1}}{\Gamma_q(\alpha + 1)} \left( \frac{b - a}{[\alpha - \beta]_q} \right)^\alpha.$$

*Proof.* Consider the integral  $I(t) = \int_a^b G_q(t, s) d_qs$ , where  $G_q(t, s)$  is defined in (6) and (9). Split the integral based on the definition of  $G_q(t, s)$ :

*Case 1:*  $a \leq t \leq s \leq b$ .

$$G_q(t, s) = \frac{1}{\Gamma_q(\alpha)} \cdot \frac{(t - a)^{\alpha-1}}{(b - a)^{\alpha-\beta-1}} (b - qs)_q^{(\alpha-\beta-1)}.$$

*Case 2:*  $a \leq s \leq t \leq b$ .

$$G_q(t, s) = \frac{1}{\Gamma_q(\alpha)} \left[ \frac{(t - a)^{\alpha-1}}{(b - a)^{\alpha-\beta-1}} (b - qs)_q^{(\alpha-\beta-1)} - (t - qs)_q^{(\alpha-1)} \right].$$

Thus,

$$I(t) = \int_a^t G_q(t, s) d_qs + \int_t^b G_q(t, s) d_qs.$$

Substitute the expression for  $G_q(t, s)$ :

$$\begin{aligned} I(t) &= \int_a^t \frac{1}{\Gamma_q(\alpha)} \left[ \frac{(t - a)^{\alpha-1}}{(b - a)^{\alpha-\beta-1}} (b - qs)_q^{(\alpha-\beta-1)} - (t - qs)_q^{(\alpha-1)} \right] d_qs \\ &\quad + \int_t^b \frac{1}{\Gamma_q(\alpha)} \cdot \frac{(t - a)^{\alpha-1}}{(b - a)^{\alpha-\beta-1}} (b - qs)_q^{(\alpha-\beta-1)} d_qs \\ &= \frac{(t - a)^{\alpha-1}}{\Gamma_q(\alpha)(b - a)^{\alpha-\beta-1}} \int_a^b (b - qs)_q^{(\alpha-\beta-1)} d_qs - \frac{1}{\Gamma_q(\alpha)} \int_a^t (t - qs)_q^{(\alpha-1)} d_qs. \end{aligned}$$

Using equation (3), under the conditions  $x = b$  or  $x = t \geq s$ , we have

$$\int_a^b (b - qs)_q^{(\alpha-\beta-1)} d_qs = \frac{(b - a)^{\alpha-\beta}}{[\alpha - \beta]_q}, \quad \int_a^t (t - qs)_q^{(\alpha-1)} d_qs = \frac{(t - a)^\alpha}{[\alpha]_q},$$

we get

$$\begin{aligned} I(t) &= \frac{(t - a)^{\alpha-1}(b - a)^{\alpha-\beta}}{\Gamma_q(\alpha)(b - a)^{\alpha-\beta-1}[\alpha - \beta]_q} - \frac{(t - a)^\alpha}{\Gamma_q(\alpha)[\alpha]_q} \\ &= \frac{(t - a)^{\alpha-1}(b - a)}{\Gamma_q(\alpha)[\alpha - \beta]_q} - \frac{(t - a)^\alpha}{\Gamma_q(\alpha)[\alpha]_q} \\ &= \frac{(t - a)^{\alpha-1}}{\Gamma_q(\alpha)} \left( \frac{b - a}{[\alpha - \beta]_q} - \frac{t - a}{[\alpha]_q} \right). \end{aligned}$$

To find the maximum, compute the  $q$ -derivative:

$$\begin{aligned} D_{q,t}I(t) &= \frac{1}{\Gamma_q(\alpha)} \left[ [\alpha - 1]_q (t - a)^{\alpha-2} \left( \frac{b-a}{[\alpha - \beta]_q} - \frac{t-a}{[\alpha]_q} \right) - (t-a)^{\alpha-1} \cdot \frac{1}{[\alpha]_q} \right] \\ &= \frac{1}{\Gamma_q(\alpha)} \left[ \frac{[\alpha - 1]_q (t-a)^{\alpha-2} (b-a)}{[\alpha - \beta]_q} - \frac{(t-a)^{\alpha-1} ([\alpha - 1]_q + 1)}{[\alpha]_q} \right] \\ &= \frac{1}{\Gamma_q(\alpha)} \left[ \frac{[\alpha - 1]_q (t-a)^{\alpha-2} (b-a)}{[\alpha - \beta]_q} - (t-a)^{\alpha-1} \right], \end{aligned}$$

where  $[\alpha - 1]_q + 1 = \frac{1-q^{\alpha-1}}{1-q} + 1 = \frac{1-q^\alpha}{1-q} = [\alpha]_q$ .

Set  $D_{q,t}I(t) = 0$ :

$$t^* = a + \frac{[\alpha - 1]_q (b-a)}{[\alpha - \beta]_q}.$$

Substitute  $t^*$  into the expression for  $I(t)$ :

$$\begin{aligned} I(t^*) &= \frac{\left( \frac{[\alpha-1]_q(b-a)}{[\alpha-\beta]_q} \right)^{\alpha-1}}{\Gamma_q(\alpha)} \left( \frac{b-a}{[\alpha-\beta]_q} - \frac{\frac{[\alpha-1]_q(b-a)}{[\alpha-\beta]_q}}{[\alpha]_q} \right) \\ &= \frac{\left( \frac{[\alpha-1]_q(b-a)}{[\alpha-\beta]_q} \right)^{\alpha-1}}{\Gamma_q(\alpha)} \left( \frac{b-a}{[\alpha-\beta]_q} \left( 1 - \frac{[\alpha-1]_q}{[\alpha]_q} \right) \right) \\ &= \frac{[\alpha-1]_q^{\alpha-1} (b-a)^{\alpha-1}}{\Gamma_q(\alpha) [\alpha-\beta]_q^{\alpha-1}} \cdot \frac{b-a}{[\alpha-\beta]_q} \cdot \frac{q^{\alpha-1}}{[\alpha]_q} \\ &= \frac{[\alpha-1]_q^{\alpha-1} (b-a)^\alpha q^{\alpha-1}}{\Gamma_q(\alpha) [\alpha-\beta]_q^\alpha [\alpha]_q}. \end{aligned}$$

The function  $I(t)$  is increasing for  $t < t^*$  ( $D_{q,t}I(t) > 0$ ) and decreasing for  $t > t^*$  ( $D_{q,t}I(t) < 0$ ), confirming the maximum at  $t^*$ .

*Theorem 4.* Let  $\mathfrak{B}_q = C_{q,\lambda}[a, b]$  denote the Banach space of functions continuous in the  $q$ -sense on the interval  $[a, b]$ , with norm

$$\|u\|_{C_{q,\lambda}} = \max_{t \in [a,b]} |u(t)|,$$

where  $[a, b] = \{a, aq, aq^2, \dots, aq^n = b\}$ . Given  $1 < \alpha \leq 2$ ,  $0 \leq \beta \leq 1$ ,  $0 < \alpha - \beta < 1$ , if the fractional  $q$ -difference boundary value problem

$$\begin{cases} D_{q,a}^\alpha u(t) + q(t)u(t) = 0, & t \in [a, b], \\ u(a) = 0, \quad D_{q,a}^\beta u(b) = 0, \end{cases} \quad (10)$$

admits a nontrivial solution  $u \in \mathfrak{B}_q$ , then the following Lyapunov-type inequality holds:

$$\int_a^b (b - qs)_q^{(\alpha-\beta-1)} |q(s)| d_qs > \frac{\Gamma_q(\alpha)}{(b-a)^\beta}. \quad (11)$$

*Proof.* Any solution  $u \in \mathfrak{B}_q$  of the boundary value problem (10) satisfies

$$u(t) = \int_a^b G_q(t, s) q(s) u(s) d_qs,$$

where  $G_q(t, s)$  is the  $q$ -Green's function given by (6).

By applying the  $C_{q,\lambda}$ -norm, we obtain

$$\begin{aligned}\|u\|_{C_{q,\lambda}} &= \max_{t \in [a,b]} \left| \int_a^b G_q(t, s) q(s) u(s) d_qs \right| \\ &\leq \max_{t \in [a,b]} \int_a^b |G_q(t, s)| |q(s)| |u(s)| d_qs \\ &\leq \|u\|_{C_{q,\lambda}} \cdot \max_{t \in [a,b]} \int_a^b |G_q(t, s)| |q(s)| d_qs.\end{aligned}$$

For a nontrivial solution ( $\|u\|_{C_{q,\lambda}} \neq 0$ ), this implies

$$1 \leq \max_{t \in [a,b]} \int_a^b |G_q(t, s)| |q(s)| d_qs.$$

By Theorem 3, the  $q$ -Green's function satisfies the bound

$$|G_q(t, s)| \leq \frac{(b-a)^\beta (b-qs)_q^{(\alpha-\beta-1)}}{\Gamma_q(\alpha)}.$$

Substituting this bound, we get

$$1 < \max_{t \in [a,b]} \int_a^b |G_q(t, s)| |q(s)| d_qs \leq \frac{(b-a)^\beta}{\Gamma_q(\alpha)} \int_a^b (b-qs)_q^{(\alpha-\beta-1)} |q(s)| d_qs.$$

Therefore, dividing both sides by  $\frac{(b-a)^\beta}{\Gamma_q(\alpha)}$ , we obtain (11).

This completes the proof.

*Corollary 5.* Let  $1 < \alpha \leq 2$ ,  $0 \leq \beta \leq 1$ , and  $1 \leq \alpha - \beta < 2$ . Suppose the fractional  $q$ -difference boundary-value problem

$$\begin{cases} D_{q,a}^\alpha u(t) + q(t)u(t) = 0, & t \in [a, b], \\ u(a) = 0, & D_{q,a}^\beta u(b) = 0, \end{cases}$$

admits a nontrivial solution  $u \in \mathfrak{B}_q = C_{q,\lambda}[a, b]$ , where  $C_{q,\lambda}[a, b]$  is the space of  $q$ -continuous functions on the  $q$ -interval  $[a, b]$  with  $0 < q < 1$ . Then the following Lyapunov-type inequality holds:

$$\int_a^b |q(s)| d_qs > \frac{\Gamma_q(\alpha)}{(b-a)^\beta b^{\alpha-\beta-1} (1-q)^{\alpha-\beta-1}}.$$

*Proof.* By Corollary 1, any solution  $u \in C_{q,\lambda}[a, b]$  to the boundary-value problem satisfies:

$$u(t) = \int_a^b G_q(t, s) q(s) u(s) d_qs,$$

where  $G_q(t, s)$  is the  $q$ -Green's function defined in (9).

Define the norm  $\|u\|_{C_{q,\lambda}} = \sup_{t \in [a,b]} |u(t)|$ . From the solution representation:

$$|u(t)| \leq \int_a^b |G_q(t, s)| |q(s)| |u(s)| d_qs \leq \|u\|_{C_{q,\lambda}} \int_a^b |G_q(t, s)| |q(s)| d_qs.$$

Taking the supremum over  $t \in [a, b]$ , we obtain

$$\|u\|_{C_{q,\lambda}} \leq \|u\|_{C_{q,\lambda}} \max_{t \in [a,b]} \int_a^b |G_q(t, s)| |q(s)| d_qs.$$

For a nontrivial solution ( $\|u\|_{C_{q,\lambda}} > 0$ ), it follows that

$$1 \leq \max_{t \in [a,b]} \int_a^b |G_q(t, s)| |q(s)| d_qs.$$

By Corollary 2,  $G_q(t, s)$  is non-negative, so  $|G_q(t, s)| = G_q(t, s)$ . By Corollary 3, the maximum of the Green's function is

$$\max_{t,s \in [a,b]} G_q(t, s) = \max_{s \in [a,b]} G_q(s, s) = \frac{(b-a)^\beta b^{\alpha-\beta-1} (1-q)^{\alpha-\beta-1}}{\Gamma_q(\alpha)}.$$

Thus,  $G_q(t, s) \leq \max_{s \in [a,b]} G_q(s, s)$ , and

$$\int_a^b G_q(t, s) |q(s)| d_qs \leq \frac{(b-a)^\beta b^{\alpha-\beta-1} (1-q)^{\alpha-\beta-1}}{\Gamma_q(\alpha)} \int_a^b |q(s)| d_qs.$$

Combining with the previous inequality, we get

$$1 \leq \frac{(b-a)^\beta b^{\alpha-\beta-1} (1-q)^{\alpha-\beta-1}}{\Gamma_q(\alpha)} \int_a^b |q(s)| d_qs.$$

Rearranging yields

$$\int_a^b |q(s)| d_qs \geq \frac{\Gamma_q(\alpha)}{(b-a)^\beta b^{\alpha-\beta-1} (1-q)^{\alpha-\beta-1}}.$$

To establish the strict inequality, suppose equality holds

$$\int_a^b |q(s)| d_qs = \frac{\Gamma_q(\alpha)}{(b-a)^\beta b^{\alpha-\beta-1} (1-q)^{\alpha-\beta-1}}.$$

This implies  $G_q(t, s) = \max_{s \in [a,b]} G_q(s, s)$  for all  $t, s \in [a, b]$  where  $q(s)u(s) \neq 0$ . By Corollary 3,  $G_q(t, s) = G_q(s, s)$  only when  $t = s$ , which has measure zero in the  $q$ -integral unless  $u \equiv 0$ . Since  $u$  is nontrivial, equality is impossible, so

$$\int_a^b |q(s)| d_qs > \frac{\Gamma_q(\alpha)}{(b-a)^\beta b^{\alpha-\beta-1} (1-q)^{\alpha-\beta-1}}.$$

### 3 Applications

In this section, we investigate two applications of Theorem 4 and Corollary 5. First, we establish lower bounds for the eigenvalues of the Riemann–Liouville type fractional  $q$ -eigenvalue problems associated with (10). Second, we utilize these findings to identify intervals where the  $q$ -analogue of the two-parameter Mittag-Leffler function has no real zeros.

*Theorem 5.* Let  $1 < \alpha \leq 2$ ,  $0 \leq \beta \leq 1$  such that  $0 < \alpha - \beta < 1$ . Assume that  $y$  is a nontrivial solution of the Riemann–Liouville type fractional  $q$ -eigenvalue problem

$$\begin{cases} D_{q,a}^\alpha u(t) + \lambda u(t) = 0, & t \in [a, b], \\ u(a) = 0, \quad D_{q,a}^\beta u(b) = 0, \end{cases} \quad (12)$$

where  $u(t) \neq 0$  for each  $t \in (a, b)$ . Then,

$$|\lambda| > \frac{[\alpha - \beta]_q \Gamma_q(\alpha)}{(b - a)^\alpha}.$$

*Corollary 6.* Let  $1 < \alpha \leq 2$ ,  $0 \leq \beta \leq 1$  such that  $1 \leq \alpha - \beta < 2$ . Assume that  $u$  is a nontrivial solution of the Riemann–Liouville type fractional  $q$ -eigenvalue problem (12), where  $u(t) \neq 0$  for each  $t \in (a, b)$ . Then,

$$|\lambda| > \frac{\Gamma_q(\alpha)}{(b - a)^\beta b^{\alpha - \beta - 1} (1 - q)^{\alpha - \beta - 1}}.$$

Consider the  $q$ -analogue of the two-parameter Mittag-Leffler function, defined as ([14]):

$$E_{q,\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma_q(k\alpha + \beta)}, \quad z, \beta \in \mathbb{C}, \quad \Re(\alpha) > 0, \quad 0 < q < 1. \quad (13)$$

We use Theorem 5 and Corollary 6 to determine intervals where the function (13) has no real zeros.

*Theorem 6.* Let  $1 < \alpha \leq 2$ ,  $0 \leq \beta \leq 1$ ,  $0 < \alpha - \beta < 1$ ,  $q \in (0, 1)$ . The  $q$ -Mittag-Leffler function

$$E_{q,\alpha,\alpha-\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma_q(k\alpha + \alpha - \beta)},$$

has no real zeros for

$$|z| \leq \frac{[\alpha - \beta]_q \Gamma_q(\alpha)}{(b - a)^\alpha}, \quad (14)$$

where  $[\alpha - \beta]_q = \frac{1 - q^{\alpha - \beta}}{1 - q}$ .

*Proof.* Consider the  $q$ -fractional eigenvalue problem

$$\begin{cases} D_{q,a}^\alpha u(t) + \lambda u(t) = 0, & t \in [a, b], \\ u(a) = 0, \quad D_{q,a}^\beta u(b) = 0, \end{cases}$$

where  $D_{q,a}^\alpha$  is the Riemann–Liouville  $q$ -fractional derivative. The general solution is

$$u(t) = c_1(t - a)^{\alpha-1} E_{q,\alpha,\alpha}(-\lambda(t - a)^\alpha) + c_2(t - a)^{\alpha-2} E_{q,\alpha,\alpha-1}(-\lambda(t - a)^\alpha).$$

Let  $g(t) = (t - a)^{\alpha-1} E_{q,\alpha,\alpha}(-\lambda(t - a)^\alpha)$ . Compute

$$D_{q,a}^\alpha g(t) = D_{q,a}^\alpha \left( \sum_{n=0}^{\infty} \frac{(-\lambda)^n (t - a)^{\alpha n + \alpha - 1}}{\Gamma_q(\alpha n + \alpha)} \right) = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{\Gamma_q(\alpha n + \alpha)} D_{q,a}^\alpha (t - a)^{\alpha n + \alpha - 1}.$$

Since  $D_{q,a}^\alpha(t-a)^{\alpha n+\alpha-1} = \frac{\Gamma_q(\alpha n+\alpha)}{\Gamma_q(\alpha n)}(t-a)^{\alpha n-1}$ , we get

$$D_{q,a}^\alpha g(t) = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{\Gamma_q(\alpha n)}(t-a)^{\alpha n-1} = -\lambda g(t).$$

The condition  $u(a) = 0$  implies  $c_2 = 0$ , since  $(t-a)^{\alpha-2} \rightarrow \infty$  as  $t \rightarrow a$ . Thus,

$$u(t) = c_1(t-a)^{\alpha-1}E_{q,\alpha,\alpha}(-\lambda(t-a)^\alpha).$$

Compute

$$D_{q,a}^\beta u(t) = c_1 \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{\Gamma_q(\alpha n + \alpha)} D_{q,a}^\beta (t-a)^{\alpha n+\alpha-1}.$$

Since  $D_{q,a}^\beta(t-a)^{\alpha n+\alpha-1} = \frac{\Gamma_q(\alpha n+\alpha)}{\Gamma_q(\alpha n+\alpha-\beta)}(t-a)^{\alpha n+\alpha-\beta-1}$ , we obtain

$$D_{q,a}^\beta u(t) = c_1(t-a)^{\alpha-\beta-1}E_{q,\alpha,\alpha-\beta}(-\lambda(t-a)^\alpha).$$

The condition  $D_{q,a}^\beta u(b) = 0$  gives

$$c_1(b-a)^{\alpha-\beta-1}E_{q,\alpha,\alpha-\beta}(-\lambda(b-a)^\alpha) = 0 \implies E_{q,\alpha,\alpha-\beta}(-\lambda(b-a)^\alpha) = 0.$$

By Theorem 5, for a nontrivial solution  $u \in \mathfrak{B}_q = C_{q,\lambda}[a, b]$ ,

$$|\lambda| > \frac{[\alpha - \beta]_q \Gamma_q(\alpha)}{(b-a)^\alpha}.$$

For  $z = -\lambda(b-a)^\alpha$ , we have

$$|z| = |\lambda|(b-a)^\alpha > [\alpha - \beta]_q \Gamma_q(\alpha).$$

Thus,  $E_{q,\alpha,\alpha-\beta}(z) \neq 0$  for (14).

*Corollary 7.* Let  $1 < \alpha \leq 2$ ,  $0 \leq \beta \leq 1$  such that  $1 \leq \alpha - \beta < 2$ . The  $q$ -Mittag-Leffler function  $E_{q,\alpha,\beta}(z)$  has no real zeros for

$$|z| \leq \frac{\Gamma_q(\alpha)}{(b-a)^\alpha}.$$

*Proof.* Following the same reasoning as in Theorem 6, suppose  $E_{q,\alpha,\beta}(\lambda) = 0$  for some real  $\lambda$ . The function  $u(t) = E_{q,\alpha,\beta}(-\lambda(t-a)^\alpha)$  satisfies the  $q$ -eigenvalue problem (12). By Corollary 6, any eigenvalue  $\lambda$  must satisfy:

$$|\lambda| > \frac{\Gamma_q(\alpha)}{(b-a)^\alpha}.$$

Hence,  $E_{q,\alpha,\beta}(z) \neq 0$ .

### Conclusion

In this study, we derived two novel Lyapunov-type inequalities for boundary value problems involving the Riemann–Liouville fractional  $q$ -derivative within the regimes  $0 < \alpha - \beta < 1$  and  $1 \leq \alpha - \beta < 2$ , thereby establishing precise estimates for eigenvalues and intervals free of zeros for  $q$ -Mittag-Leffler functions. By employing an analysis of the  $q$ -Green's function, we determined lower bounds for the eigenvalues of the problem  $D_{q,a}^\alpha u + \lambda u = 0$  and identified regions devoid of real zeros for  $q$ -analogues of Mittag-Leffler functions, which holds significant importance for discrete systems with memory, such as viscoelastic lattices and quantum circuits. This work extends classical inequalities to the realm of  $q$ -calculus, thereby bridging continuous and discrete fractional analysis, and paves the way for further research on Caputo  $q$ -fractional derivatives and multidimensional  $q$ -lattices.

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### Author Contributions

All authors contributed equally to this work.

### Conflict of Interest

The authors declare no conflict of interest.

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