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On M-term approximations of the Nikol'skii - Besov class

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Abstract

In this paper, we consider a Lebesgue space with a mixed norm of periodic functions of many variables. We obtain the exact estimation of the best M-term approximations of Nikol'skii's and Besov's classes in the Lebesgue space with the mixed norm.

Keywords: Lebesgue space, Nikol'skii - Besov class, approximation.

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1. Introduction

Let $\overline{x} = (x_1, ..., x_m) \in \mathbb{T}^m = [0, 2\pi)^m$ and $p_j \in [1, +\infty)$, $j = 1, ..., m$. $L_{\bar{p}}(\mathbb{T}^m)$ denotes the space of Lebesgue measureable functions $f(\bar{x})$ defined on $\mathbb{R}^m$, which have 2π period with respect to each variable such that

$$\|f\|_{\bar{p}} = \left[\int_0^{2\pi} \left[\cdots \left[\int_0^{2\pi} |f(\bar{x})|^{p_1} dx_1\right]^{\frac{p_2}{p_1}} \cdots \right]^{\frac{p_m}{p_{m-1}}} dx_m\right]^{\frac{1}{p_m}} < +\infty,$$

where $\overline{p} = (p_1, ..., p_m)$, $1 \le p_j < +\infty$, $j = 1, ..., m$ (see [18], p. 128, [4], p. 54). In the case $p_1 = ... = p_m = p$, we write $L_p(\mathbb{T}^m)$.

Any function $f \in L_1(\mathbb{T}^m) = L(\mathbb{T}^m)$ can be expanded to the Fourier series

$$\sum_{\overline{n} \in \mathbb{Z}^m} a_{\overline{n}}(f)\, e^{i\langle \overline{n}, \overline{x}\rangle},$$

where $\{a_{\overline{n}}(f)\}$ are Fourier coefficients of a function $f \in L_1(\mathbb{T}^m)$ with respect to a multiple trigonometric system $\{e^{i\langle \overline{n}, \overline{x}\rangle}\}_{\bar{n} \in \mathbb{Z}^m}$ and $\mathbb{Z}^m$ is the space of points in $\mathbb{R}^m$ with integer coordinates.

For a function $f \in L(\mathbb{T}^m)$ and a number $s \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, let us introduce the notation

$$\delta_0(f, \bar{x}) = a_0(f)$$

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and
$$\delta_s(f,\overline{x}) = \sum_{\overline{n}\in\rho(s)} a_{\overline{n}}(f) e^{i\langle\overline{n},\overline{x}\rangle},$$
where $\langle \bar{y}, \bar{x}\rangle = \sum\limits_{j=1}^{m} y_j x_j$ and
$$\rho(s) = \left\{\overline{k} = (k_1, ..., k_m) \in \mathbb{Z}^m : \quad [2^{s-1}] \le \max_{j=1,...,m} |k_j| < 2^s\right\},$$
where $[a]$ is the integer part of the number a.

Let us consider Nikol'skii's and Besov's classes ([4, 7, 18]). Let $1 < p_j < +\infty$, $j = 1, ..., m$, $1 \le \theta \le \infty$, $r > 0$, and
$$H^r_{\overline{p}} = \left\{f \in L_{\overline{p}}(\mathbb{T}^m) : \sup_{s\in\mathbb{Z}_+} 2^{sr} \|\delta_s(f)\|_{\overline{p}} \le 1\right\},$$
$$B^r_{\overline{p},\theta} = \left\{f \in L_{\overline{p}}(\mathbb{T}^m) : \Big(\sum_{s\in\mathbb{Z}_+} 2^{sr\theta} \|\delta_s(f)\|^{\theta}_{\overline{p}}\Big)^{\frac{1}{\theta}} \le 1\right\}.$$
It is known that for $1 \le \theta \le \theta_1 \le \infty$ the following holds
$$B^r_{\bar{p},1} \subset B^r_{\bar{p},\theta} \subset B^r_{\bar{p},\theta_1} \subset B^r_{\bar{p},\infty} = H^r_{\bar{p}}.$$

Let $f \in L_{\bar{p}}(\mathbb{T}^m)$ and $\left\{\bar{k}^{(j)}\right\}_{j=1}^{M}$ be a system of vectors $\bar{k}^{(j)} = (k_1^{(j)}, ..., k_m^{(j)})$ with integer coordinates. Consider the quantity
$$e_M(f)_{\bar{p}} = \inf_{\bar{k}^{(j)},b_j} \left\| f - \sum_{j=1}^{M} b_j e^{i\langle\bar{k}^{(j)},\bar{x}\rangle}\right\|_{\bar{p}},$$
where b_j is an arbitrary number. The quantity $e_M(f)_{\bar{p}}$ is called the best M-term approximation of a function $f \in L_{\bar{p}}(\mathbb{T}^m)$. For a given class $F \subset L_{\bar{p}}(\mathbb{T}^m)$ let
$$e_M(F)_{\bar{p}} = \sup_{f\in F} e_M(f)_{\bar{p}}.$$
The best M-term approximation was defined by S.B. Stechkin [22]. Estimations of M-term approximations of different classes were provided by R.S. Ismagilov [13], E.S. Belinsky [6], V.E. Maiorov [17], B.S. Kashin [14], R. DeVore [8], V.N. Temlyakov [23], A.S. Romanyuk [19], Dinh Dung [10], D.B. Bazarkhanov [5], L. Duan [11], M. Hansen and W. Sickel [12], S.A. Stasyuk [20, 21], and others (see bibliography in [1], [2], [8], [21], [23]).

For the case $p_1 = ... = p_m = p$ and $q_1 = ... = q_m = q$, R.A. De Vore and V.N. Temlyakov [9] proved the following theorem.

1.1. Theorem. *(see [9]). Let $1 \le p, q, \theta \le \infty$, $r(p,q) = m\left(\frac{1}{p} - \frac{1}{q}\right)_+$ if $1 \le p \le q \le 2$ or $1 \le q \le p < \infty$ and $r(p,q) = \max\left\{\frac{m}{p}, \frac{m}{2}\right\}$ in other cases. Then, for $r > r(p,q)$, the following relation holds*
$$e_M(B^r_{p,\theta})_q \asymp M^{-\frac{r}{m}+\left(\frac{1}{p}-\max\left\{\frac{1}{q},\frac{1}{2}\right\}\right)_+},$$
where $a_+ = \max\{a; 0\}$.

Moreover, in the case of $m\left(\frac{1}{p} - \frac{1}{q}\right) < r < \frac{m}{p}$ and $1 < p \le 2 < q < \infty$, S.A. Stasyuk [20, 21] proved that $e_M(B^r_{p,\theta})_q \asymp M^{-\frac{q}{2}\left(\frac{r}{m}-\left(\frac{1}{p}-\frac{1}{q}\right)\right)}$.

The main goal of the present paper is to find the order of the quantity $e_M(F)_{\bar{q}}$ for the class $F = B^r_{\bar{p},\theta}$.

Let us denote by $C(p,q,r,y)$ positive quantities, which depend on the parameters in the parentheses, such that the parameters, in general, are distinct in distinct formulas. $A(y) \asymp B(y)$ means that there are positive numbers C_1 and C_2 such that $C_1 \cdot A(y) \leq B(y) \leq C_2 \cdot A(y)$.

To prove the main results, we need the following auxiliary results.

1.2. Theorem. *(see [24]). Let* $\bar{n} = (n_1, ..., n_m)$, $n_j \in \mathbb{N}$, $j = 1, ..., m$, *and*

$$T_{\bar{n}}(\bar{x}) = \sum_{|k_j| \leq n_j, j=1,\dots,m} c_{\bar{k}} e^{i\langle \bar{k}, \bar{x}\rangle}.$$

Then, for $1 \leq p_j < q_j \leq \infty$, $j = 1, ...m$, *the following inequality holds*

$$\|T_{\bar{n}}\|_{\bar{q}} \leq 2^m \prod_{j=1}^{m} n_j^{\frac{1}{p_j} - \frac{1}{q_j}} \|T_{\bar{n}}\|_{\bar{p}}.$$

1.3. Theorem. *(see [16]). Let* $p \in (1, \infty)$. *Then there exist positive constants* $C_1(p)$ *and* $C_2(p)$ *such that for each function* $f \in L_p(\mathbb{T}^m)$ *the following estimation is valid*

$$C_1(p)\|f\|_p \leq \left\| \Big(\sum_{s=0}^{\infty} |\delta_s(f)|^2 \Big)^{\frac{1}{2}} \right\|_p \leq C_2(p)\|f\|_p.$$

Let Ω_M be a set containing no more than M vectors $\bar{k} = (k_1, ..., k_m)$ with integer coordinates and $P(\Omega_M, \bar{x})$ be any trigonometric polynomial, which consists of harmonics with “indices“ in Ω_M.

1.4. Lemma. *(see [2]). Let* $2 < q_j < +\infty$ *and* $j = 1, ..., m$. *Then, for any trigonometric polynomial* $P(\Omega_N)$ *and for any natural number* $M < N$, *there exists a trigonometric polynomial* $P(\Omega_M)$ *such that the following estimation holds*

$$\|P(\Omega_N) - P(\Omega_M)\|_{\bar{q}} \leq C_1 (NM^{-1})^{\frac{1}{2}} \|P(\Omega_N)\|_2,$$

and, moreover, $\Omega_M \subset \Omega_N$.

2. Main results

Let us prove the main results.

2.1. Theorem. *Let* $\bar{p} = (p_1, ..., p_m)$, $\bar{q} = (q_1, ..., q_m)$, $1 < p_j \leq 2 < q_j < \infty$, *and* $1 \leq \theta \leq \infty$.

1. If $\sum_{j=1}^{m} \left(\frac{1}{p_j} - \frac{1}{q_j}\right) < r < \sum_{j=1}^{m} \frac{1}{p_j}$, *then*

$$e_M \left(B^r_{\bar{p},\theta}\right)_{\bar{q}} \asymp M^{-\left(2\sum_{j=1}^{m} \frac{1}{q_j}\right)^{-1} \left(r - \sum_{j=1}^{m}\left(\frac{1}{p_j} - \frac{1}{q_j}\right)\right)}.$$

2. If $r = \sum_{j=1}^{m} \frac{1}{p_j}$, *then*

$$e_M \left(B^r_{\bar{p},\theta}\right)_{\bar{q}} \asymp M^{-\frac{1}{2}} (\log(1+M))^{1-\frac{1}{\theta}}.$$

3. If $r > \sum_{j=1}^{m} \frac{1}{p_j}$, *then*

$$e_M \left(B^r_{\bar{p},\theta}\right)_{\bar{q}} \asymp M^{-\frac{1}{m}\left(r + \sum_{j=1}^{m}\left(\frac{1}{2} - \frac{1}{p_j}\right)\right)}.$$

Proof. Firstly, we are going to consider the upper bound in the first item. Taking into account the inclusion $B^r_{\bar{p},\theta} \subset H^r_{\bar{p}}$, $1 \le \theta < +\infty$, it suffices to prove it for the class $H^r_{\bar{p}}$.

Let $1 \le p_j < q_j < \infty$ and $\mathbb{N}$ be the set of natural numbers. For a number $M \in \mathbb{N}$ choose a natural number n such that $2^{nm} < M \le 2^{(n+1)m}$. For a function $f \in H^r_{\bar{p}}$, it is known that

$$f(\bar{x}) = \sum_{s=0}^{\infty} \delta_s(f, \bar{x})$$

and

$$\|\delta_s(f)\|_{\bar{p}} \le 2^{-sr}, \quad 1 < p_j < \infty, \quad j = 1, ..., m.$$

We will seek an approximation polynomial $P(\Omega_M, \bar{x})$ in the form

$$P(\Omega_M, \bar{x}) = \sum_{s=0}^{n-1} \delta_s(f, \bar{x}) + \sum_{n \le s < \alpha n} P(\Omega_{N_s}, \bar{x}), \tag{1}$$

where the polynomials $P(\Omega_{N_s}, \bar{x})$ will be constructed for each $\delta_s(f, \bar{x})$ in accordance with Lemma 1.4 and the number $\alpha > 1$ will be chosen during the construction.

Let $\sum\limits_{j=1}^{m} (\frac{1}{p_j} - \frac{1}{q_j}) < r < \sum\limits_{j=1}^{m} \frac{1}{p_j}$. Suppose

$$N_s = \left[2^{nm} 2^{s\left(\sum\limits_{j=1}^{m} \frac{1}{p_j} - r\right)} 2^{-n\alpha\left(\sum\limits_{j=1}^{m} \frac{1}{p_j} - r\right)} \right] + 1,$$

where $[y]$ is the integer part of the number y.

Now we are going to show that the polynomials (1) have no more than M harmonics (in terms of order). By the definition of the number N_s, we have

$$\sum_{s=0}^{n-1} \sharp\{\bar{k} = (k_1, ..., k_m) : [2^{s-1}] \le \max_{j=1,...,m} |k_j| < 2^s\} + \sum_{n \le s < \alpha n} N_s \le C 2^{nm} +$$

$$+ \sum_{n \le s < \alpha n} \left(2^{nm} 2^{s\left(\sum\limits_{j=1}^{m} \frac{1}{p_j} - r\right)} 2^{-n\alpha\left(\sum\limits_{j=1}^{m} \frac{1}{p_j} - r\right)} + 1 \right) \le C 2^{nm} + (\alpha - 1)\, n \le C 2^{nm} \asymp M,$$

where $\sharp A$ denotes the number of elements in the set A.

Next, by the property of the norm, we have

$$\|f - P(\Omega_M)\|_{\bar{q}} \le C \left\| \sum_{n \le s < \alpha n} (\delta_s(f) - P(\Omega_{N_s})) \right\|_{\bar{q}} +$$

$$+ \left\| \sum_{\alpha n \le s < +\infty} \delta_s(f) \right\|_{\bar{q}} = J_1(n) + J_2(n). \tag{2}$$

Let us estimate $J_2(n)$. Applying the inequality of different metrics for trigonometric polynomials (Theorem 1.2), we can obtain

$$J_2(n) \le \sum_{\alpha n \le s < +\infty} \|\delta_s(f)\|_{\bar{q}} \le C \sum_{\alpha n \le s < +\infty} 2^{s \sum\limits_{j=1}^{m} \left(\frac{1}{p_j} - \frac{1}{q_j}\right)} \|\delta_s(f)\|_{\bar{p}}.$$

Therefore, taking into account $f \in H^r_{\bar p}$ and $\sum\limits_{j=1}^m (\frac{1}{p_j} - \frac{1}{q_j}) < r$, we get

$$J_2(n) \le C \sum_{\alpha n \le s < +\infty} 2^{-s\left(r - \sum\limits_{j=1}^m \left(\frac{1}{p_j} - \frac{1}{q_j}\right)\right)} \le C 2^{-n\alpha\left(r - \sum\limits_{j=1}^m \left(\frac{1}{p_j} - \frac{1}{q_j}\right)\right)}. \qquad (3)$$

Let us estimate $J_1(n)$. Using the property of the norm, Lemma 1.4 and the inequality of different metrics (Theorem 1.2), we get

$$J_1(n) \le \sum_{n \le s < \alpha n} \|\delta_s(f) - P(\Omega_{N_s})\|_{\bar q} \le C \sum_{n \le s < \alpha n} \left(N_s^{-1} 2^{sm}\right)^{\frac{1}{2}} \|\delta_s(f)\|_2 \le$$

$$\le C \sum_{n \le s < \alpha n} \left(N_s^{-1} 2^{sm}\right)^{\frac{1}{2}} 2^{s \sum\limits_{j=1}^m \left(\frac{1}{p_j} - \frac{1}{2}\right)} \|\delta_s(f)\|_{\bar p} \le$$

$$\le C \sum_{n \le s < \alpha n} N_s^{-\frac{1}{2}} 2^{s \sum\limits_{j=1}^m \frac{1}{p_j}} 2^{-sr} \le$$

$$\le C 2^{-\frac{nm}{2}} 2^{\frac{n\alpha}{2}\left(\sum\limits_{j=1}^m \frac{1}{p_j} - r\right)} \sum_{n \le s < \alpha n} 2^{s\left(\sum\limits_{j=1}^m \frac{1}{p_j} - r\right)\frac{1}{2}} \le C 2^{-\frac{nm}{2}} 2^{\frac{n\alpha}{2}\left(\sum\limits_{j=1}^m \frac{1}{p_j} - r\right)}. \qquad (4)$$

Suppose $\alpha = m\left(2\sum\limits_{j=1}^m \frac{1}{q_j}\right)^{-1}$. Then, from the inequality (4), we get

$$J_1(n) \le C 2^{-nm\left(2\sum\limits_{j=1}^m \frac{1}{q_j}\right)^{-1}\left(r - \sum\limits_{j=1}^m \left(\frac{1}{p_j} - \frac{1}{q_j}\right)\right)} \asymp M^{-\left(2\sum\limits_{j=1}^m \frac{1}{q_j}\right)^{-1}\left(r - \sum\limits_{j=1}^m \left(\frac{1}{p_j} - \frac{1}{q_j}\right)\right)}. \qquad (5)$$

For $\alpha = m\left(2\sum\limits_{j=1}^m \frac{1}{q_j}\right)^{-1}$, using the inequality (3) and taking into account $2^{nm} \asymp M$, we obtain

$$J_2(n) \le C M^{-\left(2\sum\limits_{j=1}^m \frac{1}{q_j}\right)^{-1}\left(r - \sum\limits_{j=1}^m \left(\frac{1}{p_j} - \frac{1}{q_j}\right)\right)}. \qquad (6)$$

By (5) and (6), we get from the inequality (2) the following

$$\|f - P(\Omega_M)\|_{\bar q} \le C M^{-\left(2\sum\limits_{j=1}^m \frac{1}{q_j}\right)^{-1}\left(r - \sum\limits_{j=1}^m \left(\frac{1}{p_j} - \frac{1}{q_j}\right)\right)},$$

for any function $f \in H^r_{\bar p}$ in the case of $\sum\limits_{j=1}^m (\frac{1}{p_j} - \frac{1}{q_j}) < r < \sum\limits_{j=1}^m \frac{1}{p_j}$.

From the inclusion $B^r_{\bar p,\theta} \subset H^r_{\bar p}$ and the definition of the M-term approximation, it follows that

$$e_M\left(B^r_{\bar p,\theta}\right)_{\bar q} \le C M^{-\left(2\sum\limits_{j=1}^m \frac{1}{q_j}\right)^{-1}\left(r - \sum\limits_{j=1}^m \left(\frac{1}{p_j} - \frac{1}{q_j}\right)\right)}$$

in the case of $\sum\limits_{j=1}^m (\frac{1}{p_j} - \frac{1}{q_j}) < r < \sum\limits_{j=1}^m \frac{1}{p_j}$.

Let us consider the lower bound. We will use the well-known formula (see [19], p. 79)

$$e_M(f)_{\bar q} = \inf_{\Omega_M} \sup_{P \in L^{\perp}_M, \|P\|_{\bar q'} \le 1} \left| \int_{\mathbb{T}^m} f(\bar x) \bar P(\bar x) d\bar x \right|, \qquad (7)$$

where $\bar q' = (q_1', ..., q_m')$, $\frac{1}{q_j} + \frac{1}{q_j'} = 1$, $j = 1, ..., m$, and $L^{\perp}_M$ is the set of functions that are orthogonal to the subspace of trigonometric polynomials with harmonics in the set Ω_M.

Consider the function

$$F_{\bar{q},n}(\bar{x}) = \sum_{\max\limits_{j=1,\dots,m}|k_j|\le 2^{\left[nm\left(2\sum\limits_{j=1}^{m}\frac{1}{q_j}\right)^{-1}\right]}} e^{i\langle\bar{k},\bar{x}\rangle}.$$

Let Ω_M be a set of M vectors with integer coordinates. Suppose

$$g(\bar{x}) = F_{\bar{q},n}(\bar{x}) - \sum_{\bar{k}\in\Omega_M}^{*} e^{i\langle\bar{k},\bar{x}\rangle},$$

where the sum $\sum\limits_{\bar{k}\in\Omega_M}^{*}$ contains those terms in the function $F_{\bar{q},n}(\bar{x})$ with indices only in Ω_M. By the inequality (see [18], p. 88)

$$\|\sum_{\max\limits_{j=1,\dots,m}|k_j|\le 2^l} e^{i\langle\bar{k},\bar{x}\rangle}\|_{\bar{p}} \le C2^{l\sum\limits_{j=1}^{m}(1-\frac{1}{p_j})} \tag{8}$$

and Perseval's equality for $1 < {q_j}' < 2$, $j = 1, ..., m$, we obtain

$$\|g\|_{\bar{q}'} \le \|F_{\bar{q},n}\|_{\bar{q}'} + (2\pi)^{\sum\limits_{j=1}^{m}(\frac{1}{q_j}-\frac{1}{2})}\|\sum_{\bar{k}\in\Omega_M}^{*} e^{i\langle\bar{k},\bar{x}\rangle}\|_2 \le C(2^{\frac{nm}{2}} + M^{\frac{1}{2}}) \le C2^{\frac{nm}{2}}. \tag{9}$$

Now we consider the function

$$P_1(\bar{x}) = C_2 2^{\left(-\frac{nm}{2}\right)} g(\bar{x}). \tag{10}$$

Then (9) implies that the function P_1 satisfies the assumptions of the formula (7) for some constant $C_2 > 0$.

Consider the function

$$f_1(\bar{x}) = C_3 2^{-nm\left(2\sum\limits_{j=1}^{m}\frac{1}{q_j}\right)^{-1}\left(r-\sum\limits_{j=1}^{m}\left(\frac{1}{p_j}-1\right)\right)} F_{\bar{q},n}(\bar{x}). \tag{11}$$

By the inequality (8), we get

$$\sum_{s=0}^{\infty} 2^{sr}\left\|\delta_s\left(f_1\right)\right\|_{\bar{p}} \le$$

$$\le C2^{-nm\left(2\sum\limits_{j=1}^{m}\frac{1}{q_j}\right)^{-1}\left(r-\sum\limits_{j=1}^{m}\left(\frac{1}{p_j}-1\right)\right)} \sum_{s=0}^{\left[nm\left(2\sum\limits_{j=1}^{m}\frac{1}{q_j}\right)^{-1}\right]} 2^{sr}2^{s\sum\limits_{j=1}^{m}\left(1-\frac{1}{p_j}\right)} \le C_3.$$

Hence $C_3^{-1} f_1 \in B^r_{\bar{p},1}$.

For the functions (10) and (11), we have, by the formula (7), the following

$$e_M(f_1)_{\bar{q}} \ge \inf_{\Omega_M}\left|\int_{\mathbb{T}^m} f_1\left(\bar{x}\right)\bar{P}_1\left(\bar{x}\right)d\bar{x}\right| \ge$$

$$\ge C2^{-nm\left(2\sum\limits_{j=1}^{m}\frac{1}{q_j}\right)^{-1}\left(r-\sum\limits_{j=1}^{m}\left(\frac{1}{p_j}-1\right)\right)} 2^{-\frac{nm}{2}}\left(\|F_{\bar{q},n}\|_2^2 - M\right) \ge$$

$$\ge C2^{-nm\left(2\sum\limits_{j=1}^{m}\frac{1}{q_j}\right)^{-1}\left(r-\sum\limits_{j=1}^{m}\left(\frac{1}{p_j}-\frac{1}{q_j}\right)\right)}. \tag{12}$$

Hence, it follows from (12) by the inclusion $B^r_{\bar{p},1} \subset B^r_{\bar{p},\theta}$ that

$$e_M(f_1)_{\bar{q}} \ge C2^{-nm\left(2\sum\limits_{j=1}^{m}\frac{1}{q_j}\right)^{-1}\left(r-\sum\limits_{j=1}^{m}\left(\frac{1}{p_j}-\frac{1}{q_j}\right)\right)}$$

in the case of $\sum\limits_{j=1}^{m}(\frac{1}{p_j}-\frac{1}{q_j})<r<\sum\limits_{j=1}^{m}\frac{1}{p_j}$. So we have proved the first item.

Now we consider the case $r=\sum\limits_{j=1}^{m}\frac{1}{p_j}$. Let $f\in B^r_{\bar{p},\theta}$. Suppose $\alpha=m\left(2\sum\limits_{j=1}^{m}\frac{1}{q_j}\right)^{-1}$ and

$$N_s=\left[2^{nm}n^{\frac{1}{\theta}-1}\left\|\delta_s(f)\right\|_{\bar{p}}2^{sr}\right]+1.$$

Then, by definition of the numbers N_s and Holder's inequality, we obtain

$$\sum_{s=0}^{n-1}\sharp\{\bar{k}=(k_1,...,k_m):[2^{s-1}]\le\max_{j=1,...,m}|k_j|<2^s\}+\sum_{n\le s<\alpha n}N_s\le$$

$$\le C2^{nm}+(\alpha-1)n+2^{nm}n^{\frac{1}{\theta}-1}((\alpha-1)n)^{\frac{1}{\theta'}}\left(\sum_{s=0}^{\infty}\left\|\delta_s(f)\right\|^{\theta}_{\bar{p}}2^{sr\theta}\right)^{\frac{1}{\theta}}\le C2^{nm}\asymp M.$$

Suppose $\beta=\max\{q_1,...,q_m\}$. Then

$$J_1(n)=\left\|\sum_{n\le s<\alpha n}(\delta_s(f)-P(\Omega_{N_s}))\right\|_{\bar{q}}\le C\left\|\sum_{n\le s<\alpha n}(\delta_s(f)-P(\Omega_{N_s}))\right\|_{\beta}.$$

Next, by Theorem 1.3, we have

$$J_1(n)\le C\left\|\left(\sum_{n\le s<\alpha n}|\delta_s(f)-P(\Omega_{N_s})|^2\right)^{\frac{1}{2}}\right\|_{\beta}.$$

Since $\beta>2$, then by applying the property of the norm, Lemma 1.4 and the inequality of different metrics for trigonometric polynomials (see Theorem 1.2), we obtain

$$J_1(n)\le\left(\sum_{n\le s<\alpha n}\left\|\delta_s(f)-P(\Omega_{N_s})\right\|^2_{\beta}\right)^{\frac{1}{2}}\le C\left(\sum_{n\le s<\alpha n}N_s^{-1}2^{sm}\left\|\delta_s(f)\right\|^2_2\right)^{\frac{1}{2}}\le$$

$$\le C\left(\sum_{n\le s<\alpha n}N_s^{-1}2^{sm}2^{2s\sum\limits_{j=1}^{m}(\frac{1}{p_j}-\frac{1}{2})}\left\|\delta_s(f)\right\|^2_{\bar{p}}\right)^{\frac{1}{2}}. \quad (13)$$

Next, since $r=\sum\limits_{j=1}^{m}\frac{1}{p_j}$, we have, by the definition of the numbers N_s and using Holder's inequality, the following

$$J_1(n)\le C(2^{-nm}n^{1-\frac{1}{\theta}})^{\frac{1}{2}}\left(\sum_{n\le s<\alpha n}2^{sr}\left\|\delta_s(f)\right\|_{\bar{p}}\right)^{\frac{1}{2}}\le$$

$$\le C(2^{-nm}n^{1-\frac{1}{\theta}})^{\frac{1}{2}}\left(\sum_{n\le s<\alpha n}2^{sr\theta}\left\|\delta_s(f)\right\|^{\theta}_{\bar{p}}\right)^{\frac{1}{2\theta}}\left(\sum_{n\le s<\alpha n}1\right)^{\frac{1}{2}(1-\frac{1}{\theta})}$$

$$\le C2^{-\frac{nm}{2}}n^{1-\frac{1}{\theta}}\asymp M^{-\frac{1}{2}}(\log(1+M))^{1-\frac{1}{\theta}}.$$

Thus,

$$J_1(n)\le CM^{-\frac{1}{2}}(\log(1+M))^{1-\frac{1}{\theta}} \quad (14)$$

in the case of $r=\sum\limits_{j=1}^{m}\frac{1}{p_j}$.

To estimate $J_2(n)$, we apply Holder's inequality, and taking into account $r=\sum\limits_{j=1}^{m}\frac{1}{p_j}$ and $\alpha=m\left(2\sum\limits_{j=1}^{m}\frac{1}{q_j}\right)^{-1}$, we obtain

$$J_2(n)\le C\sum_{n\alpha\le s<+\infty}2^{s\sum\limits_{j=1}^{m}(\frac{1}{p_j}-\frac{1}{q_j})}\left\|\delta_s(f)\right\|_{\bar{p}}\le \tag{15}$$

$$\le C\left(\sum_{s=0}^{\infty}2^{sr\theta}\left\|\delta_s(f)\right\|_{\bar{p}}^{\theta}\right)^{\frac{1}{\theta}}\left(\sum_{n\alpha\le s<+\infty}2^{-s\theta'\sum\limits_{j=1}^{m}\frac{1}{q_j}}\right)^{\frac{1}{\theta'}}\le C2^{-n\alpha\sum\limits_{j=1}^{m}\frac{1}{q_j}}=C2^{-\frac{nm}{2}}\asymp M^{-\frac{1}{2}}.$$

By (14) and (15), the inequality (2) implies that

$$\|f-P(\Omega_M)\|_{\bar{q}}\le CM^{-\frac{1}{2}}(\log(1+M))^{1-\frac{1}{\theta}}$$

in the case of $r=\sum\limits_{j=1}^{m}\frac{1}{p_j}$. It proves the upper bound estimation in the second item.

Let $r>\sum\limits_{j=1}^{m}\frac{1}{p_j}$. Suppose

$$N_s=\left[2^{n\left(r-\sum\limits_{j=1}^{m}\left(\frac{1}{p_j}-1\right)\right)}2^{-s\left(r-\sum\limits_{j=1}^{m}\frac{1}{p_j}\right)}\right]+1.$$

Then

$$\sum_{s=0}^{n-1}\sharp\{\bar{k}=(k_1,...,k_m):[2^{s-1}]\le\max_{j=1,...,m}|k_j|<2^{s}\}+\sum_{n\le s<\alpha n}N_s\le$$
$$\le C2^{nm}+(\alpha-1)n\le C2^{nm}\le CM.$$

If $f\in H_{\bar{p}}^{r}$, then, by using the definition of the numbers N_s and $r>\sum\limits_{j=1}^{m}\frac{1}{p_j}$, we obtain from (13) the following

$$J_1(n)\le\left(\sum_{n\le s<\alpha n}N_s^{-1}2^{sm}2^{2s\sum\limits_{j=1}^{m}\left(\frac{1}{p_j}-\frac{1}{2}\right)}\left\|\delta_s(f)\right\|_{\bar{p}}^{2}\right)^{\frac{1}{2}}\le$$

$$\le C2^{-\frac{n}{2}\left(r-\sum\limits_{j=1}^{m}\left(\frac{1}{p_j}-1\right)\right)}\left(\sum_{n\le s<\alpha n}2^{-s\left(r-\sum\limits_{j=1}^{m}\frac{1}{p_j}\right)}\right)^{\frac{1}{2}}\le C2^{-n\left(r+\sum\limits_{j=1}^{m}\left(\frac{1}{2}-\frac{1}{p_j}\right)\right)}.$$

Thus,

$$J_1(n)\le CM^{-\frac{1}{m}\left(r+\sum\limits_{j=1}^{m}(\frac{1}{2}-\frac{1}{p_j})\right)} \tag{16}$$

in the case of $r>\sum\limits_{j=1}^{m}\frac{1}{p_j}$.

To estimate $J_2(n)$, we suppose $\alpha=\left(r+\sum\limits_{j=1}^{m}\left(\frac{1}{2}-\frac{1}{p_j}\right)\right)\left(r+\sum\limits_{j=1}^{m}\left(\frac{1}{q_j}-\frac{1}{p_j}\right)\right)^{-1}$ and get

$$J_2(n)\le C\sum_{n\alpha\le s<\infty}2^{-s\left(r+\sum\limits_{j=1}^{m}\left(\frac{1}{p_j}-\frac{1}{q_j}\right)\right)}\le C2^{-n\left(r+\sum\limits_{j=1}^{m}\left(\frac{1}{2}-\frac{1}{p_j}\right)\right)}\le$$

$$\le CM^{-\frac{1}{m}\left(r+\sum\limits_{j=1}^{m}\left(\frac{1}{2}-\frac{1}{p_j}\right)\right)} \tag{17}$$

for a function $f \in H^r_{\bar p}$. By (16) and (17), it follows from (2) that

$$\|f - P(\Omega_M)\|_{\bar q} \le CM^{-\frac{1}{m}\left(r + \sum\limits_{j=1}^{m}\left(\frac{1}{2} - \frac{1}{p_j}\right)\right)}$$

for any function $f \in H^r_{\bar p}$ in the case of $r > \sum\limits_{j=1}^{m} \frac{1}{p_j}$.

It follows from $B^r_{\bar p,\theta} \subset H^r_{\bar p}$ that

$$e_M\left(B^r_{\bar p,\theta}\right)_{\bar q} \le e_M\left(H^r_{\bar p}\right)_{\bar q} \le CM^{-\frac{1}{m}\left(r + \sum\limits_{j=1}^{m}\left(\frac{1}{2} - \frac{1}{p_j}\right)\right)}$$

in the case of $r > \sum\limits_{j=1}^{m} \frac{1}{p_j}$. It proves the upper bound estimation in the item 3.

Let us consider the lower bound estimation in the case of $r = \sum\limits_{j=1}^{m} \frac{1}{p_j}$. Consider the function

$$g_1(\bar x) = \sum_{s=1}^{n} \sum_{\bar k \in \rho(s)} \prod_{j=1}^{m} k_j^{-1} \cos k_j x_j. \tag{18}$$

Then

$$\delta_s(g_1, \bar x) = \sum_{\bar k \in \rho(s)} \prod_{j=1}^{m} k_j^{-1} \cos k_j x_j.$$

It is known that for a function $d_s(\bar x) = \sum\limits_{\bar k \in \rho(s)} \prod\limits_{j=1}^{m} \cos k_j x_j$ the following relation holds

$$\|d_s\|_{\bar p} \asymp 2^{s\sum\limits_{j=1}^{m}(1-\frac{1}{p_j})}, \ \ 1 < p_j < +\infty, \ \ j = 1, ..., m.$$

Therefore, by the Marcinkiewicz theorem on multipliers (see [18]), we have

$$\|\delta_s(g_1)\|_{\bar p} \le C2^{-sm}\|d_s\|_{\bar p} \le C2^{-s\sum\limits_{j=1}^{m}\frac{1}{p_j}}.$$

Hence, since $r = \sum\limits_{j=1}^{m} \frac{1}{p_j}$, we obtain

$$\left(\sum_{s=0}^{\infty} 2^{sr\theta}\|\delta_s(g_1)\|^{\theta}_{\bar p}\right)^{\frac{1}{\theta}} \le C_1 n^{\frac{1}{\theta}}.$$

Therefore, the function $f_2(\bar x) = C_1^{-1} n^{-\frac{1}{\theta}} g_1(\bar x)$ belongs to the class $B^r_{\bar p,\theta}$, $1 < p_j < +\infty$, $j = 1, ..., m$.

Now, we are going to construct a function P_1, which satisfies the conditions of the formula (7). Let

$$v_1(\bar x) = \sum_{s=1}^{n} \sum_{\bar k \in \rho(s)} \prod_{j=1}^{m} \cos k_j x_j$$

and Ω_M be an arbitrary set of M vectors $\bar k = (k_1, ..., k_m)$ with integer coordinates. Consider the function

$$u_1(\bar x) = \sum_{\bar k \in \Omega_M}^{*} \prod_{j=1}^{m} \cos k_j x_j$$

which contains only those terms in (18) with indices in Ω_M. Suppose $w_1(\bar x) = v_1(\bar x) - u_1(\bar x)$. Then, since $1 < {q_j}' < 2$, $j = 1, ..., m$, we obtain, by Perseval's equality, the following

$$\|w_1\|_{\bar q'} \le \|v_1\|_{\bar q'} + \|u_1\|_2 \le \|v_1\|_{\bar q'} + CM^{\frac{1}{2}}.$$

By the property of the norm and the estimation of the norm of the Dirichlet kernel in the Lebesgue space, we have

$$\|v_1\|_{\bar{q}'} \leq \sum_{s=1}^{n} \|\delta_s(v_1)\|_{\bar{q}'} \leq$$

$$\leq C\sum_{s=1}^{n} 2^{s\sum\limits_{j=1}^{m}\left(1-\frac{1}{q_j'}\right)} \leq C2^{n\sum\limits_{j=1}^{m}\frac{1}{q_j}}.$$

Therefore, taking into account $\frac{1}{q_j} < \frac{1}{2}$, $j = 1, ..., m$, we get

$$\|w_1\|_{\bar{q}'} \leq C(2^{\frac{nm}{2}} + M^{\frac{1}{2}}) \leq C_2 2^{\frac{nm}{2}}.$$

Hence, the function

$$P_1(\bar{x}) = C_2^{-1}2^{-\frac{nm}{2}}w_1(\bar{x})$$

satisfies the conditions of the formula (7). Then, by substituting the functions f_2 and P_1 into (7) and by orthogonality of the trigonometric system, we obtain

$$e_M\left(f_2\right)_{\bar{q}} \geq C\sum_{n_1\leq s<n}\sum_{\bar{k}\in\rho(s)}\prod_{j=1}^{m} k_j^{-1}2^{-\frac{nm}{2}}n^{-\frac{1}{\theta}} \geq$$

$$\geq C(\ln 2)^m \sum_{n_1\leq s<n} 2^{-\frac{nm}{2}}n^{-\frac{1}{\theta}} = C(\ln 2)^m 2^{-\frac{nm}{2}}n^{-\frac{1}{\theta}}(n-n_1) \geq$$

$$\geq C(\ln 2)^m 2^{-\frac{nm}{2}}n^{1-\frac{1}{\theta}} \asymp M^{-\frac{1}{2}}(\log(1+M))^{1-\frac{1}{\theta}},$$

where n_1 is a natural number such that $n_1 \leq \frac{n}{2}$.

So, for the function $f_2 \in B^r_{\bar{p},\theta}$, it has been proved that

$$e_M\left(f_2\right)_{\bar{q}} \geq CM^{-\frac{1}{2}}(\log(1+M))^{1-\frac{1}{\theta}}$$

in the case of $r = \sum\limits_{j=1}^{m}\frac{1}{p_j}$. Hence

$$e_M\left(B^r_{\bar{p},\theta}\right)_{\bar{q}} \geq CM^{-\frac{1}{2}}(\log(1+M))^{1-\frac{1}{\theta}}$$

in the case of $r = \sum\limits_{j=1}^{m}\frac{1}{p_j}$. It proves the lower bound estimation in the second item.

Let us prove the lower bound estimation for the case $r > \sum\limits_{j=1}^{m}\frac{1}{p_j}$. Since in this case an upper bound estimation of the quantity $e_M\left(B^r_{\bar{p},\theta}\right)_{\bar{q}}$ does not depend on θ and $B^r_{\bar{p},1} \subset B^r_{\bar{p},\theta}$, $1 < \theta \leq +\infty$, it suffices to prove the lower bound estimation for $B^r_{\bar{p},1}$.

For a number $M \in \mathbb{N}$, we choose a natural number n such that $2^{nm} < M \leq 2^{(n+1)m}$ and $2M \leq \sharp\rho(n)$, where $\sharp\rho(n)$ denotes the number of elements in the set $\rho(n)$.

Consider the following function

$$f_3(\bar{x}) = 2^{-n\left(r+\sum\limits_{j=1}^{m}\left(1-\frac{1}{p_j}\right)\right)}\sum_{\bar{k}\in\rho(n)} e^{i\langle\bar{k},\bar{x}\rangle}.$$

Then $\|\delta_s(f_3)\|_{\bar{p}} = 0$ provided $s \neq n$ and

$$\|\delta_n(f_3)\|_{\bar{p}} = 2^{-n\left(r+\sum\limits_{j=1}^{m}\left(1-\frac{1}{p_j}\right)\right)}\prod_{j=1}^{m}\Big\|\sum_{k_j=2^{n-1}}^{2^n-1} e^{ik_jx_j}\Big\|_{p_j}.$$

By the estimation of the norm of the Dirichlet kernel (see [18], p. 181), we have

$$\Big\| \sum_{k_j=2^{n-1}}^{2^n-1} e^{ik_jx_j} \Big\|_{p_j} \leq C2^{n(1-\frac{1}{p_j})},$$

for $p_j \in (1,\infty)$, $j=1,...,m$. Therefore

$$\|\delta_n(f_3)\|_{\bar{p}} \leq C2^{-nr}.$$

Hence

$$\sum_{s=0}^{\infty} 2^{sr} \|\delta_s(f_3)\|_{\bar{p}} \leq C_3,$$

i.e. the function $C_3^{-1} f_3 \in B^r_{\bar{p},1}$. Next, we consider the functions

$$v_2(\bar{x}) = \sum_{\bar{k}\in\rho(n)} e^{i\langle \bar{k},\bar{x}\rangle}$$

and

$$u_2(\bar{x}) = \sum_{\bar{k}\in\rho(n)\cap\Omega_M} e^{i\langle \bar{k},\bar{x}\rangle}.$$

Suppose $w_2(\bar{x}) = v_2(\bar{x}) - u_2(\bar{x})$. By Perseval's equality,

$$\|u_2\|_2 \leq M^{\frac{1}{2}}, \quad \|v_2\|_2 \leq C2^{\frac{nm}{2}}.$$

From these relations, we obtain, by the properties of the norm, the following

$$\|w_2\|_2 \leq \|v_2\|_2 + \|u_2\|_2 \leq C_4 2^{\frac{nm}{2}}.$$

Therefore, the function $P_2(\bar{x}) = C_4^{-1} 2^{-\frac{nm}{2}} w_2(\bar{x})$ satisfies the conditions of the formula (7). Since $2 < q_j < \infty$, $j=1,...,m$, we have $e_M(f_3)_2 \leq Ce_M(f_3)_{\bar{q}}$. Now, by the formula (7), we get

$$e_M(f_3)_{\bar{q}} \geq Ce_M(f_3)_2 \geq$$

$$\geq C \inf_{\Omega_M} \int_{\mathbb{T}^m} f_3(\bar{x})\bar{P}_2(\bar{x})d\bar{x} =$$

$$= C_2^{-1} 2^{-\frac{nm}{2}} 2^{-n\left(r+\sum\limits_{j=1}^m\left(1-\frac{1}{p_j}\right)\right)} \inf_{\Omega_M} [\sharp\rho(n) - \sharp(\rho(n)\cap\Omega_M)] \geq$$

$$\geq C2^{-\frac{nm}{2}} 2^{-n\left(r+\sum\limits_{j=1}^m\left(1-\frac{1}{p_j}\right)\right)} [\sharp\rho(n) - M] \geq$$

$$\geq C2^{-\frac{nm}{2}} 2^{-n\left(r+\sum\limits_{j=1}^m\left(1-\frac{1}{p_j}\right)\right)} [\sharp\rho(n) - \frac{\sharp\rho(n)}{2}] \geq$$

$$\geq C2^{-\frac{nm}{2}} 2^{-n\left(r-\sum\limits_{j=1}^m \frac{1}{p_j}\right)}.$$

It follows from the relation $2^{nm} \asymp M$ that

$$e_M(f_3)_{\bar{q}} \geq CM^{-\frac{1}{m}\left(r+\sum\limits_{j=1}^m\left(\frac{1}{2}-\frac{1}{p_j}\right)\right)}$$

in the case of $r > \sum\limits_{j=1}^m \frac{1}{p_j}$ for the function $C_3^{-1} f_3 \in B^r_{\bar{p},1}$. Hence

$$e_M\left(B^r_{\bar{p},1}\right)_{\bar{q}} \geq CM^{-\frac{1}{m}\left(r+\sum\limits_{j=1}^m\left(\frac{1}{2}-\frac{1}{p_j}\right)\right)}.$$

Therefore,

$$e_M\left(B^r_{\bar p,\theta}\right)_{\bar q} \ge CM^{-\frac{1}{m}\left(r+\sum\limits_{j=1}^{m}\left(\frac{1}{2}-\frac{1}{p_j}\right)\right)}$$

in the case of $r > \sum\limits_{j=1}^{m} \frac{1}{p_j}$. So Theorem 2.1 has been proved.

2.2. Theorem. *Let $\bar p = (p_1, ..., p_m)$, $\bar q = (q_1, ..., q_m)$, $1 < p_j < q_j \le 2$, and $1 \le \theta \le +\infty$. If $r > \sum\limits_{j=1}^{m}(\frac{1}{p_j} - \frac{1}{q_j})$, then*

$$e_M\left(B^r_{\bar p,\theta}\right)_{\bar q} \asymp M^{-\frac{1}{m}\left(r-\sum\limits_{j=1}^{m}\left(\frac{1}{p_j}-\frac{1}{q_j}\right)\right)}.$$

Proof. For a number $M \in \mathbb{N}$, we choose a natural number n such that $M \asymp 2^{nm}$. By the inequality of distinct metrics (see Theorem 1.2) and by Holder's inequality, we have

$$\|f - \sum_{s=0}^{n}\delta_s(f)\|_{\bar q} \le \sum_{s=n}^{\infty}\|\delta_s(f)\|_{\bar q} \le$$

$$\le \left[\sum_{s=0}^{\infty} 2^{sr\theta}\|\delta_s(f)\|^{\theta}_{\bar q}\right]^{\frac{1}{\theta}}\left[\sum_{s=n}^{\infty} 2^{s\theta'\left(r-\sum\limits_{j=1}^{m}\left(\frac{1}{p_j}-\frac{1}{q_j}\right)\right)}\right]^{\frac{1}{\theta'}} \le$$

$$\le C2^{n\left(r-\sum\limits_{j=1}^{m}\left(\frac{1}{p_j}-\frac{1}{q_j}\right)\right)} \le CM^{-\frac{1}{m}\left(r-\sum\limits_{j=1}^{m}\left(\frac{1}{p_j}-\frac{1}{q_j}\right)\right)}$$

for $f \in B^r_{\bar p,\theta}$, $\frac{1}{\theta} + \frac{1}{\theta'} = 1$. Therefore

$$e_M(f)_{\bar q} \le \|f - \sum_{s=0}^{n}\delta_s(f)\|_{\bar q} \le CM^{-\frac{1}{m}\left(r-\sum\limits_{j=1}^{m}\left(\frac{1}{p_j}-\frac{1}{q_j}\right)\right)}.$$

Hence

$$e_M\left(B^r_{\bar p,\theta}\right)_{\bar q} \le CM^{-\frac{1}{m}\left(r-\sum\limits_{j=1}^{m}\left(\frac{1}{p_j}-\frac{1}{q_j}\right)\right)}.$$

It proves the upper bound estimation.

For the lower bound estimation, let us consider the function

$$f_4(\bar x) = n^{-r+\sum\limits_{j=1}^{m}\left(\frac{1}{p_j}-1\right)} V_n(\bar x),$$

where $V_n(\bar x)$ is a multiple of the Valle-Poisson sum.

Next, following the proof in [9] (pp. 46-47) and applying Theorem 1.2, we obtain the lower bound estimation of the quantity $e_M\left(B^r_{\bar p,\theta}\right)_{\bar q}$.

2.3. Theorem. *Let $\bar p = (p_1, ..., p_m)$, $\bar q = (q_1, ..., q_m)$, $2 \le p_j < q_j < \infty$, $j = 1, ..., m$, and $1 \le \theta \le +\infty$. If $r > \frac{m}{2}$, then*

$$e_M\left(B^r_{\bar p,\theta}\right)_{\bar q} \asymp M^{-\frac{r}{m}}.$$

Proof. By the inclusion $B^r_{\bar p,\theta} \subset B^r_{2,\theta} \subset H^r_2$, we have

$$e_M\left(B^r_{\bar p,\theta}\right)_{\bar q} \le e_M\left(B^r_{2,\theta}\right)_{\bar q} \le e_M\left(H^r_2\right)_{\bar q}.$$

By Theorem 2.1,

$$e_M\left(H^r_2\right)_{\bar q} \le CM^{-\frac{r}{m}},$$

for $p_j = 2$, $j = 1, ..., m$. Hence

$$e_M\left(B^r_{\bar p,\theta}\right)_{\bar q} \le CM^{-\frac{r}{m}}.$$

It proves the upper bound estimation.

Let us consider the lower bound estimation. Consider Rudin-Shapiro's polynomial (see [15], p. 155) of the type

$$R_s(x) = \sum_{s=2^{s-1}}^{2^s} \varepsilon_k e^{ikx}, \;\; x \in [0, 2\pi], \;\; \varepsilon_k = \pm 1.$$

It is known that $\|R_s\|_\infty = \max\limits_{x\in[0,2\pi]} |R_s(x)| \le C 2^{\frac{s}{2}}$ (see [15], p. 155). For a given number M choose a number n such that $M \asymp 2^{nm}$. Now we consider the function

$$f_5(\bar{x}) = 2^{-n(\frac{m}{2}+r)} \sum_{s=1}^{n} \prod_{1}^{m} R_s(x_j).$$

Then, by the continuity, we have $f_5 \in L_{\bar{p}}(\mathbb{T}^m)$ and

$$\sum_{s=0}^{\infty} 2^{s\theta r} \|\delta_s(f_5)\|_{\bar{p}}^{\theta} = 2^{-n(\frac{m}{2}+r)} \sum_{s=1}^{n} 2^{s\theta r} \|\prod_{1}^{m} R_s(x_j)\|_{\bar{p}}^{\theta} \le$$

$$\le 2^{-n(\frac{m}{2}+r)} \sum_{s=1}^{n} 2^{s(\frac{m}{2}+r)\theta} \le C_5.$$

Hence, the function $C_5^{-1} f_5 \in B^r_{\bar{p},\theta}$. Now, we construct a function $P(\bar{x})$, which satisfies the conditions in the formula (7). Suppose

$$v_3(\bar{x}) = \sum_{s=1}^{n} \prod_{1}^{m} R_s(x_j), \;\; u_3(\bar{x}) = \sum_{s}^{*} \prod_{1}^{m} R_s(x_j),$$

where the sign $*$ means that the polynomial $u_3(\bar{x})$ contains only those harmonics of v_3, which have indices in Ω_M. Suppose $w_3(\bar{x}) = v_3(\bar{x}) - u_3(\bar{x})$. Then, since $1 < {q_j}' = \frac{q_j}{q_j - 1} < 2$, $j = 1, ..., m$, we have the following (by Perseval's equality)

$$\|w_3\|_{\bar{q}'} \le \|w_3\|_2 \le C_1 2^{\frac{nm}{2}}.$$

Therefore, for the function $P_3(\bar{x}) = C_1^{-1} 2^{-\frac{nm}{2}} w_3(\bar{x})$ the inequality $\|P_3\|_{\bar{q}'} \le 1$ holds. Now, using the formula (7), we obtain

$$e_M\left(B^r_{\bar{p},\theta}\right)_{\bar{q}} \ge e_M(f_3)_{\bar{q}} \ge 2^{-n(\frac{m}{2}+r)} 2^{-\frac{nm}{2}} (2^{nm} - M) \ge C 2^{-n(m+r)} 2^{nm} \ge C M^{-\frac{r}{m}}.$$

So

$$e_M\left(B^r_{\bar{p},\theta}\right)_{\bar{q}} \ge C M^{-\frac{r}{m}}.$$

It proves Theorem 2.3.

Remark. In the case $p_j = p$, $q_j = q$, $j = 1, ..., m$, and $r > m(\frac{1}{p} - \frac{1}{q})$, the results of R.A. DeVore and V.N. Temlyakov [9] follow from Theorem 2.1 - 2.3. If $1 < p \le 2 < q < \infty$ and $m(\frac{1}{p} - \frac{1}{q}) < r \le \frac{m}{p}$, the results of S.A. Stasyuk [20, 21] follow from the first and second items of Theorem 2.1. Theorem 2.1 - 2.3 were announced in [3].

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