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Inverse problem for an inhomogeneous fractional equation with the quadrat of pseudoparabolic operator

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Abstract. In this paper the issues of unique solvability and construction of a solution of an inverse boundary value problem for an inhomogeneous differential equation containing a quadrat of Hilfer fractional analogue of the pseudoparabolic operator are studied. The spectral problem is studied, eigenvalues and eigenfunctions are found. The solution of the direct and inverse boundary value problems are obtained in the form of Fourier series. Sufficient coefficient conditions for unique solvability of these problems are established. Theorems on absolute and uniform convergence of Fourier series are proved.

Key Words and Phrases: Boundary value problem, inverse problem, Hilfer fractional differential equation, quadrat of the pseudoparabolic operator, unique solvability, absolute and uniform convergence of Fourier series.

2010 Mathematics Subject Classifications: Primary 34B08, 34B10; Secondary 35G05.

1. Formulation of the problem statement

The theory of the boundary value problems is currently one of the most important directions of the modern theory of differential equations of mathematical physics and mechanics. Studies of many problems of gas dynamics, theory of elasticity, theory of plates and shells are described by higher-order partial differential equations.

In recent years the interest to study the differential equations with nonlocal boundary conditions is increasing. Inverse problems for differential equations find many applications in modern science and technology. Therefore, a large number of research works are devoted to the study of various kinds of inverse problems (see, for example [1]–[14]).

Fractional calculus plays an important role in mathematical modeling of many applied problems (see, [13]–[17]). Interesting results in investigations of fractional differential equations are obtained in the works [18]–[20].

In this paper, we study an inverse boundary value problem for an inhomogeneous partial differential equation with Hilfer fractional operator, with quadrat of pseudoparabolic operator and with redefinition function at the end point of the given segment. The questions of the existence and uniqueness of the solution to the inverse boundary value problem are investigated.

In the domain $\Omega \equiv (0,T) \times (0,l)$ we consider the equation

$$\left(D^{\alpha,\gamma} - D^{\alpha,\gamma}\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial x^2}\right)^2 u(t,x) = f(t,x) \tag{1}$$

with boundary conditions

$$\lim_{t\to+0} I_{0t}^{1-\gamma} u(t,x) = \varphi_1(x), \quad u(T,x) = \varphi_2(x), \tag{2}$$

$$u(t,0) = u(t,l) = u_{xx}(t,0) = u_{xx}(t,l) = 0, \tag{3}$$

where $\varphi_\kappa(x) \in C\,(\Omega_l)$, $\varphi_\kappa(0) = \varphi_\kappa(l) = \varphi_\kappa''(0) = \varphi_\kappa''(l) = 0$, $\kappa = 1,2$, $D^{\alpha,\gamma} = I_{0t}^{\gamma-\alpha}\frac{d}{dt}I_{0t}^{1-\gamma}$ is Hilfer fractional operator, $0 < \alpha \le \gamma \le 1$, $f(t,x) \in C(\Omega_T \times \Omega_l)$, $f(t,0) = f(t,l) = f_{xx}(t,0) = f_{xx}(t,l) = 0$, $I_{0t}^{\alpha}\psi(t) = \frac{1}{\Gamma(\alpha)}\int\limits_0^t \frac{\psi(s)ds}{(t-s)^{1-\alpha}}$ is Riemann-Liouville integral operator, $\Omega_T \equiv [0,T]$, $\Omega_l \equiv [0,l]$, $0 < T < \infty$, $0 < l < \infty$.

2. Direct boundary value problem

Problem 1. It is required to find a function $u(t,x)$, that satisfies the fractional pseudoparabolic differential equation (1), the two-point boundary value conditions (2), the Dirichlet type conditions (3) and belongs to the class of functions

$$t^2 D^{\alpha,\gamma} u \in C\,(\overline{\Omega}), \quad t^2 D^{\alpha,\gamma} u_{xxxx} \in C\,(\Omega), \quad t^2 u_{xxxx} \in C\,(\Omega), \tag{4}$$

where $\bar{\Omega} \equiv \Omega_T \times \Omega_l$.

We suppose that:

$$u(t,x) = \sum_{n=1}^{\infty} u_n(t) \cdot \vartheta_n(x) \tag{5}$$

and

$$f(t,x) = \sum_{n=1}^{\infty} f_n(t) \cdot \vartheta_n(x), \quad (t,x) \in \Omega, \tag{6}$$

where

$$u_n(t) = \int\limits_0^l u(t,y)\vartheta_n(y)dy, \quad f_n(t) = \int\limits_0^l f(t,y)\vartheta_n(y)dy, \tag{7}$$

$$\vartheta_n(x) = \sqrt{\frac{2}{l}}\sin\lambda_n x, \ \ \lambda_n = \frac{n\,\pi}{l}, \ \ n = 1,\, 2,\, ... \tag{8}$$

The system of functions $\{\vartheta_n(x)\}_{n=1}^{\infty}$ in (8) forms a complete system of orthonormal functions in $L_2(\Omega_l)$ and satisfies the boundary conditions:

$$\vartheta_n(0) = \vartheta_n(l) = \vartheta_n''(0) = \vartheta_n''(l) = 0.$$

Consequently, the functions defined by Fourier series (5) formally satisfy conditions (3). We rewrite equation (1) in the following form

$$\Bigg[(D^{\alpha,\gamma})^2 - 2\,(D^{\alpha,\gamma})^2\,\frac{\partial^2}{\partial\,x^2} + (D^{\,\alpha,\gamma})^2\,\frac{\partial^4}{\partial\,x^4} - 2D^{\alpha,\gamma}\frac{\partial^2}{\partial\,x^2} +$$

$$+ 2D^{\alpha,\gamma}\frac{\partial^4}{\partial\,x^4} + \frac{\partial^4}{\partial\,x^4}\Bigg]\,u(t,x) = f(t,x).$$

Substituting the Fourier series (5) and (6) into the differential equation (1), we obtain

$$\sum_{n=1}^{\infty}(D^{\alpha,\gamma})^2\,u_n(t)\,\vartheta_n(x) - 2\sum_{n=1}^{\infty}(D^{\alpha,\gamma})^2\,\vartheta_n''(x) + \sum_{n=1}^{\infty}(D^{\alpha,\gamma})^2\,\vartheta_n^{(IV)}(x) -$$

$$-2\sum_{n=1}^{\infty}D^{\alpha,\gamma}u_n(t)\,\vartheta_n''(x) + 2\sum_{n=1}^{\infty}D^{\alpha,\gamma}u_n(t)\,\vartheta_n^{(IV)}(x) + \sum_{n=1}^{\infty}u_n(t)\,\vartheta_n^{(IV)}(x) = \sum_{n=1}^{\infty}f_n(t)\,\vartheta_n(x).$$

Taking into account the properties of the eigenfunctions in (8), from the last relation we have

$$\sum_{n=1}^{\infty}\left[1 + 2\lambda_n^2 + \lambda_n^4\right](D^{\alpha,\gamma})^2\,u_n(t)\vartheta_n(x) + 2\sum_{n=1}^{\infty}\lambda_n^2\left[1 + \lambda_n^2\right]D^{\alpha,\gamma}u_n(t)\,\vartheta_n(x) +$$

$$+\sum_{n=1}^{\infty}u_n(t)\,\lambda_n^4\vartheta_n(x) = \sum_{n=1}^{\infty}\int\limits_0^l f(t,y)\,\vartheta_n(y)dy\cdot\vartheta_n(x).$$

Hence, scalarly multiplying each term by eigenfunctions $\vartheta_m(x)$, with respect to the Fourier coefficients in (7), we arrive at a countable system of ordinary differential equations with Hilfer operator

$$\left[\left(1 + \lambda_n^2\right)D^{\alpha,\gamma} + \lambda_n^2\right]^2 u_n(t) = \int\limits_0^l f(t,y)\,\vartheta_n(y)dy$$

or

$$\left[D^{\alpha,\gamma}+\mu_n\right]^2 u_n(t)=g_n(t), \tag{9}$$

where

$$\mu_n=\frac{\lambda_n^2}{1+\lambda_n^2},\quad g_n(t)=\frac{1}{1+2\lambda_n^2+\lambda_n^4}\int\limits_0^l f(t,y)\,\vartheta_n(y)dy.$$

We solve the countable system of differential equations (9) and, using the first formula in (7), we determine from (2) the boundary conditions

$$\lim_{t\to+0} I_{0t}^{1-\gamma}u_n(t)=\lim_{t\to+0} I_{0t}^{1-\gamma}\int\limits_0^l u(t,y)\vartheta_n(y)dy=\int\limits_0^l \varphi_1(y)\vartheta_n(y)dy=\varphi_{1,n}, \tag{10}$$

$$u_n(T)=\int\limits_0^l u(T,y)\vartheta_n(y)dy=\int\limits_0^l \varphi_2(y)\vartheta_n(y)dy=\varphi_{2,n}. \tag{11}$$

By virtue of initial value condition (10), we integrate the countable system of differential equations (9) [42-44]:

$$u_n(t)=\left(\varphi_{1,n}t^{\gamma-1}+C_nt\right)t^{\gamma-1}E_{\alpha,\gamma}\left(-\mu_nt^\alpha\right)+\int\limits_0^t (t-s)^\alpha E_{\alpha,\alpha}\left(-\mu_n(t-s)^\alpha\right)g_n(s)ds, \tag{12}$$

where C_n is arbitrary number,

$$E_{\alpha,\gamma}(z)=\sum_{m=0}^{\infty}\frac{z^m}{\Gamma(\alpha\,m+\gamma)},\quad z,\alpha,\gamma\in\mathbb{C},\quad \mathrm{Re}\,(\alpha)>0$$

is Mittag-Leffler function.

To find C_n, we use the final point condition (11)

$$C_n=\varphi_{2,n}\frac{T^{-\gamma}}{E_{\alpha,\gamma}\left(-\mu_nT^\alpha\right)}-\varphi_{1,n}T^{\gamma-1}-$$

$$-\int\limits_0^T \frac{T^{-\gamma}(T-s)^\alpha}{E_{\alpha,\gamma}\left(-\mu_nT^\alpha\right)}E_{\alpha,\alpha}\left(-\mu_n(T-s)^\alpha\right)g_n(s)ds.$$

Substituting the last presentation into the equation (12), we obtain

$$t^2u_n(t)=\varphi_{1,n}P(t)+\varphi_{2,n}Q(t)+$$

$$+\frac{1}{1+2\lambda_n^2+\lambda_n^4}\int\limits_0^T K(t,s)\int\limits_0^l f(s,y)\,\vartheta_n(y)dyds, \qquad (13)$$

where we made some designations:

$$P(t)=\left[1-\left(\frac{t}{T}\right)^{\gamma-1}\right]t^{2\gamma}E_{\alpha,\gamma}\left(-\mu_n t^{\alpha}\right),\quad Q(t)=\frac{E_{\alpha,\gamma}\left(-\mu_n t^{\alpha}\right)}{E_{\alpha,\gamma}\left(-\mu_n T^{\alpha}\right)}t^2\left(\frac{t}{T}\right)^{\gamma},$$

$$K(t,s)=\begin{cases} -t^2\left(\frac{t}{T}\right)^{\gamma}(T-s)^{\alpha}\frac{E_{\alpha,\gamma}(-\mu_n t^{\alpha})}{E_{\alpha,\gamma}(-\mu_n T^{\alpha})}E_{\alpha,\alpha}\left(-\mu_n (T-s)^{\alpha}\right), & t<s\le T,\\ t^2(t-s)^{\alpha}E_{\alpha,\alpha}\left(-\mu_n(t-s)^{\alpha}\right)- & \\ -t^2\left(\frac{t}{T}\right)^{\gamma}(T-s)^{\alpha}\frac{E_{\alpha,\gamma}(-\mu_n t^{\alpha})}{E_{\alpha,\gamma}(-\mu_n T^{\alpha})}E_{\alpha,\alpha}\left(-\mu_n (T-s)^{\alpha}\right), & 0\le s<t. \end{cases}$$

We note that the functions $P(t),\ Q(t),\ K(t,s)$ are smooth. Substituting the presentation (13) into the Fourier series (5), we obtain

$$t^2u(t,x)=\sum_{n=1}^{\infty}\vartheta_n(x)\Bigg[\varphi_{1,n}t^2P(t)+\varphi_{2,n}t^2Q(t)+$$

$$+\frac{1}{1+2\lambda_n^2+\lambda_n^4}\int\limits_0^T t^2K(t,s)\int\limits_0^l f(s,y)\,\vartheta_n(y)dyds\Bigg]. \qquad (14)$$

To establish the uniqueness of the solution to the boundary value problem (1)-(4), we show that a homogeneous boundary value problem $\left[t^2u(t,x)\right]_{t=0}^{t=T}\equiv 0$ has only a trivial solution. So, we suppose that $f(t,x)\equiv 0,\ \forall t\in\Omega_T$ and $\varphi_{\kappa}(x)\equiv 0$. Then $f_n(t)\equiv 0,\ \forall t\in\Omega_T$ and $\varphi_{\kappa,n}=0$. Taking into account formula (6) and conditions (10) and (11) from the representation (14) it follows that

$$t^2u_n(t)=\int\limits_0^l t^2u(t,y)\vartheta_n(y)dy\equiv 0,\ \ n=1,2,\,...$$

Hence, due to the completeness of the systems of eigenfunctions $\{\vartheta_n(x)\}_{n=1}^{\infty}$ in the space $L_2(\Omega_l)$, we conclude that $t^2u(t,x)\equiv 0$ for all $x\in\Omega_l$ and $t\in\Omega_T$. Consequently, if boundary value problem (1)-(4) has a solution, then this solution is unique in the domain Ω.

Smoothness conditions 1. We need in some assumptions in proving theorems. Let be $\varphi_{\kappa}(x)\in C^6\left(\Omega_l\right)$ and $f(t,x)\in C^2\left(\Omega_l\right)$ for fixed values of t. Then, we integrate by parts $\varphi_{\kappa,n}=\int\limits_0^l\varphi_{\kappa}(y)\,\vartheta_n(y)dy$ and $\int\limits_0^l f(t,x)\,\vartheta_n(y)dy$ sixes and two times on the variable x and t we obtain, respectively, the results

$$|\,\varphi_{\kappa,n}\,| \leq \frac{\left|\,\varphi_{\kappa,n}^{(VI)}\,\right|}{\lambda_n^6},\ \ |\,f_n(t)\,| \leq \frac{|\,f_n''(t)\,|}{\lambda_n^2}, \tag{15}$$

where

$$\varphi_{\kappa,n}^{(VI)} = \int\limits_0^l \frac{\partial^6 \varphi_\kappa(y)}{\partial y^6}\,\vartheta_n(y)dy, \quad \kappa = 1, 2, \quad f_n''(t) = \int\limits_0^l \frac{\partial^2 f(t,y)}{\partial y^2}\,\vartheta_n(y)dy.$$

We need also in Bessel's inequalities:

$$\left\|\,\vec{\varphi}_\kappa^{(VI)}\,\right\|_{\ell_2} \leq \left(\frac{2}{l}\right)^3 \left\|\,\frac{\partial^6 \varphi_\kappa(x)}{\partial x^6}\,\right\|_{L_2(\Omega_l)},\ \kappa = 1, 2, \tag{16}$$

$$\left\|\,\vec{f}''(t)\,\right\|_{B_2(\Omega_T)} \leq \frac{2}{l}\left\|\,\frac{\partial^2 f(t,x)}{\partial x^2}\,\right\|_{L_2(\Omega_l)}. \tag{17}$$

We take into account that a Mittag-Leffler function for all $\alpha \in (0,1)$, $\alpha < \gamma \leq 1,\ 0 < \mu_n < 1$ is bounded

$$|\,E_{\alpha,\gamma}\left(-\mu_n t^\alpha\right)| < \infty.$$

We find out the conditions under which the formal solution (14) will be a solution to the direct boundary value problem (1)-(4). Let us show the absolute and uniform convergence of the Fourier series (14). Taking into account the formulas (15), (16), we have

$$t^2\,|\,u(t,x)\,| \leq \sqrt{\frac{2}{l}}\left[\sum_{n=1}^\infty |\,\varphi_{1,n}\,|\,|\,P(t)\,| + \sum_{n=1}^\infty |\,\varphi_{2,n}\,|\,|\,Q(t)\,| + \right.$$

$$\left. + \sum_{n=1}^\infty \frac{1}{\lambda_n^4}\int\limits_0^T |\,K(t,s)\,|\left|\,\int\limits_0^l f(s,y)\,\vartheta_n(y)\,dy\,\right| ds\right] \leq$$

$$\leq \sqrt{\frac{2}{l}}\left[\sum_{n=1}^\infty \frac{\left|\,\varphi_{1,n}^{(VI)}\,\right|}{\lambda_n^6}\max_{0\leq t\leq T}|\,P(t)\,| + \sum_{n=1}^\infty \frac{\left|\,\varphi_{2,n}^{(VI)}\,\right|}{\lambda_n^6}\max_{0\leq t\leq T}|\,Q(t)\,| + \right.$$

$$\left. + \sum_{n=1}^\infty \frac{1}{\lambda_n^4}\max_{0\leq t\leq T}\int\limits_0^T |\,K(t,s)\,|\left|\,\int\limits_0^l f(s,y)\,\vartheta_n(y)\,dy\,\right| ds\right] \leq$$

$$\leq \sqrt{\frac{2}{l}}\left[M_1\sqrt{\sum_{n=1}^{\infty}\frac{1}{\lambda_n^{12}}}\left\|\vec{\varphi}_1^{(VI)}\right\|_{\ell_2}+M_2\sqrt{\sum_{n=1}^{\infty}\frac{1}{\lambda_n^{12}}}\left\|\vec{\varphi}_2^{(VI)}\right\|_{\ell_2}+\right.$$

$$\left.+M_3\sqrt{\sum_{n=1}^{\infty}\frac{1}{\lambda_n^{8}}}\left\|\vec{f}(t)\right\|_{B_2(\Omega_T)}\right]\leq$$

$$\leq \sqrt{\frac{2}{l}}\left[M_1\chi_1\left(\frac{2}{l}\right)^3\left\|\frac{\partial^6\varphi_1(x)}{\partial x^6}\right\|_{L_2(\Omega_l)}+M_2\chi_1\left(\frac{2}{l}\right)^3\left\|\frac{\partial^6\varphi_2(x)}{\partial x^6}\right\|_{L_2(\Omega_l)}+\right.$$

$$\left.+M_3\chi_2\sqrt{\frac{2}{l}}\max_{0\leq t\leq T}\|\,f(t,x)\,\|_{L_2(\Omega_l)}\right]<\infty, \tag{18}$$

where

$$M_1=\max_{0\leq t\leq T}|\,P(t)\,|,\quad M_2=\max_{0\leq t\leq T}|\,Q(t)\,|,\quad M_3=\max_{0\leq t\leq T}\int_0^T|\,K(t,s)\,|\,ds,$$

$$\chi_1=\sqrt{\sum_{n=1}^{\infty}\frac{1}{\lambda_n^{12}}},\quad \chi_2=\sqrt{\sum_{n=1}^{\infty}\frac{1}{\lambda_n^{8}}}.$$

From (18) implies the convergence of series (14).

Now we will show that the function (14) belongs to the class of functions (4). We differentiate the series (14) over t:

$$t^2D^{\alpha,\gamma}u(t,x)=\sum_{n=1}^{\infty}\vartheta_n(x)\left[\varphi_{1,n}D^{\alpha,\gamma}P(t)+\varphi_{2,n}D^{\alpha,\gamma}Q(t)+\right.$$

$$\left.+\frac{1}{1+2\lambda_n^2+\lambda_n^4}\int_0^T D^{\alpha,\gamma}K(t,s)\int_0^l f(s,y)\,\vartheta_n(y)dyds\right]. \tag{19}$$

Now differentiate functions (14) and (19) on x the required number of times:

$$t^2u_{xxxx}(t,x)=\sum_{n=1}^{\infty}\lambda_n^4\vartheta_n(x)\left[\varphi_{1,n}P(t)+\varphi_{2,n}Q(t)+\right.$$

$$\left.+\frac{1}{1+2\lambda_n^2+\lambda_n^4}\int_0^T K(t,s)\int_0^l f(s,y)\,\vartheta_n(y)dyds\right], \tag{20}$$

$$t^2 D^{\alpha,\gamma} u_{xxxx}(t,x)=\sum_{n=1}^{\infty}\lambda_n^4\vartheta_n(x)\Big[\varphi_{1,n}D^{\alpha,\gamma}P(t)+\varphi_{2,n}D^{\alpha,\gamma}Q(t)+$$

$$+\frac{1}{1+2\lambda_n^2+\lambda_n^4}\int\limits_0^T D^{\alpha,\gamma}K(t,s)\int\limits_0^l f(s,y)\,\vartheta_n(y)dyds\Big]. \qquad (21)$$

The proofs of absolute and uniform convergence of series (19)-(21) are similar. Therefore, we will provide a proof of convergence only for series (20). So, by virtue of (15)-(17), for the series (20) we have

$$t^2\,|\,u_{xxxx}(t,x)\,|\le\sqrt{\frac{2}{l}}\left[\sum_{n=1}^{\infty}\lambda_n^4\,|\,\varphi_{1,n}\,|\,|\,P(t)\,|+\sum_{n=1}^{\infty}\lambda_n^4\,|\,\varphi_{2,n}\,|\,|\,Q(t)\,|+\right.$$

$$\left.+\sum_{n=1}^{\infty}\int\limits_0^T|\,K(t,s)\,|\,|\,f_n(s)\,|\,ds\right]\le$$

$$\le\sqrt{\frac{2}{l}}\left[\sum_{n=1}^{\infty}\frac{\left|\varphi_{1,n}^{(VI)}\right|}{\lambda_n^2}\max_{0\le t\le T}|\,P(t)\,|+\sum_{n=1}^{\infty}\frac{\left|\varphi_{2,n}^{(VI)}\right|}{\lambda_n^2}\max_{0\le t\le T}|\,Q(t)\,|+\right.$$

$$\left.+\sum_{n=1}^{\infty}\max_{0\le t\le T}\int\limits_0^T|\,K(t,s)\,|\frac{|\,f_n''(s)\,|}{\lambda_n^2}ds\right]\le$$

$$\le\sqrt{\frac{2}{l}}\sqrt{\sum_{n=1}^{\infty}\frac{1}{\lambda_n^4}}\left[M_1\left\|\vec{\varphi}_1^{(VI)}\right\|_{\ell_2}+M_2\left\|\vec{\varphi}_2^{(VI)}\right\|_{\ell_2}+M_3\left\|\vec{f}''(t)\right\|_{B_2(\Omega_T)}\right]\le$$

$$\le\chi_3\left(\sqrt{\frac{2}{l}}\right)^3\left[M_1\left(\frac{2}{l}\right)^2\left\|\frac{\partial^6\varphi_1(x)}{\partial x^6}\right\|_{L_2(\Omega_l)}+M_2\left(\frac{2}{l}\right)^2\left\|\frac{\partial^6\varphi_2(x)}{\partial x^6}\right\|_{L_2(\Omega_l)}+\right.$$

$$\left.+M_3\max_{0\le t\le T}\left\|\frac{\partial^2 f(t,x)}{\partial x^2}\right\|_{L_2(\Omega_l)}\right]<\infty,\qquad \chi_3=\sqrt{\sum_{n=1}^{\infty}\frac{1}{\lambda_n^4}}.$$

So, we are proved that the following theorem is true.

Theorem 1. *Let the smoothness conditions be fulfilled. Then the direct problem* (1)-(4) *has a unique solution in the form of Fourier series* (14)*.*

3. Inverse boundary value problem

We suppose that in the conditions (2) the function $\varphi_2(x)$ is redefinition function and we study the following problem.

Problem 2. It is required to find a pair of functions $\{u(t,x),\,\varphi_2(x)\}$, that first of them satisfies the differential equation (1), the boundary value conditions (2), the Dirichlet type conditions (3) and belongs to the class (4). Moreover, to find the redefinition function $\varphi_2(x)$ is given the following additional condition

$$u(\bar{t},x)=\psi(x),\quad 0<\bar{t}<T,\quad 0<x<l. \tag{22}$$

For the function $\psi(x)$ is fulfilled the conditions

$$\psi(0)=\psi(l)\;=\psi''(0)=\psi''(l)\;=0.$$

By virtue of additional condition (22), from the Fourier series we obtain

$$\varphi_{2,n}=\psi_n Q^{-1}(\bar{t})-\varphi_{1,n}P(\bar{t})Q^{-1}(\bar{t})-$$

$$-\frac{1}{1+2\lambda_n^2+\lambda_n^4}\int\limits_0^T Q^{-1}(\bar{t})K(\bar{t},s)\int\limits_0^l f(s,y)\,\vartheta_n(y)dyds, \tag{23}$$

where

$$\psi_n=\int\limits_0^l \psi(y)\vartheta_n(y)dy.$$

From (23) we obtain the Fourier series for the redefinition function

$$\varphi_2(x)=\sum_{n=1}^{\infty}\vartheta_n(x)\left[\psi_n Q^{-1}(\bar{t})-\varphi_{1,n}P(\bar{t})Q^{-1}(\bar{t})-\right.$$

$$\left.-\frac{1}{1+2\lambda_n^2+\lambda_n^4}\int\limits_0^T Q^{-1}(\bar{t})K(\bar{t},s)\int_0^l f(s,y)\,\vartheta_n(y)dyds\right]. \tag{24}$$

Smoothness condition 2. Let be $\psi(x)\in C^6\left(\Omega_l\right)$. Then, we integrate by parts $\psi_n=\int\limits_0^l\psi(y)\,\vartheta_n(y)dy$ sixes times on the variable x and we obtain

$$|\,\psi_n\,|\le\lambda_n^{-6}\left|\,\psi_n^{(VI)}\,\right|, \tag{25}$$

where

$$\psi_n^{(VI)} = \int\limits_0^l \frac{\partial^6 \psi(y)}{\partial y^6}\, \vartheta_n(y) dy.$$

Here it is not difficult to prove that Bessel's inequalities are valid:

$$\left\| \vec{\psi} \right\|_{\ell_2} \le \left(\frac{2}{l}\right)^3 \left\| \frac{\partial^6 \psi(x)}{\partial x^6} \right\|_{L_2(\Omega_l)}. \tag{26}$$

Theorem 2. *Let the smoothness conditions 1 and 2 be fulfilled. Then the Fourier series* (24) *absolute and uniform converges on the interval* Ω_l.

Proof. Taking into account the formulas (25), (26), from the series (24) we have

$$|\,\varphi_2(x)\,| \le \sqrt{\frac{2}{l}} \left[\sum_{n=1}^{\infty} |\,\psi_n\,|\,|\,Q^{-1}(\bar{t})\,| + \sum_{n=1}^{\infty} |\,\varphi_{1,n}\,|\,|\,P(\bar{t})\,|\,|\,Q^{-1}(\bar{t})\,| + \right.$$

$$\left. + \sum_{n=1}^{\infty} \frac{1}{\lambda_n^4} \int\limits_0^T |\,Q^{-1}(\bar{t})\,|\,|\,K(\bar{t},s)\,| \left| \int\limits_0^l f(s,y)\, \vartheta_n(y) dy \right| ds \right] \le$$

$$\le \sqrt{\frac{2}{l}} \left[\sum_{n=1}^{\infty} \lambda_n^{-6} \left|\,\psi_n^{(VI)}\,\right| |\,Q^{-1}(\bar{t})\,| + \sum_{n=1}^{\infty} \lambda_n^{-6} \left|\,\varphi_{1,n}^{(VI)}\,\right| |\,P(\bar{t})\,|\,|\,Q^{-1}(\bar{t})\,| + \right.$$

$$\left. + \sum_{n=1}^{\infty} \lambda_n^{-4} \int\limits_0^T |\,Q^{-1}(\bar{t})\,|\,|\,K(\bar{t},s)\,| \left| \int\limits_0^l f(s,y)\, \vartheta_n(y) dy \right| ds \right] \le$$

$$\le \sqrt{\frac{2}{l}} \left[\bar{M}_1 \sqrt{\sum_{n=1}^{\infty} \lambda_n^{-12}} \left\| \vec{\psi}^{(VI)} \right\|_{\ell_2} + \bar{M}_2 \sqrt{\sum_{n=1}^{\infty} \lambda_n^{-12}} \left\| \vec{\varphi}_1^{(VI)} \right\|_{\ell_2} + \right.$$

$$\left. + \bar{M}_3 \sqrt{\sum_{n=1}^{\infty} \lambda_n^{-8}} \left\| \vec{f}(t) \right\|_{B_2(\Omega_T)} \right] \le$$

$$\le \sqrt{\frac{2}{l}} \left[\bar{M}_1 \chi_1 \left(\frac{2}{l}\right)^3 \left\| \frac{\partial^6 \psi(x)}{\partial x^6} \right\|_{L_2(\Omega_l)} + \bar{M}_2 \chi_1 \left(\frac{2}{l}\right)^3 \left\| \frac{\partial^6 \varphi_1(x)}{\partial x^6} \right\|_{L_2(\Omega_l)} + \right.$$

$$\left. + \bar{M}_3 \chi_2 \sqrt{\frac{2}{l}} \max_{0 \le t \le T} \|\, f(t,x)\,\|_{L_2(\Omega_l)} \right] < \infty, \tag{27}$$

where

$$\bar{M}_1 = \left| Q^{-1}(\bar{t}) \right|, \quad \bar{M}_2 = \left| P(t) \right| \left| Q^{-1}(\bar{t}) \right|, \quad \bar{M}_3 = T \left| Q^{-1}(\bar{t}) \right| \left| K(\bar{t}, s) \right|.$$

From (27) implies the convergence of series (14). Theorem 2 is proved.

Substituting the presentation (23) into the Fourier series (14), we obtain

$$t^2 u(t,x) = \sum_{n=1}^{\infty} \vartheta_n(x) \Bigg[\varphi_{1,n} P_1(t) + \psi_n Q_1(t) +$$

$$+ \frac{1}{1 + 2\lambda_n^2 + \lambda_n^4} \int\limits_0^T K_1(t,s) \int\limits_0^l f(s,y)\, \vartheta_n(y) dy ds \Bigg], \tag{28}$$

where

$$P_1(t) = t^2 P(t) - t^2 Q(t) P(\bar{t}) Q^{-1}(\bar{t}), \quad Q_1(t) = t^2 Q^{-1}(\bar{t}) Q(t),$$

$$K_1(t,s) = t^2 K(t,s) - t^2 Q(t) Q^{-1}(\bar{t}) K(\bar{t}, s).$$

Theorem 3. *Let the smoothness conditions 1 and 2 be fulfilled. Then the Fourier series* (28) *absolute and uniform converges on the domain* Ω*.*

The *proof* of the theorem 3 is similar to the proof of the theorem 1 and 2.

Conclusion

It were found the functions $u(t,x)$ and $\phi_2(x)$, that first function $u(t,x)$ satisfies the fractional pseudoparabolic differential equation (1), the two-point boundary value conditions (2), the Dirichlet type conditions (3), belongs to the class of functions (4) and additonal condition (5). The functions $\phi_1(x)$, $\phi_2(x)$, $f(t,s)$ satisfy the first smoothness conditions and $\psi(x)$ satisfies second smoothness condition.

The unique solvability theorems and ways of construction of a solution of an inverse boundary value problem for an inhomogeneous differential equation containing a quadrat of Hilfer fractional analogue of the pseudoparabolic operator are studied. The solution of the direct and inverse boundary value problems are obtained in the form of Fourier series. Sufficient coefficient conditions for unique solvability of these problems are established. Theorems on absolute and uniform convergence of Fourier series (14), (19)-(21), (24), (28), are proved.

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