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Estimates for the Kolmogorov Widths of the Nikol'skii–Besov–Amanov Classes in the Lorentz Space

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Abstract—The Lorentz space with anisotropic norm of periodic functions of several variables is considered. Estimates for the Kolmogorov widths of the Nikol'skii–Besov–Amanov classes in the Lorentz space with anisotropic norm are found.

Keywords: Lorentz space, Nikol'skii–Besov class, Kolmogorov width.

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INTRODUCTION

Let $\bar{x} = (x_1, \ldots, x_m) \in I^m = [0, 2\pi)^m$ and $\theta_j, q_j \in [1, +\infty)$, $j = 1, \ldots, m$.

Denote by $S_0(\mathbb{R}^m)$ the class of all Lebesgue measurable almost everywhere finite functions f on $\mathbb{R}^m$ such that, for any number $y > 0$, the Lebesgue measure of the set $\{\bar{x} = (x_1, \ldots, x_m) \in \mathbb{R}^m : |f(\bar{x})| > y\}$ is finite; the measure of this set is called the *distribution function* and is denoted by $\lambda_f(y)$ (see, for example, [1, 2]). Functions $f, g \in S_0(\mathbb{R}^m)$ are called *equimeasurable* if their distribution functions are equal.

The *nonincreasing permutation of a function* $f \in S_0(\mathbb{R}^m)$ is the function f^* nonincreasing on the half-line $(0, +\infty)$ and equimeasurable with $|f|$; it is defined by the formula $f^*(t) = \inf\{y > 0 : \lambda_f(y) < t\}$, $t > 0$ (see, for example, [1, 2]).

Now, consider repeated permutations of a function. Let $f \in S_0(\mathbb{R}^m)$. For fixed values $x_2, \ldots, x_m$, regarding $|f(x_1, x_2, \ldots, x_m)|$ as a function of x_1, we can define its nonincreasing permutation $f^{*_1}(t_1, x_2, \ldots, x_m)$ in x_1. Further, for fixed $t_1, x_3, \ldots, x_m$, for the function $f^{*_1}(t_1, x_2, x_3 \ldots, x_m)$, we define its nonincreasing permutation $f^{*_1,*_2}(t_1, t_2, x_3, \ldots, x_m)$ in x_2. Continuing this process with fixed $t_1, t_2, \ldots, t_{m-1}$, we define (see [2, 3]) for the function $f^{*_1,\ldots,*_{m-1}}(t_1, t_2, t_3 \ldots, t_{m-1}, x_m)$ its nonincreasing permutation $f^{*_1,*_2,\ldots,*_{m-1},*_m}(t_1, t_2, \ldots, t_m)$ in x_m.

Denote by $L^*_{\bar{q},\bar{\theta}}(I^m)$ the space of all Lebesgue measurable functions $f(\bar{x})$ that are 2π-periodic in each variable and have a finite value of the norm (see [3])

$$\|f\|^*_{\bar{q},\bar{\theta}} = \left[\int_0^{2\pi} t_m^{\frac{\theta_m}{q_m}-1}\left[\cdots\left[\int_0^{2\pi}\Big(f^{*_1,\ldots,*_m}(t_1,\ldots,t_m)\Big)^{\theta_1} t_1^{\frac{\theta_1}{q_1}-1}dt_1\right]^{\frac{\theta_2}{\theta_1}}\cdots\right]^{\frac{\theta_m}{\theta_{m-1}}}dt_m\right]^{\frac{1}{\theta_m}}.$$

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In the case $q_1 = \cdots = q_m = \theta_1 = \cdots = \theta_m = q$, the space $L^*_{\overline{q},\overline{\theta}}(I^m)$ coincides with the Lebesgue space $L_q(I^m)$ (see [4, Ch. 1, Sect. 1.1]). In what follows, $\overset{\circ}{L}{}^*_{\overline{q},\overline{\theta}}(I^m)$ denotes the set of all functions $f \in L^*_{\overline{q},\overline{\theta}}(I^m)$ such that $\int_0^{2\pi} f(\overline{x})\,dx_j = 0$ for $j = 1,\ldots,m$.

For a function $f \in L_1(I^m) = L(I^m)$, we consider its Fourier series $\sum_{\overline{n}\in\mathbb{Z}^m} a_{\overline{n}}(f)\, e^{i\langle\overline{n},\overline{x}\rangle}$ in the multiple trigonometric system $\{e^{i\langle\overline{n},\overline{x}\rangle}\}_{\mathbb{Z}^m}$, where $\mathbb{Z}^m$ is the integer lattice in $\mathbb{R}^m$. Define

$$\delta_{\overline{s}}(f,\overline{x}) = \sum_{\overline{n}\in\rho(\overline{s})} a_{\overline{n}}(f)\, e^{i\langle\overline{n},\overline{x}\rangle}, \quad \text{where} \quad \langle\bar{y},\bar{x}\rangle = \sum_{j=1}^m y_j x_j, \quad s_j = 1,2,\ldots,$$

$$\rho(\bar{s}) = \left\{\overline{k} = (k_1,\ldots,k_m) \in \mathbb{Z}^m\colon 2^{s_j-1} \le |k_j| < 2^{s_j},\ j = 1,\ldots,m\right\}.$$

Let l_p^m for $1 \le p \le \infty$ denote the space $\mathbb{R}^m$ with the norm

$$\|\bar{x}\|_{l_p^m} = \left(\sum_{j=1}^m |x_j|^p\right)^{1/p} \quad \text{for} \quad 1 \le p < +\infty, \quad \|\bar{x}\|_{l_\infty^m} = \max_{j=1,\ldots,m} |x_j|.$$

Denote by U_p^m the unit ball in l_p^m.

A numerical sequence $\{a_{\overline{n}}\}_{\overline{n}\in\mathbb{Z}^m}$ is an element of $l_{\overline{p}}$ if

$$\left\|\{a_{\overline{n}}\}_{\overline{n}\in\mathbb{Z}^m}\right\|_{l_{\overline{p}}} = \left\{\sum_{n_m=-\infty}^{\infty}\left[\cdots\left[\sum_{n_1=-\infty}^{\infty}|a_{\overline{n}}|^{p_1}\right]^{\frac{p_2}{p_1}}\cdots\right]^{\frac{p_m}{p_{m-1}}}\right\}^{\frac{1}{p_m}} < +\infty,$$

where $\overline{p} = (p_1,\ldots,p_m)$ with $1 \le p_j < +\infty$ for $j = 1,2,\ldots,m$.

We denote by $C(p,q,r,y)$ positive values depending on the parameters specified in the brackets; these values can be different in different formulas. For positive values $A(y)$ and $B(y)$, we will write $A(y) \asymp B(y)$ if there exist positive numbers C_1 and C_2 such that $C_1 A(y) \le B(y) \le C_2 A(y)$.

Spaces of functions with dominating mixed derivatives $S_p^{\bar{r}}H$ and $S_{p,\theta}^{\bar{r}}B$ were defined by Nikol'skii [5] and Amanov [6], respectively, in terms of mixed moduli of smoothness of the corresponding mixed derivatives (see also [4, 7]).

Lizorkin and Nikol'skii [8, Theorem 5.3] studied the decompositional expansion of elements of the space $\overset{\circ}{S}{}^{\overline{r}}_{p,\theta}B$ and proved that the value

$$\left\{\sum_{\bar{s}\in\mathbb{Z}_+^m} 2^{\langle\overline{s},\overline{r}\rangle\theta}\|\delta_{\overline{s}}(f)\|_p^\theta\right\}^{1/\theta}$$

is a norm of the space $\overset{\circ}{S}{}^{\bar{r}}_{p,\theta}B$ equivalent to the original norm for $1 < p < +\infty$ and $1 \le \theta \le +\infty$. Here, $\overset{\circ}{S}{}^{\bar{r}}_{p,\theta}B = \overset{\circ}{L}_p(I^m) \cap S^{\bar{r}}_{p,\theta}B$.

Let us consider the similar space in the anisotropic Lorentz space $\overset{\circ}{L}{}^*_{\overline{p},\overline{\theta}}(I^m)$. Denote by $\overset{\circ}{S}{}^{\overline{r}}_{\overline{p},\overline{\theta},\overline{\tau}}B$ the space of all functions $f \in \overset{\circ}{L}{}^*_{\overline{p},\overline{\theta}}(I^m)$ for which

$$\|f\|_{\overset{\circ}{S}{}^{\overline{r}}_{\overline{p},\overline{\theta},\overline{\tau}}B} = \left\|\left\{2^{\langle\overline{s},\overline{r}\rangle}\|\delta_{\overline{s}}(f)\|^*_{\overline{p},\overline{\theta}}\right\}_{\bar{s}\in\mathbb{Z}_+^m}\right\|_{l_{\overline{\tau}}} < \infty,$$

where $\overline{p} = (p_1, \ldots, p_m)$, $\overline{\theta} = (\theta_1, \ldots, \theta_m)$, $\overline{\tau} = (\tau_1, \ldots, \tau_m)$, and $\overline{r} = (r_1, \ldots, r_m)$ with $1 < p_j < \infty$, $1 \le \theta_j < \infty$, $1 \le \tau_j \le +\infty$, and $r_j > 0$ for $j = 1, \ldots, m$. In this space, we consider (with the same notation) the class

$$\overset{\circ}{S}{}^{\overline{r}}_{\overline{p},\overline{\theta},\overline{\tau}}B = \Big\{ f \in \overset{\circ}{L}{}^{*}_{\overline{p},\overline{\theta}}(I^m) : \ \|f\|_{\overset{\circ}{S}{}^{\overline{r}}_{\overline{p},\overline{\theta},\overline{\tau}}B} = \Big\| \Big\{ 2^{\langle \overline{s}, \overline{r}\rangle} \|\delta_{\overline{s}}(f)\|^{*}_{\overline{p},\overline{\theta}} \Big\}_{\overline{s}\in\mathbb{Z}^m_+} \Big\|_{l_{\overline{\tau}}} \le 1 \Big\}.$$

The notion of width as an approximation characteristic of subsets of a Banach space was introduced by Kolmogorov [9].

Definition. Let W be a centrally symmetric set in a Banach space. The Kolmogorov M-width of W is the value

$$d_M(W, X) = \inf_{P} \sup_{f \in W} \|f - Pf\|_X,$$

where the infimum is taken over all mappings P acting in X to linear subspaces of dimension at most M.

In the scalar case, the following bound is known for a Sobolev class W^r_p under the condition $r > (1/p - 1/q)_+$:

$$d_M(W^r_p, L_q) \asymp \begin{cases} M^{-r}, & 1 \le q \le p \le \infty, \ r > 0, \\ M^{-r+\frac{1}{p}-\frac{1}{q}}, & 1 < p \le q < 2, \ r > 1/p - 1/q, \\ M^{-r+\frac{1}{p}-\frac{1}{2}}, & 1 < p \le 2 \le q \le \infty, \ r > 1/p, \\ M^{-r}, & 2 \le p \le q < \infty, \ r > \dfrac{1/p - 1/q}{1 - 2/q}. \end{cases}$$

For $p = q = 2$, the exact values of $d_M(W^r_p, L_q)$ were obtained by Kolmogorov [9]; in the case $q = p = 2$ and $r = 1$, the order was found by Rudin [10]; and, for the cases $p = 1$ and $q = 2$ and $p = q = \infty$, the orders were found by Stechkin [11]. For $p = q = \infty$, the exact value was obtained by Tikhomirov [12]; for $1 \le p = q < \infty$, the orders were found by Babadzhanov and Tikhomirov [13]; and, for $p = q = 1$ and odd M, the exact value was established by Subbotin [14,15]. For $1 \le p \le q \le 2$, the orders were found by Ismagilov [16, 17], Gluskin [18] (the case $p = 1$ and $2 < q \le \infty$), Kashin [19] (the case $1 \le p \le q \le \infty$ and $q \ge 2$), and Makovoz [20] (for $1 \le q \le p$). Kashin [21] and Kulanin [22] found bounds for the widths of Sobolev classes of small smoothness.

Bounds for the Kolmogorov widths of classes of smooth functions of several variables were established by Babenko [23], Mityagin [24], Telyakovskii [25], Bugrov [26], Galeev [27–29], Temlyakov [30, 31], Din' Zung [32], Belinskii [33, 34], Romanyuk [35–37], Heping Wang and Yongsheng Sun [38], Bazarkhanov [39], and others.

For the Besov class $S^r_{p,\theta}B$, Romanyuk [35] proved the following theorem.

Theorem A. *Let $r_1 = \ldots = r_\nu < r_{\nu+1} < \ldots < r_m$ and $1 \le \theta \le \infty$.*

1. *If $2 \le p < q < \infty$ and $r_1 > \beta = (1/p - 1/q)/(1 - 2/q)$, then*

$$d_M(S^r_{p,\theta}B, L_q) \asymp \Big(\frac{\log^{\nu-1} M}{M}\Big)^{r_1} (\log M)^{(\nu-1)\left(\frac{1}{2}-\frac{1}{\theta}\right)_+}.$$

2. *If $1 < p \le 2 < q < \infty$ and $r_1 > 1/p$, then*

$$d_M(S^r_{p,\theta}B, L_q) \asymp \Big(\frac{\log^{\nu-1} M}{M}\Big)^{r_1-\frac{1}{p}+\frac{1}{2}} (\log M)^{(\nu-1)\left(\frac{1}{2}-\frac{1}{\theta}\right)_+}.$$

Hereinafter, $\log M$ is the logarithm to base 2 of the number $M > 1$.

For other relations between p and q, bounds for $d_M(S^r_{p,\theta}B, L_q)$ were obtained by Galeev [29].

The aim of this paper is to find bounds for the Kolmogorov width of the class $\mathring{S}^{\bar r}_{\bar p,\bar\theta,\bar\tau}B$ in the space $L^*_{\bar q,\bar\theta}(I^m)$.

1. AUXILIARY STATEMENTS

Let us present some additional notation and auxiliary statements.

Suppose that $\bar s \in \mathbb{Z}^m_+$ and $\Im(\rho(\bar s))$ is the set of functions of the form $f(\bar x) = \sum_{\bar n \in \rho(\bar s)} a_{\bar n} e^{i\langle \bar n, \bar x\rangle}$.

Lemma 1 [29, Lemma D]. *Suppose that $\bar s \in \mathbb{N}^m$, $f \in \Im(\rho(\bar s))$, $M_{\bar s} \in Z_+$, $M_{\bar s} \le 2^{\langle \bar s, \bar 1\rangle} = \prod_{j=1}^m 2^{s_j}$, $\bar\alpha = (\alpha_1, \dots, \alpha_m)$, $\alpha_j = r_j + 1/q - 1/p$ for $j = 1, \dots, m$, and $1 < p, q < +\infty$. Then there exists a linear subspace $\mathbb{L}_{M_s} \subset \Im(\rho(\bar s))$ of dimension $M_{\bar s}$ and an operator $P_{M_{\bar s}} : \Im(\rho(\bar s)) \to \mathbb{L}_{M_s}$ such that*

$$\|\delta_{\bar s}(f) - P_{M_{\bar s}}\delta_{\bar s}(f)\|_q \asymp d_{M_{\bar s}}\Big(U_p^{2^{\langle \bar s, \bar 1\rangle}}, l_q^{2^{\langle \bar s, \bar 1\rangle}}\Big) 2^{\langle \bar s, -\bar\alpha\rangle} \|\delta_{\bar s}(f^{(r)})\|_p.$$

For a vector $\bar\gamma = (\gamma_1, \dots, \gamma_m)$ with $\gamma_j > 0$ for $j = 1, \dots, m$, define

$$Y^m(\overline{\gamma}, n) = \left\{\overline{s} = (s_1, \dots, s_m) \in \mathbb{Z}^m_+ : \langle \overline{x}, \overline{\gamma}\rangle \ge n\right\}.$$

Lemma 2 (see [40]). *Let $\alpha \in (0, +\infty)$, and let vectors $\overline{\gamma} = (\gamma_1, \dots, \gamma_m)$, $\overline{\gamma}' = (\gamma_1', \dots, \gamma_m')$, and $\overline{\theta} = (\theta_1, \dots, \theta_m)$ be such that $\theta_j \in [1, +\infty)$ for $j = 1, \dots, m$, $1 = \gamma_1 = \dots = \gamma_\nu < \gamma_{\nu+1} \le \dots \le \gamma_m$, $1 = \gamma_j' = \gamma_j$ for $j = 1, \dots, \nu$, and $1 = \gamma_j' < \gamma_j$ for $j = \nu + 1, \dots, m$. Then*

$$\left\|\left\{2^{-\alpha\langle \overline{s}, \overline{\gamma}\rangle}\right\}_{\overline{s} \in Y^m(\overline{\gamma}', n)}\right\|_{l_{\overline{\theta}}} \asymp 2^{-n\alpha} n^{\sum_{j=2}^{\nu} \frac{1}{\theta_j}}.$$

2. MAIN RESULTS

In this section we prove bounds for the Kolmogorov width in the space $L^*_{\bar p,\bar q}(I^m)$ of the Nikol'skii–Besov–Amanov class $\mathring{S}^{\overline{r}}_{\overline{p},\overline{\theta},\overline{\tau}}B$.

Theorem 1. *Suppose that $1 < p_j < q_j \le 2$, $1 < \theta_j^{(1)}, \theta_j^{(2)} < \infty$, and $1 \le \tau_j \le \infty$ for $j = 1, \dots m$. Suppose also that $0 < r_1 + 1/q_1 - 1/p_1 = \dots = r_\nu + 1/q_\nu - 1/p_\nu < r_{\nu+1} + 1/q_{\nu+1} - 1/p_{\nu+1} \le \dots \le r_m + 1/q_m - 1/p_m$. Then*

$$d_M\Big(\mathring{S}^{\overline{r}}_{\overline{p},\overline{\theta}^{(1)},\overline{\tau}}B,\ L^*_{\bar q,\bar\theta^{(2)}}\Big) \le C\Big(\frac{\log^{\nu-1} M}{M}\Big)^{\left(r_1 + \frac{1}{q_1} - \frac{1}{p_1}\right)} (\log M)^{\sum_{j=2}^{\nu}\left(\frac{1}{2} - \frac{1}{\tau_j}\right)_+}.$$

Proof. Assume that $f \in \mathring{S}^{\overline{r}}_{\overline{p},\overline{\theta}^{(1)},\overline{\tau}}B$ and $\gamma_j = (r_j + 1/q_j - 1/p_j)/(r_1 + 1/q_1 - 1/p_1)$ for $j = 1, \dots, m$. Then it follows from the conditions of the theorem that $1 = \gamma_1 = \dots = \gamma_\nu < \gamma_{\nu+1} \le \dots \le \gamma_m$. Choose numbers γ_j' such that $\gamma_j' = \gamma_j$ for $j = 1, \dots, \nu$ and $\gamma_j' < \gamma_j$ for $j = \nu + 1, \dots, m$. Consider the partial sum

$$S_n^{\gamma'}(f, \bar x) = \sum_{\langle \bar s, \bar\gamma'\rangle < n} \delta_{\bar s}(f, \bar x)$$

of the Fourier series of the function $f \in \overset{\circ}{S}{}^{\overline{r}}_{\overline{p},\overline{\theta}^{(1)},\overline{\tau}}B$ over the stepwise hyperbolic cross.

In [40] it was proved that

$$\sup_{f\in \overset{\circ}{S}{}^{\bar r}_{\bar p,\bar\theta^{(1)},\bar\tau}B} \|f - S_n^{\gamma'}(f)\|^*_{\bar q,\bar\theta^{(2)}} \le C\, 2^{-n\left(r_1+\frac{1}{q_1}-\frac{1}{p_1}\right)} n^{\sum_{j=2}^{\nu}\left(\frac{1}{\theta_j^{(2)}}-\frac{1}{\tau_j}\right)}, \quad \theta_j^{(2)} < \tau_j, \quad j=1,\dots,m,$$

$$\sup_{f\in \overset{\circ}{S}{}^{\bar r}_{\bar p,\bar\theta^{(1)},\bar\tau}B} \|f - S_n^{\gamma'}(f)\|^*_{\bar q,\bar\theta^{(2)}} \le C\, 2^{-n\left(r_1+\frac{1}{q_1}-\frac{1}{p_1}\right)}, \quad \tau_j \le \theta_j^{(2)}, \quad j=1,\dots,m.$$

Choose a positive integer n such that $M \asymp 2^n n^{\nu-1}$. Then

$$d_M\Big(\overset{\circ}{S}{}^{\overline{r}}_{\overline{p},\overline{\theta}^{(1)},\overline{\tau}}B,\ L^*_{\bar q,\bar\theta^{(2)}}\Big) \le \sup_{f\in \overset{\circ}{S}{}^{\bar r}_{\bar p,\bar\theta^{(1)},\bar\tau}B} \|f - S_n^{\gamma'}(f)\|^*_{\bar q,\bar\theta^{(2)}}$$

$$\le C\, 2^{-n\left(r_1+\frac{1}{q_1}-\frac{1}{p_1}\right)} n^{\sum_{j=2}^{\nu}\left(\frac{1}{\theta_j^{(2)}}-\frac{1}{\tau_j}\right)} \le C\Big(\frac{\log^{\nu-1}M}{M}\Big)^{\left(r_1+\frac{1}{q_1}-\frac{1}{p_1}\right)} (\log M)^{\sum_{j=2}^{\nu}\left(\frac{1}{2}-\frac{1}{\tau_j}\right)}$$

if $\theta_j^{(2)} < \tau_j$ for $j=1,\dots,m$, and

$$d_M\Big(\overset{\circ}{S}{}^{\overline{r}}_{\overline{p},\overline{\theta}^{(1)},\overline{\tau}}B,\ L^*_{\bar q,\bar\theta^{(2)}}\Big) \le C\, 2^{-n\left(r_1+\frac{1}{q_1}-\frac{1}{p_1}\right)} \le C\Big(\frac{\log^{\nu-1}M}{M}\Big)^{\left(r_1+\frac{1}{q_1}-\frac{1}{p_1}\right)}$$

if $\tau_j \le \theta_j^{(2)}$ for $j=1,\dots,m$. Theorem 1 is proved. □

Theorem 2. *Suppose that* $1 < \theta_j^{(1)}, \theta_j^{(2)} < \infty$ *and* $1 \le \tau_j \le \infty$ *for* $j=1,\dots m$. *Suppose also that* $0 < r_1+1/q_1-1/p_1 = \dots = r_\nu+1/q_\nu-1/p_\nu < r_{\nu+1}+1/q_{\nu+1}-1/p_{\nu+1} \le \dots \le r_m+1/q_m-1/p_m$.

1. *If* $2 \le p_j < q \le \theta_j^{(2)} < \infty$ *for* $j=1,\dots,m$ *and* $r_1 > \beta = (1/p_1-1/q)/(1-2/q)$, *then*

$$d_M\Big(\overset{\circ}{S}{}^{\overline{r}}_{\overline{p},\overline{\theta}^{(1)},\overline{\tau}}B,\ L^*_{q,\bar\theta^{(2)}}\Big) \le C\Big(\frac{\log^{\nu-1}M}{M}\Big)^{r_1} (\log M)^{\sum_{j=2}^{\nu}\left(\frac{1}{2}-\frac{1}{\tau_j}\right)_+}.$$

2. *If* $1 < p_j \le 2 < q \le \theta_j^{(2)}$ *and* $r_j > 1/p_j$ *for* $j=1,\dots,m$, *then*

$$d_M\Big(\overset{\circ}{S}{}^{\overline{r}}_{\overline{p},\overline{\theta}^{(1)},\overline{\tau}}B, L^*_{q,\bar\theta^{(2)}}\Big) \le C\Big(\frac{\log^{\nu-1}M}{M}\Big)^{\left(r_1+\frac{1}{2}-\frac{1}{p_1}\right)} (\log M)^{\sum_{j=2}^{\nu}\left(\frac{1}{2}-\frac{1}{\tau_j}\right)_+}.$$

Proof. Let us prove the first statement of the theorem. Let $f \in \overset{\circ}{S}{}^{\bar r}_{\bar p,\bar\theta^{(1)},\bar\tau}B$. Since $q \le \theta_j^{(2)}$ for $j=1,\dots,m$, we have $L_q(I^m) \subset L^*_{\bar q,\bar\theta^{(2)}}(I^m)$ and $\|f\|^*_{q,\bar\theta^{(2)}} \le C\|f\|_q$.

Since $2 < q < \infty$, we have $0 < 1-2/q < 1$. Hence, $r_1 > \beta = (1/p_1-1/q)(1-2/q) > 1/p_1-1/q$. Therefore, the condition $0 < r_1+1/q_1-1/p_1 = \dots = r_\nu+1/q_\nu-1/p_\nu < r_{\nu+1}+1/q_{\nu+1} - 1/p_{\nu+1} \le \dots \le r_m+1/q_m-1/p_m$ implies that $r_j > 1/p_j - 1/q_j$ for $j=1,\dots,m$. Consequently, $\overset{\circ}{S}{}^{\overline{r}}_{\overline{p},\overline{\theta}^{(1)},\overline{\tau}}B \subset L_q(I^m)$ (see [40]); i.e., $f \in L_q(I^m)$.

As in [29, p. 608], we consider the operator $P_M : L_q(I^m) \to \mathbb{L}_M$ acting on a function

$$f(\bar x) = \sum_{\bar s\in\mathbb{Z}^m_+} \delta_{\bar s}(f,\bar x) \quad \text{by the rule} \quad (P_M f)(\bar x) = \sum_{\bar s\in\mathbb{Z}^m_+} P_{M_{\bar s}}\delta_{\bar s}(f,\bar x),$$

where $P_{M_{\bar{s}}}$ and $\mathbb{L}_{M_{\bar{s}}}$ are the operators and subspaces from Lemma 1 and $\mathbb{L}_M = \bigcup_{\bar{s}\in\mathbb{Z}_+^m} \mathbb{L}_{M_{\bar{s}}}$.

Define $\rho_j = r_j + 1/q - 1/p_j$ and $\gamma_j = \rho_j/\rho_1$. Then it is easily seen from the condition of the theorem that $\gamma_1 = \dots = \gamma_\nu < \gamma_{\nu+1} \le \dots \le \gamma_m$.

It is known that there exist numbers $\rho_j^{'}$ such that $\rho_j^{'} = \rho_j = \rho_1$ for $j = 1, \dots, \nu$, $\rho_1 < \rho_j^{'}$ for $j = \nu+1, \dots, m$, and $1 < \rho_{\nu+1}/\rho_{\nu+1}^{'} < \dots < \rho_m/\rho_m^{'}$. Define $\gamma_j^{'} = \rho_j^{'}/\rho_1^{'}$ for $j = 1, \dots, m$. Then $\gamma_j^{'} = \gamma_j$ for $j = 1, \dots, \nu$ and $\gamma_j^{'} < \gamma_j$ for $j = \nu+1, \dots, m$.

For a positive integer M, we choose a number $n \in \mathbb{N}$ such that $M \asymp 2^n n^{\nu-1}$. For each $\bar{s} \in \mathbb{Z}_+^m$, we define the number

$$M_{\bar{s}} = \begin{cases} 2^{\langle \bar{s}, \bar{1}\rangle} & \text{if } \langle \bar{s}, \bar{\gamma}'\rangle \le n, \\ 2^{n(1+\varepsilon\rho_1) - \varepsilon\langle \bar{s}, \bar{\rho}\rangle} & \text{if } \langle \bar{s}, \bar{\gamma}'\rangle > n, \end{cases}$$

where the positive number ε will be chosen later and $[a]$ is the integer part of a. Then, using Lemmas C and D from [30], we find that $\sum_{\bar{s}\in\mathbb{Z}_+^m} M_{\bar{s}} \le C\, 2^n n^{\nu-1} \le CM$. Thus, the dimension of the subspace $\mathbb{L}_M$ has order $2^n n^{\nu-1}$.

Applying the inequality $\|f\|_{q,\bar{\theta}^{(2)}}^* \le C\|f\|_q$ to the function $f - P_M f \in L_q(I^m)$ and using the Littlewood–Paley theorem in the Lebesgue space $L_q(I^m)$ (see [4, p. 54]), we get

$$\|f - P_M f\|_{q,\bar{\theta}^{(2)}}^* \le C\|f - P_M f\|_q$$

$$\le C\bigg\| \bigg(\sum_{\bar{s}\in\mathbb{Z}_+^m} |\delta_{\bar{s}}(f) - P_{M_{\bar{s}}}\delta_{\bar{s}}(f)|^2 \bigg)^{1/2} \bigg\|_q \le C\bigg(\sum_{\bar{s}\in\mathbb{Z}_+^m} \|\delta_{\bar{s}}(f) - P_{M_{\bar{s}}}\delta_{\bar{s}}(f)\|_q^2 \bigg)^{1/2}. \tag{2.1}$$

Choose a number $p_0 > \max_{j=1,\dots,m} p_j$. Then, by Lemma 1, using Bernstein's inequality for trigonometric polynomials (see [4, p. 98]), we obtain

$$\|\delta_{\bar{s}}(f) - P_{M_{\bar{s}}}\delta_{\bar{s}}(f)\|_q \asymp d_{M_{\bar{s}}}\Big(B_{p_0}^{2^{\langle \bar{s},\bar{1}\rangle}}, l_q^{2^{\langle \bar{s},\bar{1}\rangle}} \Big) \prod_{j=1}^m 2^{s_j\left(\frac{1}{p_0} - \frac{1}{q}\right)} \|\delta_{\bar{s}}(f)\|_{p_0}. \tag{2.2}$$

Since $p_j < p_0$ for $j = 1, \dots, m$, using Nilkol'skii's inequality for different metrics of trigonometric polynomials (see [41, 42]), we get

$$\|\delta_{\bar{s}}(f)\|_{p_0} \le C \prod_{j=1}^m 2^{s_j\left(\frac{1}{p_j} - \frac{1}{p_0}\right)} \|\delta_{\bar{s}}(f)\|_{\bar{p},\bar{\theta}^{(1)}}^*. \tag{2.3}$$

It follows from inequalities (2.1)–(2.3) that

$$\|f - P_M f\|_{q,\bar{\theta}^{(2)}}^* \le C\bigg(\sum_{\bar{s}\in\mathbb{Z}_+^m} \bigg(\prod_{j=1}^m 2^{s_j\left(\frac{1}{p_j} - \frac{1}{q}\right)} \|\delta_{\bar{s}}(f^{(r)})\|_{\bar{p},\bar{\theta}^{(1)}}^* \bigg)^2 d_{M_{\bar{s}}}^2\Big(B_{p_0}^{2^{\langle \bar{s},\bar{1}\rangle}}, l_q^{2^{\langle \bar{s},\bar{1}\rangle}} \Big) \bigg)^{1/2}. \tag{2.4}$$

If $\tau_j \le 2$ for $j = 1, \dots, m$, then, using Jensen's inequality (see [4, p. 125]), we find from (2.4) that

$$\|f - P_M f\|_{q,\bar{\theta}^{(2)}}^* \le C\bigg\| \bigg\{ \prod_{j=1}^m 2^{s_j\left(\frac{1}{p_j} - \frac{1}{q}\right)} d_{M_{\bar{s}}}\Big(B_{p_0}^{2^{\langle \bar{s},\bar{1}\rangle}}, l_q^{2^{\langle \bar{s},\bar{1}\rangle}} \Big) \|\delta_{\bar{s}}(f)\|_{\bar{p},\bar{\theta}^{(1)}}^* \bigg\}_{\bar{s}\in\mathbb{Z}_+^m} \bigg\|_{\bar{\tau}}. \tag{2.5}$$

If $2 < \tau_j < +\infty$ for $j = 1, \dots, m$, then, applying Hölder's inequality (for $\alpha_j = \tau_j/2 > 1$, $j = 1, \dots, m$), we find from (2.4) that

$$\|f - P_M f\|_{q,\bar{\theta}^{(2)}}^* \le C\Big\| \Big\{ 2^{\langle \bar{s},\bar{r}\rangle} \|\delta_{\bar{s}}(f)\|_{\bar{p},\bar{\theta}^{(1)}}^* \Big\}_{\bar{s}\in\mathbb{Z}_+^m} \Big\|_{\bar{\tau}}$$

$$\times \left\| \left\{ \prod_{j=1}^m 2^{-s_j\left(r_j+\frac{1}{q}-\frac{1}{p_j}\right)} d_{M_{\bar s}}\left(B_{p_0}^{2^{\langle \bar s,\bar 1\rangle}}, l_q^{2^{\langle \bar s,\bar 1\rangle}}\right)\right\}_{\langle \bar s,\gamma'\rangle>n}\right\|_{\bar\varepsilon}, \tag{2.6}$$

where $\bar\varepsilon=(\varepsilon_1,\dots,\varepsilon_m)$ and $\varepsilon_j=2\tau_j/(\tau_j-2)$ for $j=1,\dots,m$.

By the definition of the Kolmogorov width, it follows from (2.5) and (2.6) that

$$d_M\left(\mathring{S}^{\bar r}_{\bar p,\bar\theta^{(1)},\bar\tau}B, L^*_{q,\bar\theta^{(2)}}\right) \le C \sup_{\bar s\in\mathbb{Z}^m_+} \prod_{j=1}^m 2^{-s_j\left(r_j+\frac{1}{q}-\frac{1}{p_j}\right)} d_{M_{\bar s}}\left(B_{p_0}^{2^{\langle \bar s,\bar 1\rangle}}, l_q^{2^{\langle \bar s,\bar 1\rangle}}\right) \tag{2.7}$$

in the case $1<\tau_j\le 2$, $j=1,\dots,m$, and

$$d_M\left(\mathring{S}^{\bar r}_{\bar p,\bar\theta^{(1)},\bar\tau}B, L^*_{q,\bar\theta^{(2)}}\right) \le C\left\| \left\{ \prod_{j=1}^m 2^{-s_j\left(r_j+\frac{1}{q}-\frac{1}{p_j}\right)} d_{M_{\bar s}}\left(B_{p_0}^{2^{\langle \bar s,\bar 1\rangle}}, l_q^{2^{\langle \bar s,\bar 1\rangle}}\right)\right\}_{\langle \bar s,\gamma'\rangle>n}\right\|_{\bar\varepsilon} \tag{2.8}$$

if $2<\tau_j$, $j=1,\dots,m$, where $\bar\varepsilon=(\varepsilon_1,\dots,\varepsilon_m)$ and $\varepsilon_j=2\tau_j/(\tau_j-2)$ for $j=1,\dots,m$.

Substituting into formulas (2.7) and (2.8) the value $d_{M_{\bar s}}\big(B_{p_0}^{2^{\langle \bar s,\bar 1\rangle}}, l_q^{2^{\langle \bar s,\bar 1\rangle}}\big)=0$ for $\langle \bar s,\bar\gamma'\rangle\le n$ and the bounds for finite-dimensional widths [43, 44]

$$d_{n_{\bar s}}(B^m_{p_0}, l^m_q) \asymp \min\left\{1, m^{\frac{2\beta}{q}} n^{-\beta}\right\}, \quad m>n,$$

for $\langle \bar s,\bar\gamma'\rangle>n$, we obtain

$$d_M\left(\mathring{S}^{\bar r}_{\bar p,\bar\theta^{(1)},\bar\tau}B, L^*_{q,\bar\theta^{(2)}}\right) \le C \sup_{\bar s\in\mathbb{Z}^m_+} \prod_{j=1}^m 2^{-s_j\left(r_j+\frac{1}{q}-\frac{1}{p_j}\right)} 2^{\frac{2\beta}{q}\langle \bar s,\bar 1\rangle} M_{\bar s}^{-\beta} \tag{2.9}$$

in the case $1\le\tau_j\le 2$, $j=1,\dots,m$, and

$$d_M\left(\mathring{S}^{\bar r}_{\bar p,\bar\theta^{(1)},\bar\tau}B, L^*_{q,\bar\theta^{(2)}}\right) \le C\left\| \left\{ \prod_{j=1}^m 2^{-s_j\left(r_j+\frac{1}{q}-\frac{1}{p_j}\right)} 2^{\frac{2\beta}{q}\langle \bar s,\bar 1\rangle} M_{\bar s}^{-\beta}\right\}_{\langle \bar s,\gamma'\rangle>n}\right\|_{\bar\varepsilon} \tag{2.10}$$

in the case $2<\tau_j\le\infty$, $j=1,\dots,m$.

Let $2<\tau_j\le\infty$ for $j=1,\dots,m$. Then, substituting the values of $M_{\bar s}$ in (2.10), we obtain

$$d_M\left(\mathring{S}^{\bar r}_{\bar p,\bar\theta^{(1)},\bar\tau}B, L^*_{q,\bar\theta^{(2)}}\right) \le C2^{-n\beta(1+\varepsilon\rho_1)}\left\| \left\{ \prod_{j=1}^m 2^{-s_j\left(r_j+\frac{1}{q}-\frac{1}{p_j}\right)} 2^{\frac{2\beta}{q}\langle \bar s,\bar 1\rangle} 2^{\varepsilon\beta\langle \bar s,\bar\rho\rangle}\right\}_{\langle \bar s,\gamma'\rangle>n}\right\|_{\bar\varepsilon}. \tag{2.11}$$

Since $\rho_j=r_j+1/q-1/p_j$ and $\gamma_j=\rho_j/\rho_1$ for $j=1,\dots,m$, it follows from the inequalities $1\le\gamma_j$ and $\beta>0$ that

$$\prod_{j=1}^m 2^{-s_j\left(r_j+\frac{1}{q}-\frac{1}{p_j}\right)} 2^{\frac{2\beta}{q}\langle \bar s,\bar 1\rangle} 2^{\varepsilon\beta\langle \bar s,\bar\rho\rangle} = 2^{-\rho_1\langle \bar s,\bar\gamma\rangle} 2^{\frac{2\beta}{q}\langle \bar s,\bar 1\rangle} 2^{\varepsilon\beta\rho_1\langle \bar s,\bar\gamma\rangle} \le 2^{-\left(\rho_1-\frac{2\beta}{q}-\varepsilon\beta\rho_1\right)\langle \bar s,\bar\gamma\rangle}. \tag{2.12}$$

In view of (2.12), we derive from (2.11) the bound

$$d_M\left(\mathring{S}^{\bar r}_{\bar p,\bar\theta^{(1)},\bar\tau}B, L^*_{q,\bar\theta^{(2)}}\right) \le C2^{-n\beta(1+\varepsilon\rho_1)}\left\| \left\{ 2^{-\left(\rho_1-\frac{2\beta}{q}-\varepsilon\beta\rho_1\right)\langle \bar s,\bar\gamma\rangle}\right\}_{\langle \bar s,\gamma'\rangle>n}\right\|_{\bar\varepsilon} \tag{2.13}$$

in the case $2<\tau_j\le\infty$, $j=1,\dots,m$.

Since $\beta = \dfrac{1/p_1 - 1/q}{1-2/q}$, it is easy to see that $\rho_1 - 2\beta/q = r_1 - \beta$. Therefore, inequality (2.13), can be written in the form

$$d_M\Big(\overset{\circ}{S}{}^{\overline{r}}_{\overline{p},\overline{\theta}^{(1)},\overline{\tau}}B, L^*_{q,\bar{\theta}^{(2)}}\Big) \le C2^{-n\beta(1+\varepsilon\rho_1)}\Big\|\Big\{2^{-(r_1-\beta-\varepsilon\beta\rho_1)\langle\bar{s},\bar{\gamma}\rangle}\Big\}_{\langle\bar{s},\gamma'\rangle>n}\Big\|_{\bar{\varepsilon}} \tag{2.14}$$

in the case $2 < \tau_j \le \infty$, $j = 1, \dots, m$.

By the condition of the theorem, $r_1 > \beta$ and $2 < q < \infty$. Then, $r_1 > 1/p_1 - 1/q$, and

$$\frac{r_1 - \beta}{\beta\Big(r_1 + \dfrac{1}{q} - \dfrac{1}{p_1}\Big)} > 0.$$

Choose a number ε such that

$$0 < \varepsilon < \frac{r_1 - \beta}{\beta\Big(r_1 + \dfrac{1}{q} - \dfrac{1}{p_1}\Big)}.$$

Then

$$r_1 - \beta - \varepsilon\beta\Big(r_1 + \frac{1}{q} - \frac{1}{p_1}\Big) = r_1 - \beta - \varepsilon\beta\rho_1 > 0.$$

Therefore, applying Lemma 2, we obtain

$$\Big\|\Big\{2^{-\big(r_1-\beta-\varepsilon\beta\rho_1\big)\langle\bar{s},\bar{\gamma}\rangle}\Big\}_{\langle\bar{s},\gamma'\rangle>n}\Big\|_{\bar{\varepsilon}} \le C2^{-n(r_1-\beta-\varepsilon\beta\rho_1)}n^{\sum_{j=2}^{\nu}\frac{1}{\varepsilon_j}}. \tag{2.15}$$

Hence, in view of the relations $1/\varepsilon_j = (\tau_j - 2)/2\tau_j$ for $j = 1, \dots, m$ and $2^n n^{\nu-1} \asymp M$, we find from inequalities (2.14) and (2.15) that

$$d_M\Big(\overset{\circ}{S}{}^{\overline{r}}_{\overline{p},\overline{\theta}^{(1)},\overline{\tau}}B, L^*_{q,\bar{\theta}^{(2)}}\Big) \le C2^{-n\beta(1+\varepsilon\rho_1)}2^{-n(r_1-\beta-\varepsilon\beta\rho_1)}n^{\sum_{j=2}^{\nu}\left(\frac{1}{2}-\frac{1}{\tau_j}\right)}$$

$$\le C2^{-nr_1}n^{\sum_{j=2}^{\nu}\left(\frac{1}{2}-\frac{1}{\tau_j}\right)} \le C\Big(\frac{\log^{\nu-1}M}{M}\Big)^{r_1}(\log M)^{\sum_{j=2}^{\nu}\left(\frac{1}{2}-\frac{1}{\tau_j}\right)}$$

in the case when $2 < \tau_j \le \infty$ and $2 \le p_j < q < \infty$ for $j = 1, \dots, m$ and $r_1 > \beta$.

Let $1 \le \tau_j \le 2$ for $j = 1, \dots, m$. Then, substituting the value $M_{\bar{s}}$ into (2.9), we get

$$d_M\Big(\overset{\circ}{S}{}^{\overline{r}}_{\overline{p},\overline{\theta}^{(1)},\overline{\tau}}B, L^*_{q,\bar{\theta}^{(2)}}\Big) \le C2^{-n\beta(1+\varepsilon\rho_1)} \sup_{\langle\bar{s},\bar{\gamma}'\rangle>n}\prod_{j=1}^{m}2^{-s_j\left(r_j+\frac{1}{q}-\frac{1}{p_j}\right)}2^{\frac{2\beta}{q}\langle\bar{s},\bar{1}\rangle}2^{\varepsilon\beta\langle\bar{s},\bar{\rho}\rangle}$$

$$\le C2^{-n\beta(1+\varepsilon\rho_1)}\sup_{\langle\bar{s},\bar{\gamma}'\rangle>n}2^{-\left(\rho_1-\frac{2\beta}{q}-\varepsilon\beta\rho_1\right)\langle\bar{s},\bar{\gamma}\rangle} = C2^{-n\beta(1+\varepsilon\rho_1)}\sup_{\langle\bar{s},\bar{\gamma}'\rangle>n}2^{-(r_1-\beta-\varepsilon\beta\rho_1)\langle\bar{s},\bar{\gamma}\rangle}$$

$$\le C2^{-nr_1} \le C\Big(\frac{\log^{\nu-1}M}{M}\Big)^{r_1}.$$

Thus,

$$d_M\Big(\overset{\circ}{S}{}^{\overline{r}}_{\overline{p},\overline{\theta}^{(1)},\overline{\tau}}B, L^*_{q,\bar{\theta}^{(2)}}\Big) \le C\Big(\frac{\log^{\nu-1}M}{M}\Big)^{r_1}$$

in the case when $1 \le \tau_j \le 2$ and $2 \le p_j < q < \infty$ for $j = 1, \dots, m$.

Let us prove the second statement of Theorem 2. Let $1 < p_j \le 2 < q < \infty$ for $j = 1, \dots m$ and $r_1 > 1/p_1$. Since $1 < p_j \le 2$ for $j = 1, \dots m$, we have $\overset{\circ\,\overline{r}}{S}_{\overline{p},\bar\theta^{(1)},\overline{\tau}}B \subset \overset{\circ\,\overline{\rho}}{S}_{2,\bar\theta^{(1)},\overline{\tau}}B$, where $\bar\rho = (\rho_1, \dots, \rho_m)$ with $\rho_j = r_j + 1/2 - 1/p_j$ for $j = 1, \dots, m$. Since $r_1 > 1/p_1$ by the condition of the theorem, we have $\rho_1 > \beta = (1/2 - 1/q)/(1 - 2/q)$.

Using statement 1 for $p_j = 2$, $j = 1, \dots, m$, and replacing r_j by ρ_j, we get

$$d_M\Big(\overset{\circ\,\overline{r}}{S}_{\overline{p},\bar\theta^{(1)},\overline{\tau}}B, L^*_{q,\bar\theta^{(2)}}\Big) \le d_M\Big(\overset{\circ\,\overline{r}}{S}_{2,\bar\theta^{(1)},\overline{\tau}}B, L^*_{q,\bar\theta^{(2)}}\Big) \le C\Big(\frac{\log^{\nu-1} M}{M}\Big)^{r_1+\frac{1}{2}-\frac{1}{p_1}}(\log M)^{\sum_{j=2}^{\nu}\left(\frac{1}{2}-\frac{1}{\tau_j}\right)}$$

in the case $1 \le \tau_j \le \infty$, $j = 1, \dots, m$. Theorem 2 is proved. □

Theorem 3. *Suppose that $1 < \theta_j^{(1)}, \theta_j^{(2)} < \infty$ for $j = 1, \dots m$, $1 \le \tau \le \infty$, $0 < r_1 = \dots = r_\nu < r_{\nu+1} \le \dots \le r_m$, and $r_j > 1/p_j$ for $j = 1, \dots, m$. Suppose also that $2 \le q_j < p_j < \infty$ for $j = 1, \dots, m$. Then*

$$d_M\Big(\overset{\circ\,\overline{r}}{S}_{\overline{p},\bar\theta^{(1)},\tau}B, L^*_{\bar q,\bar\theta^{(2)}}\Big) \ge C\Big(\frac{\log^{\nu-1} M}{M}\Big)^{r_1}(\log M)^{(\nu-1)\left(\frac{1}{2}-\frac{1}{\tau}\right)_+}.$$

Proof. Choose a number $p_0 > \max_{j=1,\dots,m} p_j$. Then $L_{p_0}(I^m) \subset L^*_{\bar p,\bar\theta^{(1)}}(I^m)$.

Using the properties of the norm, Nikol'skii's inequality for different metrics of trigonometric polynomials (see [41, 42]), and Hölder's inequality $(1/\tau + 1/\tau' = 1)$, we find that

$$\|f\|_{p_0} \le \sum_{\bar s\in\mathbb{Z}_+^m}\prod_{j=1}^{m} 2^{s_j(\frac{1}{p_j}-\frac{1}{p_0})}\|\delta_{\bar s}(f)\|^*_{\bar p,\bar\theta^{(1)}}$$

$$\le \Big\{\sum_{\bar s\in\mathbb{Z}_+^m} 2^{\langle\bar s,\bar r\rangle\tau}\big(\|\delta_{\bar s}(f)\|^*_{\bar p,\bar\theta^{(1)}}\big)^{\tau}\Big\}^{1/\tau}\Big\{\sum_{\bar s\in\mathbb{Z}_+^m}\prod_{j=1}^{m} 2^{-s_j(r_j-\frac{1}{p_j}+\frac{1}{p_0})\tau'}\Big\}^{1/\tau'}.$$

Since $r_j > 1/p_j > 1/p_j - 1/p_0$ for $j = 1, \dots, m$, we obtain $\overset{\circ\,\overline{r}}{S}_{\overline{p},\bar\theta^{(1)},\tau}B \subset L_{p_0}(I^m)$.

On the other hand, since $p_0 > \max_{j=1,\dots,m} p_j$, we have $\|\delta_{\bar s}(f)\|^*_{\bar p,\bar\theta^{(1)}} \le C\|\delta_{\bar s}(f)\|_{p_0}$. Therefore, $\overset{\circ\,\overline{r}}{S}_{p_0,\tau}B \subset \overset{\circ\,\overline{r}}{S}_{\overline{p},\bar\theta^{(1)},\tau}B$. Consequently, $d_M\Big(\overset{\circ\,\overline{r}}{S}_{\overline{p},\bar\theta^{(1)},\tau}B, L^*_{\bar q,\bar\theta^{(2)}}\Big) \ge d_M\Big(\overset{\circ\,\overline{r}}{S}_{p_0,\tau}B, L^*_{\bar q,\bar\theta^{(2)}}\Big)$. Since $q_j \ge 2$ for $j = 1, \dots, m$ by the condition of the theorem, we obtain

$$d_M\Big(\overset{\circ\,\overline{r}}{S}_{\overline{p},\bar\theta^{(1)},\tau}B, L^*_{\bar q,\bar\theta^{(2)}}\Big) \ge d_M\Big(\overset{\circ\,\overline{r}}{S}_{p_0,\tau}B, L^*_{\bar q,\bar\theta^{(2)}}\Big) \ge d_M\Big(\overset{\circ\,\overline{r}}{S}_{p_0,\tau}B, L_2\Big). \tag{2.16}$$

Galeev [27] proved the bound

$$d_M\big(\overset{\circ\,\overline{r}}{S}_{p_0,\tau}B, L_2\big) \ge C\Big(\frac{\log^{\nu-1} M}{M}\Big)^{r_1}(\log M)^{(\nu-1)\left(\frac{1}{2}-\frac{1}{\tau}\right)_+}. \tag{2.17}$$

It follows from (2.16) and (2.17) that

$$d_M\Big(\overset{\circ\,\overline{r}}{S}_{\overline{p},\bar\theta^{(1)},\tau}B, L^*_{\bar q,\bar\theta^{(2)}}\Big) \ge C\Big(\frac{\log^{\nu-1} M}{M}\Big)^{r_1}(\log M)^{(\nu-1)\left(\frac{1}{2}-\frac{1}{\tau}\right)_+}.$$

Theorem 3 is proved. □

Theorem 4. *Suppose that $1 < \theta_j^{(1)}, \theta_j^{(2)} < \infty$ for $j = 1, \dots m$, $1 \le \tau \le \infty$, $0 < r_1 = \dots = r_\nu < r_{\nu+1} \le \dots \le r_m$, and $r_j > 1/p_j$ for $j = 1, \dots, m$. Then the following statements hold.*

1. *If* $2 \le p_j < q_j < \infty$ *and* $r_j > 1/p_j$ *for* $j = 1, \dots, m$, *then*

$$d_M\Big(\overset{\circ}{S}{}^{\overline{r}}_{\overline{p},\bar{\theta}^{(1)},\tau}B, L^*_{\bar{q},\bar{\theta}^{(2)}}\Big) \ge C\Big(\frac{\log^{\nu-1} M}{M}\Big)^{r_1}(\log M)^{(\nu-1)\left(\frac{1}{2}-\frac{1}{\tau}\right)_+}.$$

2. *If* $1 < p_j \le 2 < q_j < \infty$ *and* $r_j > 1/p_j$ *for* $j = 1, \dots, m$ *and* $p_1 > p_j$ *for* $j = 2, \dots, m$, *then*

$$d_M\Big(\overset{\circ}{S}{}^{\overline{r}}_{\overline{p},\bar{\theta}^{(1)},\tau}B, L^*_{\bar{q},\bar{\theta}^{(2)}}\Big) \ge C\Big(\frac{\log^{\nu-1} M}{M}\Big)^{r_1+\frac{1}{2}-\frac{1}{p_1}}(\log M)^{(\nu-1)\left(\frac{1}{2}-\frac{1}{\tau}\right)}.$$

Proof. Let $2 \le p_j < q_j < \infty$ for $j = 1, \dots m$. Since $L^*_{\bar{q},\bar{\theta}^{(2)}}(I^m) \subset L_2(I^m)$, we find by Theorem 3 (for $q_j = \theta_j^{(2)} = 2$, $j = 1, \dots, m$) that

$$d_M\Big(\overset{\circ}{S}{}^{\overline{r}}_{\overline{p},\bar{\theta}^{(1)},\tau}B, L^*_{\bar{q},\bar{\theta}^{(2)}}\Big) \ge d_M\Big(\overset{\circ}{S}{}^{\overline{r}}_{\overline{p},\bar{\theta}^{(1)},\tau}B, L_2\Big) \ge C\Big(\frac{\log^{\nu-1} M}{M}\Big)^{r_1}(\log M)^{(\nu-1)\left(\frac{1}{2}-\frac{1}{\tau}\right)_+}.$$

Let us prove the second statement. Since $2 < q_j < \infty$ for $j = 1, \dots m$, we have $L^*_{\bar{q},\bar{\theta}^{(2)}}(I^m) \subset L_2(I^m)$. Therefore,

$$d_M\Big(\overset{\circ}{S}{}^{\overline{r}}_{\overline{p},\bar{\theta}^{(1)},\tau}B, L^*_{\bar{q},\bar{\theta}^{(2)}}\Big) \ge d_M\Big(\overset{\circ}{S}{}^{\overline{r}}_{\overline{p},\bar{\theta}^{(1)},\tau}B, L_2\Big). \tag{2.18}$$

It follows from the condition $p_1 > p_j$, $j = 2, \dots, m$, that $S^{\overline{r}}_{p_1,\tau}B \subset S^{\overline{r}}_{\overline{p},\bar{\theta}^{(1)},\tau}B$. Then

$$d_M\Big(\overset{\circ}{S}{}^{\overline{r}}_{\overline{p},\bar{\theta}^{(1)},\tau}B, L_2\Big) \ge d_M\big(S^{\overline{r}}_{p_1,\tau}B, L_2\big). \tag{2.19}$$

It follows from inequalities (2.18) and (2.19) that $d_M\Big(\overset{\circ}{S}{}^{\overline{r}}_{\overline{p},\bar{\theta}^{(1)},\tau}B, L^*_{\bar{q},\bar{\theta}^{(2)}}\Big) \ge d_M\big(S^{\overline{r}}_{p_1,\tau}B, L_2\big)$. Therefore, using statement (d) of the theorem from [29, p. 608], we get

$$d_M\Big(\overset{\circ}{S}{}^{\overline{r}}_{\overline{p},\bar{\theta}^{(1)},\tau}B, L^*_{\bar{q},\bar{\theta}^{(2)}}\Big) \ge C\Big(\frac{\log^{\nu-1} M}{M}\Big)^{r_1+\frac{1}{2}-\frac{1}{p_1}}(\log M)^{\sum_{j=2}^{\nu}\left(\frac{1}{2}-\frac{1}{\tau}\right)}.$$

Theorem 4 is proved. □

Remark. In the case $\tau_1 = \dots \tau_m = \tau$ and $q_1 = \dots = q_m = q \le \theta_j^{(2)}$ for $j = 1, \dots, m$, it follows from statements 2 of Theorems 2 and 4 that the following bound holds for $p_1 > p_j$, $j = 1, \dots, m$:

$$d_M\Big(\overset{\circ}{S}{}^{\overline{r}}_{\overline{p},\bar{\theta}^{(1)},\tau}B, L^*_{q,\bar{\theta}^{(2)}}\Big) \asymp \Big(\frac{\log^{\nu-1} M}{M}\Big)^{r_1+\frac{1}{2}-\frac{1}{p_1}}(\log M)^{(\nu-1)\left(\frac{1}{2}-\frac{1}{\tau}\right)_+}.$$

It follows from statements 2 of Theorems 2 and 4 that

$$d_M\Big(\overset{\circ}{S}{}^{\overline{r}}_{\overline{p},\bar{\theta}^{(1)},\tau}B, L^*_{q,\bar{\theta}^{(2)}}\Big) \asymp \Big(\frac{\log^{\nu-1} M}{M}\Big)^{r_1}(\log M)^{(\nu-1)\left(\frac{1}{2}-\frac{1}{\tau}\right)_+}$$

for $\tau_1 = \dots = \tau_m = \tau$ and $q_1 = \dots = q_m = q \le \theta_j^{(2)}$, $j = 1, \dots, m$.

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REFERENCES

1. S. G. Krein, Yu. I. Petunin, and E. M. Semenov, *Interpolation of Linear Operators* (Nauka, Moscow, 1978; Amer. Math. Soc., Providence, RI, 1982).
2. V. I. Kolyada, "Embeddings of fractional Sobolev spaces and estimates of Fourier transforms," Sb. Math. **192** (7), 979–1000 (2001).
3. A. P. Blozinski, "Multivariate rearrangements and Banach function spaces with mixed norms," Trans. Amer. Math. Soc. **263** (1), 149–167 (1981).
4. S. M. Nikol'skii, *Approximation of Functions of Several Variables and Imbedding Theorems* (Nauka, Moscow, 1977) [in Russian].
5. S. M. Nikol'skii, "Functions with a dominant mixed derivative satisfying the multiple Hölder condition," Sib. Mat. Zh. **4** (6), 1342–1364 (1963).
6. T. I. Amanov, "Representation and embedding theorems for function spaces SB," Trudy Mat. Inst. Steklova **77**, 5–34 (1965).
7. T. I. Amanov, *Spaces of Differentiable Functions with Dominating Mixed Derivative* (Nauka, Alma-Ata, 1976) [in Russian].
8. P. I. Lizorkin and S. M. Nikol'skii, "Function spaces of mixed smoothness from the decomposition point of view," Trudy Mat. Inst. Steklova **187**, 143–161 (1989).
9. A. Kolmogoroff, "Über die beste Annaherung von Funktionen einer gegeben Funktionenklasse," Ann. Math. **37**, 107–111 (1936).
10. W. Rudin, "L_2-approximation by partial sums of orthogonal developments," Duke Math. J. **19**, 1–4 (1952).
11. S. B. Stechkin, "On the best approximations of given classes of functions by arbitrary polynomials," Uspekhi Mat. Nauk **9** (1), 133–134 (1954).
12. V. M. Tikhomirov, "Diameters of sets in functional spaces and the theory of best approximations," Russ. Math. Surv. **15** (3), 75–111 (1960).
13. S. B. Babadzhanov and V. M. Tikhomirov, "On widths of one class in the space L^p," Izv. AN Uzb. SSR, Ser. Fiz.-Mat., No. 2, 24–30 (1967).
14. Yu. N. Subbotin, "Diameter of class W^rL in $L(0,2\pi)$ and spline function approximation," Math. Notes **7** (1), 27–32 (1970).
15. Yu. N. Subbotin, "Approximation by spline functions and estimates of widths," Trudy Mat. Inst. Steklova **109**, 35–60 (1971).
16. R. S. Ismagilov, "On n–dimensional widths of compacta in a Hilbert space," Funkts. Anal. Prilozh. **2** (2), 32–39 (1968).
17. R. S. Ismagilov, "Diameters of sets in normed linear spaces and the approximation of functions by trigonometric polynomials," Russ. Math. Surv. **29** (3), 169–181 (1974).
18. E. D. Gluskin, "On a problem concerning widths," Dokl. AN SSSR **219** (3), 527–530 (1974).
19. B. S. Kashin, "Widths of some finite-dimensional sets and classes of smooth functions," Izv. AN SSSR, Ser. Mat. **41** (2), 334–351 (1977).
20. Y. Makovoz, "On trigonometric n-widths and their generalization," J. Approx. Theory **41** (4), 361–366 (1984).
21. B. S. Kashin, "Widths of Sobolev classes of small-order smoothness," Vestn. Mosk. Gos. Univ., Ser. Mat. Mekh., No. 5, 50–54 (1981).
22. E. D. Kulanin, "Estimates for diameters of Sobolev classes of small-order smoothness," Vestn. Mosk. Gos. Univ., Ser. Mat. Mekh., No. 2, 24–30 (1983).
23. K. I. Babenko, "Approximation by trigonometric polynomials in a certain class of periodic functions of several variables," Soviet Math. Dokl. **1**, 672–675 (1960).

24. B. S. Mityagin, "Approximation of functions in the spaces L^p and C on a torus," Mat. Sb. **58** (4), 397–414 (1962).
25. S. A. Telyakovskii, "Some bounds for trigonometric series with quasi-convex coefficients," Mat. Sb. **63** (3), 426–444 (1964).
26. Ya. S. Bugrov, "Approximation of classes of functions with dominating mixed derivative," Mat. Sb. **64** (3), 410–418 (1964).
27. E. M. Galeev, "Kolmogorov widths in the space $\tilde{L}_q$ of the classes $\tilde{W}_p^{\overline{\alpha}}$ and $\tilde{H}_p^{\overline{\alpha}}$ of periodic functions of several variables," Math. USSR Izv. **27** (2), 219–237 (1986).
28. E. M. Galeev, "Kolmogorov widths of classes of periodic functions of one and several variables," Math. USSR Izv. **36** (2), 435–448 (1991).
29. E. M. Galeev, "Widths of the Besov classes $B_{p,\theta}^r(\mathbb{T}^d)$," Math. Notes **69** (5–6), 605–613 (2001).
30. V. N. Temlyakov, "Approximations of functions with bounded mixed derivative," Proc. Steklov Inst. Math. **178**, 1–121 (1989).
31. V. N. Temlyakov, "Estimates for the asymptotic characteristics of classes of functions with bounded mixed derivative or difference," Trudy Mat. Inst. Steklova **189**, 161–197 (1990).
32. Din' Zung, "Approximation by trigonometric polynomials of functions of several variables on the torus," Math. USSR Sb. **59** (1), 247–267 (1986).
33. E. S. Belinskii, "Approximation by a 'floating' system of exponents on classes of periodic functions with bounded mixed derivative," in *Studies on the Theory of Functions of Several Real Variables* (Yaroslavsk. Gos. Univ., Yaroslavl, 1988), pp. 16–33 [in Russian].
34. E. S. Belinskii, "Estimates for the Kolmogorov diameters of classes of functions with conditions on the mixed difference in the uniform metric," Math. Notes **50** (5), 1198–2000 (1991).
35. A. S. Romanyuk, "The best trigonometric approximations and the Kolmogorov diameters of the Besov classes of functions of many variables," Ukr. Math. J. **45** (5), 724–738 (1993).
36. A. S. Romanyuk, "Kolmogorov widths of the Besov classes $B_{p,\theta}^r$ in the metric of the space L_∞," Ukr. Math. Bull. **2** (2), 205–222 (2005).
37. A. S. Romanyuk, "Kolmogorov and trigonometric widths of the Besov classes $B_{p,\theta}^r$ of multivariate periodic functions," Sb. Math. **197** (1), 69–94 (2006).
38. Heping Wang and Yongsheng Sun, "Kolmogorov widths between the anisotropic space and the space of functions with mixed smoothness," J. Approx. Theory **126** (1), 52–59 (2004).
39. D. B. Bazarkhanov, "Estimates for certain approximation characteristics of Nikol'skii–Besov spaces with generalized mixed smoothness," Dokl. Math. **79** (3), 305–308 (2009).
40. G. Akishev, "On approximation of function classes in Lorentz spaces with anisotropic norm," Anal. Theory Appl. **29** (4), 358–372 (2013).
41. G. Akishev, "Inequalities of different metrics for trigonometric polynomials," in *Ualikhanov Readings: Conference Proceedings, Kokshetau, Kazakhstan, 2004*, pp. 3–6.
42. E. D. Nursultanov, "Nikol'skii's inequality for different metrics and properties of the sequence of norms of the Fourier sums of a function in the Lorentz space," Proc. Steklov Inst. Math. **255** (1), 185–202 (2006).
43. B. S. Kashin, "Some properties of matrices of bounded operators from l_2^n to l_2^m," Izv. Akad. Nauk Armyan. SSR, Ser. Mat. **15** (5), 379–394 (1980).
44. E. D. Gluskin, "Norms of random matrices and widths of finite-dimensional sets," Math. USSR Sb. **48** (1), 173–182 (1984).

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