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Numerical solution of the boundary value problems for the parabolic equation with involution

In this work, we study two boundary value problems for involutory parabolic equation with the first and second kind conditions. We propose absolute stable difference schemes for numerical solutions of these boundary value problems. Actually the stability estimates for solutions of difference schemes are proved. Later error analysis for the numerical solution of both difference schemes are illustrated by test examples.

Keywords: involution, parabolic, finite difference scheme, stability estimate, boundary value problem.

Introduction

It is well known that various models in physics can be reduced to a parabolic equation with delay and involution. Time delay and involutory parabolic equations with local and non local boundary conditions have been investigated by several researchers [1–17].

1 Finite differences for involutory parabolic equation with Dirichlet condition

We consider boundary value problem for parabolic equation with involution and Dirichlet condition as follows

$$\left\{ \begin{array}{l} u_t(t, x) - (a(x)u_x(t, x))_x + \delta u(t, x) + q(- (a(x)u_t(-t, x))_x + \delta u(-t, x)) = f(t, x), \\ t \in I, x \in (0, l), \\ u(0, x) = \varphi(x), x \in [0, l], \\ u(t, 0) = 0, u(t, l) = 0, t \in I. \end{array} \right. \quad (1)$$

Here and in future a, φ and f are given smooth functions and δ and q are known numbers such that $a(x) \geq a_0 > 0, \forall x \in (0, l), \delta, |q| < 1, I = (-\infty, \infty)$.

1.1 Stability of differential problem

Denote by $W_2^2(0, l)$, the Sobolev space of all functions $v(x)$ defined on $[0, l]$ equipped with norm

$$\|v\|_{W_2^2(0, l)} = \left(\int_0^l |v(x)|^2 dx \right)^{\frac{1}{2}} + \left(\int_0^l |v''(x)|^2 dx \right)^{\frac{1}{2}}.$$

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Theorem 1. Let $\varphi \in W_2^2(0, l)$ and $f(t, x)$ be continuously differentiable on $I \times [0, l]$. Then, for the solution of initial boundary value problem (1) the stability estimates

$$\begin{aligned} \sup_{t \in I} \|u(t, \cdot)\|_{L_2(0, l)} &\leq M(\delta) \left[\|\varphi\|_{L_2(0, l)} + \int_{-\infty}^{\infty} \|f(s, \cdot)\|_{L_2(0, l)} ds \right], \\ \sup_{t \in I} \|u_t(t, \cdot)\|_{L_2(0, l)} + \sup_{t \in I} \|u_t(t, \cdot)\|_{W_2^2(0, l)} &\leq \\ &\leq M(\delta) \left[\|\varphi\|_{W_2^2(0, l)} + \|f(0, \cdot)\|_{L_2(0, l)} + \int_{-\infty}^{\infty} \|f'(s, \cdot)\|_{L_2(0, l)} ds \right] \end{aligned} \quad (2)$$

hold, where $M(\delta)$ does not depend on both functions φ and f .

Proof. One can write problem (1) in the abstract initial value problem

$$\begin{cases} u_t(t) + Au(t) + q Au(-t) = f(t), & t \in I, \\ u(0) = \varphi. \end{cases} \quad (3)$$

Here $A = A^x$ is a self adjoint positive definite operator in $H = L_2(0, l)$ which is defined by formula

$$Au(x) = -(a(x)u_x(x))_x + \delta u(x) \quad (4)$$

with the domain $D(A) = \{u \in W_2^2(0, l) \mid u(0) = 0, u(l) = 0\}$, $\varphi = \varphi(x)$ is given element of H and $f(t) = f(t, x)$ is a given abstract function. The proof of Theorem 1 is based on the stability of abstract problem (3) and positiveness and self-adjointness of the abstract operator A defined by (4).

1.2 Stability of difference problems

Let $[0, l]_h = \{x_n = nh, 0 \leq n \leq M\}$ be grid space. Denote by $L_{2h} = L_2[0, l]_h$, Hilbert space of grid functions $\rho^h(x) = \{\rho^n\}_0^M$ defined on $[0, l]_h$ equipped with norm

$$\|\rho^h\|_{L_{2h}} = \left(\sum_{x \in [0, l]_h} |\rho^h(x)|^2 \right)^{\frac{1}{2}}.$$

To the operator (4) we assign the difference operator by formula

$$A_h^x \rho^h(x) = - \left(a(x) \rho_x^h(x) \right)_x + \delta \rho^h(x)$$

acting in the space of grid functions $\rho^h(x) = \{\rho^n(x)\}_0^M$ and satisfying the conditions $\rho^0 = 0$, $\rho^M = 0$, where

$$\rho_x^i = \frac{\rho^i - \rho^{i-1}}{h}, \quad 1 \leq i \leq M, \quad \rho_x^k = \frac{\rho^{k+1} - \rho^k}{h}, \quad 0 \leq k \leq M - 1.$$

In the first step of discretization we get the following Dirichlet problem

$$\begin{cases} u_t^h(t, x) + A_h^x u^h(t, x) + q A_h^x u^h(-t, x) = f^h(t, x), & t \in I, \\ u^h(0, x) = \varphi^h(x), & x \in [0, l]_h. \end{cases}$$

In the second step of discretization one can construct the first order of accuracy difference scheme

$$\begin{cases} \frac{u_k^h(x)-u_{k-1}^h(x)}{\tau} + A_h^x u_k^h(x) + qA_h^x u_{-k}^h(x) = f_k^h(x), & f_k^h(x) = f^h(t_k, x), \\ t_k = k\tau, & k \in Z, x \in [0, l]_h, \\ u_0^h(x) = \varphi^h(x), & x \in [0, l]_h \end{cases} \quad (5)$$

and the second order of accuracy difference scheme

$$\begin{cases} \frac{u_k^h(x)-u_{k-1}^h(x)}{\tau} + \frac{1}{2} (A_h^x u_k^h(x) + qA_h^x u_{-k}^h(x)) + \frac{1}{2} (A_h^x u_{k-1}^h(x) + qA_h^x u_{-k+1}^h(x)) = \\ = f_{k+\frac{1}{2}}^h(x) = f^h(t_{k+\frac{1}{2}}, x), & t_{k+\frac{1}{2}} = (k + \frac{1}{2}) \tau, \\ t_k = k\tau, & k \in Z, x \in [0, l]_h, \\ u_0^h(x) = \varphi^h(x), & x \in [0, l]_h. \end{cases} \quad (6)$$

Theorem 2. Let τ and h be sufficiently small positive numbers. Then, for the solution $\{u_k^h(x)\}_{-\infty}^{\infty}$ of difference schemes (5) and (6) the stability estimates

$$\begin{aligned} \sup_{k \in Z} \|u_k^h\|_{L_{2h}} &\leq M(\delta) \left[\|\varphi^h\|_{L_{2h}} + \sum_{k=-\infty}^{\infty} \|f_k^h\|_{L_{2h}} \right], \\ \sup_{k \in Z} \left\| \frac{u_k^h - u_{k-1}^h}{\tau} \right\|_{L_{2h}} + \sup_{k \in Z} \|u_k^h\|_{W_{2h}^2} &\leq \\ &\leq M(\delta) \left[\|\varphi^h\|_{W_{2h}^2} + \|f_0^h\|_{L_{2h}} + \sum_{k=-\infty}^{\infty} \left\| \frac{f_k^h - f_{k-1}^h}{\tau} \right\|_{L_{2h}} \right] \end{aligned} \quad (7)$$

are valid, where $M(\delta)$ does not depend on τ, h, φ^h and f_k^h .

Proof. Difference schemes (5) and (6) can be rewritten as the following abstract difference schemes

$$\begin{cases} \frac{u_k^h - u_{k-1}^h}{\tau} + A_h u_k^h + qA_h u_{-k}^h = f_k^h, & k \in Z, \\ u_0^h = \varphi^h \end{cases} \quad (8)$$

and

$$\begin{cases} \frac{u_k^h - u_{k-1}^h}{\tau} + \frac{1}{2} (A_h u_k^h + qA_h u_{-k}^h) + \frac{1}{2} (A_h u_{k-1}^h + qA_h u_{-k+1}^h) = f_{k+\frac{1}{2}}^h, \\ k \in Z, u_0^h = \varphi^h, \end{cases} \quad (9)$$

correspondingly. So, the proof of Theorem 2 is based on the stability of the difference schemes (8) and (9) on the positive definiteness and self-adjointness of the operator A^h in the Hilbert space L_{2h} .

2 Finite differences for involutory parabolic equation with Neumann condition

Let us take boundary value problem for parabolic equation with involution and Neumann condition as follows

$$\left\{ \begin{array}{l} u_t(t, x) - (a(x)u_x(t, x))_x + \delta u(t, x) + q(- (a(x)u_t(-t, x))_x + \delta u(-t, x)) = f(t, x), \\ t \in I, x \in (0, l), \\ u(0, x) = \varphi(x), x \in [0, l], \\ u_x(t, 0) = 0, u_x(t, l) = 0, t \in I. \end{array} \right. \quad (10)$$

Theorem 3. Let $\varphi \in W_2^2(0, l)$ and $f(t, x)$ be continuously differentiable on $I \times [0, l]$. Then, for the solution of initial boundary value problem (10) the stability estimates (2) hold.

Proof. One can write problem (10) in the abstract initial value problem (3), where $A = A^x$ is a self adjoint positive definite operator in $H = L_2(0, l)$ which is defined by formula (4) with the domain $D(A) = \{u \in W_2^2(0, l) \mid u_x(0) = 0, u_x(l) = 0\}$. So, the proof of Theorem 1 is based on the stability of abstract problem (3) and positiveness and self-adjointness of the abstract operator A defined by (4).

To the operator (4) we assign the difference operator by formula

$$A_h^x \rho^h(x) = - \left(a(x) \rho_x^h(x) \right)_x + \delta \rho^h(x),$$

acting in the space of grid functions $\rho^h(x) = \{\rho^n(x)\}_0^M$ and satisfying the conditions $\rho^2 = 4\rho^1 - 3\rho^0$, $\rho^{M-2} = 4\rho^{M-1} - 3\rho^M$, where

$$\rho_x^i = \frac{\rho^i - \rho^{i-1}}{h}, \quad 1 \leq i \leq M, \quad \rho_x^k = \frac{\rho^{k+1} - \rho^k}{h}, \quad 0 \leq k \leq M - 1.$$

After discretization one can construct the following difference schemes

$$\left\{ \begin{array}{l} \frac{u_k^h(x) - u_{k-1}^h(x)}{\tau} + A_h^x u_k^h(x) + q A_h^x u_{-k}^h(x) = f_k^h(x), \quad f_k^h(x) = f^h(t_k, x), \\ t_k = k\tau, k \in Z, x \in [0, l]_h, \\ u_0^h(x) = \varphi^h(x), x \in [0, l]_h \end{array} \right. \quad (11)$$

and

$$\left\{ \begin{array}{l} \frac{u_k^h(x) - u_{k-1}^h(x)}{\tau} + \frac{1}{2} (A_h^x u_k^h(x) + q A_h^x u_{-k}^h(x)) + \frac{1}{2} (A_h^x u_{k-1}^h(x) + q A_h^x u_{-k+1}^h(x)) = \\ = f_{k+\frac{1}{2}}^h(x) = f^h(t_{k+\frac{1}{2}}, x), \quad t_{k+\frac{1}{2}} = (k + \frac{1}{2}) \tau, \\ t_k = k\tau, k \in Z, x \in [0, l]_h, \\ u_0^h(x) = \varphi^h(x), x \in [0, l]_h. \end{array} \right. \quad (12)$$

Theorem 4. Let τ and h be sufficiently small positive numbers. Then, for the solution $\{u_k^h(x)\}_{-\infty}^{\infty}$ of difference schemes (11) and (12) the stability estimates (7) are valid.

Proof. Difference schemes (11) and (12) can be rewritten as the abstract difference schemes (8) and (9), correspondingly. So, the proof of Theorem 2 is based on the stability of the difference schemes (8) and (9) on the positive definiteness and self-adjointness of the operator A^h in the Hilbert space L_{2h} .

3 Numerical implementation

In this section we will consider test examples with the first and second kind boundary conditions.

3.1 Test example for first kind boundary condition

Test example 1. Consider boundary value problem for parabolic equation with involution and Dirichlet condition

$$\begin{cases} u_t(t, x) - ((2 + \cos(x))u_x(t, x))_x + u(t, x) - ((2 + \cos(x))u_x(-t, x))_x + u(-t, x) = f(t, x), \\ f(t, x) = \cos t \sin x, \quad t \in (-\pi, \pi), \quad x \in (0, \pi), \\ u(0, x) = 0, \quad x \in [0, \pi], \\ u(t, 0) = 0, \quad u(t, \pi) = 0, \quad t \in [-\pi, \pi]. \end{cases} \tag{13}$$

Here and in future we define sets of grid points as follows

$$[-\pi, \pi]_\tau \times [0, \pi]_h = \{(t_k, x_i) : t_k = k\tau, \quad -N \leq k \leq N, \quad x_i = ih, \quad 0 \leq i \leq M, \quad N\tau = \pi, \quad Mh = \pi\}.$$

By using Taylor decomposition in two points

$$u(t_k) - u(t_{k-1}) = \tau u'(t_k) + o(\tau^2), \tag{14}$$

$$u(t_k) - u(t_{k-1}) = \frac{\tau}{2} u'(t_k) + \frac{\tau}{2} u'(t_{k-1}) + o(\tau^3), \tag{15}$$

$$u''(x_n) = \frac{u(x_{n+1}) - 2u(x_n) + u(x_{n-1}))}{h^2} + o(h^2), \tag{16}$$

we present the first order of accuracy difference scheme in t

$$\begin{cases} \frac{u_k^n - u_{k-1}^n}{\tau} - (2 + \cos(x_n)) \frac{u_k^{n+1} - 2u_k^n + u_k^{n-1}}{h^2} + \sin(x_n) \frac{u_k^{n+1} - u_k^{n-1}}{2h} + u_k^n - \\ - (2 + \cos(x_n)) \frac{u_{-k}^{n+1} - 2u_{-k}^n + u_{-k}^{n-1}}{h^2} + \sin(x_n) \frac{u_{-k}^{n+1} - u_{-k}^{n-1}}{2h} + u_{-k}^n = f_k^n, \quad f_k^n = f(t_k, x_n), \\ t_k = k\tau, \quad -N + 1 \leq k \leq N, \\ u_k^0 = 0, \quad u_k^M = 0, \quad k = 0, \pm 1, \pm 2, \dots, \pm N, \\ u_0^n = 0, \quad x_n = nh, \quad n = 0, 1, \dots, M \end{cases} \tag{17}$$

and the second order of accuracy difference scheme in t

$$\left\{ \begin{array}{l} \frac{u_k^n - u_{k-1}^n}{\tau} - \frac{(2 + \cos(x_n))}{2} \left(\frac{u_k^{n+1} - 2u_k^n + u_k^{n-1}}{h^2} + \frac{u_{-k}^{n+1} - 2u_{-k}^n + u_{-k}^{n-1}}{h^2} \right) + \frac{u_k^n + u_{-k}^n}{2} + \\ + \frac{\sin(x_n)}{2} \left(\frac{u_k^{n+1} - u_k^{n-1}}{2h} + \frac{u_{-k}^{n+1} - u_{-k}^{n-1}}{2h} \right) - \\ - \frac{(2 + \cos(x_n))}{2} \left(\frac{u_{k-1}^{n+1} - 2u_{k-1}^n + u_k^{n-1}}{h^2} + \frac{u_{-k+1}^{n+1} - 2u_{-k+1}^n + u_{-k+1}^{n-1}}{h^2} \right) + \\ + \frac{\sin(x_n)}{2} \left(\frac{u_{k-1}^{n+1} - u_{k-1}^{n-1}}{2h} + \frac{u_{-k+1}^{n+1} - u_{-k+1}^{n-1}}{2h} \right) + \frac{u_{k-1}^n + u_{-k+1}^n}{2} = \\ = f^h(t_{k+\frac{1}{2}}, x), \quad t_{k+\frac{1}{2}} = (k + \frac{1}{2})\tau, \quad t_k = k\tau, \quad -N + 1 \leq k \leq N, \\ u_k^0 = 0, \quad u_k^M = 0, \quad k = 0, \pm 1, \pm 2, \dots, \pm N, \\ u_0^n = 0, \quad x_n = nh, \quad n = 0, 1, \dots, M. \end{array} \right. \quad (18)$$

Later, system of equations (17) and (18) can be rewritten in the matrix form as follows

$$\left\{ \begin{array}{l} A_n U_{n-1} + B_n U_n + C_n U_{n+1} = R\varphi_n, \quad n = 1, \dots, M - 1, \\ U_0 = \vec{0}, \quad U_M = \vec{0}. \end{array} \right. \quad (19)$$

Here R is identity matrix. For solving (19) we apply modified Gauss elimination method by formula

$$U_n = \alpha_n U_{n+1} + \beta_n, \quad n = M - 1, \dots, 1, 0, \quad (20)$$

where α_0 is matrix with zero elements and vector β_0 with zero elements, matrices α_n and vectors β_n are defined recurrently by

$$\left\{ \begin{array}{l} \alpha_n = (B_n + A_n \alpha_{n-1})^{-1} A_n, \\ \beta_n = (B_n + A_n \alpha_{n-1})^{-1} (R\varphi_n - A_n \beta_{n-1}), \\ n = 1, \dots, M - 1. \end{array} \right. \quad (21)$$

Error is calculated by formula

$$Error(N, M) = \max_{k=0, \pm 1, \dots, \pm N, i=1, \dots, M} |u_k^i - u(t_k, x_i)|, \quad (22)$$

where u_k^i and $u(t_k, x_i)$ ($k = 0, \pm 1, \dots, \pm N$, $i = 1, \dots, M$) are values of solution of difference scheme and differential problem at point correspondingly. Table 1 shows that if numbers N and M increase by factor 2 then the values of errors decreases by a factor of approximately $\frac{1}{2}$ for the first order difference scheme (17) and $\frac{1}{4}$ for the second order of accuracy difference scheme (18).

Table 1

Error analysis for test example (13) with Dirichlet condition

$N = M$	1st order difference scheme	2nd order difference scheme
10	0.531	$8,35 \times 10^{-2}$
20	0.351	$2,88 \times 10^{-2}$
40	0.208	$7,36 \times 10^{-3}$
80	0.115	$1,92 \times 10^{-3}$
160	$6,09 \times 10^{-2}$	$5,01 \times 10^{-4}$
320	$3,13 \times 10^{-2}$	$1,28 \times 10^{-4}$

3.2 Test example for second kind boundary condition

Test example 2. Consider boundary problem for involutory parabolic equation with Neumann condition

$$\left\{ \begin{array}{l} u_t(t, x) - ((2 + \cos(x))u_x(t, x))_x + u(t, x) - ((2 + \cos(x))u_x(-t, x))_x + u(-t, x) = f(t, x), \\ f(t, x) = \cos t \cos x, t \in (-\pi, \pi), x \in (0, \pi), \\ u(0, x) = 0, x \in [0, \pi], \\ u_x(t, 0) = 0, u_x(t, l) = 0, t \in [-\pi, \pi]. \end{array} \right. \quad (23)$$

By using (14), (15), (16) and

$$u'(0) = \frac{-u(x_2) + 4u(x_1) - 3u(x_0)}{h^2} + o(h^2),$$

$$u'(\pi) = \frac{u(x_{M-2}) - 4u(x_{M-1}) + 3u(x_M)}{h^2} + o(h^2),$$

one can get the first order of accuracy difference scheme in t

$$\left\{ \begin{array}{l} \frac{u_k^n - u_{k-1}^n}{\tau} - (2 + \cos(x_n)) \frac{u_k^{n+1} - 2u_k^n + u_k^{n-1}}{h^2} + \sin(x_n) \frac{u_k^{n+1} - u_k^{n-1}}{2h} + u_k^n - \\ - (2 + \cos(x_n)) \frac{u_{-k}^{n+1} - 2u_{-k}^n + u_{-k}^{n-1}}{h^2} + \sin(x_n) \frac{u_{-k}^{n+1} - u_{-k}^{n-1}}{2h} + u_{-k}^n = f_k^n, \\ f_k^n = f(t_k, x_n), t_k = k\tau, -N + 1 \leq k \leq N, \\ u_k^0 = u_k^1, u_k^M = u_k^{M-1}, k = 0, \pm 1, \pm 2, \dots, \pm N, \\ u_0^n = 0, x_n = nh, n = 0, 1, \dots, M \end{array} \right. \quad (24)$$

and the second order of accuracy difference scheme in t

$$\left\{ \begin{aligned} & \frac{u_k^n - u_{k-1}^n}{\tau} - \frac{(2 + \cos(x_n))}{2} \left(\frac{u_k^{n+1} - 2u_k^n + u_k^{n-1}}{h^2} + \frac{u_{-k}^{n+1} - 2u_{-k}^n + u_{-k}^{n-1}}{h^2} \right) u_k^n + \\ & + \frac{\sin(x_n)}{2} \left(\frac{u_k^{n+1} - u_k^{n-1}}{2h} + \frac{u_{-k}^{n+1} - u_{-k}^{n-1}}{2h} \right) + \frac{u_k^n + u_{-k}^n}{2} + \\ & + \frac{(2 + \cos(x_n))}{2} \left(\frac{u_{k-1}^{n+1} - 2u_{k-1}^n + u_{k-1}^{n-1}}{h^2} + \frac{u_{-k+1}^{n+1} - 2u_{-k+1}^n + u_{-k+1}^{n-1}}{h^2} \right) + \\ & + \frac{(2 - \sin(x_n))}{2} \left(\frac{u_{k-1}^{n+1} - u_{k-1}^{n-1}}{2h} + \frac{u_{-k+1}^{n+1} - u_{-k+1}^{n-1}}{2h} \right) + \frac{u_{k-1}^n + u_{-k+1}^n}{2} = \\ & = f^h(t_{k+\frac{1}{2}}, x), \quad t_{k+\frac{1}{2}} = (k + \frac{1}{2})\tau, \quad t_k = k\tau, \quad -N + 1 \leq k \leq N, \\ & 3u_k^0 = 4u_k^1 - u_k^2, \quad 3u_k^M = 4u_k^{M-1} - u_k^{M-2}, \quad k = 0, \pm 1, \pm 2, \dots, \pm N, \\ & u_0^n = 0, \quad x_n = nh, \quad n = 0, 1, \dots, M. \end{aligned} \right. \quad (25)$$

System of equations (24) and (25) can be written in the following forms

$$\left\{ \begin{aligned} & A_n U_{n-1} + B_n U_n + C_n U_{n+1} = R\varphi_n, \quad n = 1, \dots, M-1, \\ & U_0 = U_1, \quad U_M = U_{M-1}, \end{aligned} \right. \quad (26)$$

and

$$\left\{ \begin{aligned} & A_n U_{n-1} + B_n U_n + C_n U_{n+1} = R\varphi_n, \quad n = 1, \dots, M-1, \\ & 3U_0 = 4U_1 - U_2, \quad 3U_M = 4U_{M-1} - U_{M-2}, \end{aligned} \right. \quad (27)$$

correspondingly. For solving (26) we use formula (20), where $\alpha_0 = R$ is identity matrix and vector β_0 has only zero elements, matrices α_n and vectors β_n are defined by (21). Errors are computed by formula (22). Let us move to (27). We seek solution (27) in the form (see [18, 19])

$$U_n = \alpha_n U_{n+1} + \beta_n U_{n+2} + \gamma_n, \quad n = M-2, M-1, \dots, 1, 0.$$

Here auxiliary matrices α_n, β_n and vector γ_n are calculated by formulas

$$\alpha_n = -D_n(A_n + C_n\beta_{n-1}), \quad \beta_n = O,$$

$$\gamma_n = D_n(R\varphi_n - C_n\gamma_{n-1}),$$

$$D_n = (B_n + C_n\alpha_{n-1})^{-1}, \quad n = 0, \dots, M-2,$$

$$\alpha_0 = \frac{4}{3}R, \quad \beta_0 = -\frac{1}{3}R, \quad \alpha_1 = \frac{8}{5}R, \quad \beta_1 = -\frac{3}{5}R, \quad \gamma_0 = \gamma_1 = \vec{0}.$$

At the same time formulas for unknown U_M and U_{M-1} are given in [19].

Table 2 shows that if numbers N and M increase by factor 2 then the values of errors decrease by a factor of approximately $\frac{1}{2}$ for difference scheme (24) and $\frac{1}{4}$ for difference scheme (25).

Error analysis for test example (23) with Neuman condition

$N = M$	1st order difference scheme	2nd order difference scheme
10	0.619	$8,41 \times 10^{-2}$
20	0.414	$2,91 \times 10^{-2}$
40	0.253	$7,41 \times 10^{-3}$
80	0.144	$1,95 \times 10^{-3}$
160	$7,77 \times 10^{-2}$	$5,03 \times 10^{-4}$
320	$4,05 \times 10^{-2}$	$1,30 \times 10^{-4}$

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Инволюциялы параболалық теңдеу үшін шеттік есептердің сандық шешімі

Мақалада бірінші және екінші типті шарттары бар эволюциялық параболалық теңдеудің екі шеттік есептерін зерттелген. Осы шеттік есептерді сандық түрде шешу үшін абсолютті тұрақты айырымдық схемалары ұсынылған. Айырымдық схемаларының шешімдерінің тұрақтылығын бағалау іс жүзінде дәлелденді. Екі айырымдық схемасының сандық шешімінің қателіктерін одан әрі талдау сынақ мысалдарымен келтірілген.

Кілт сөздер: инволюция, параболалық, ақырлы-айырымдық схемасы, тұрақтылықты бағалау, шеттік есеп.

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Численное решение краевых задач для параболического уравнения с инволюцией

В статье исследованы две краевые задачи для эволюционного параболического уравнения с условиями первого и второго рода. Предложены абсолютно устойчивые разностные схемы для численного решения этих краевых задач. Фактически доказаны оценки устойчивости решений разностных схем. Дальнейший анализ погрешностей численного решения обеих разностных схем проиллюстрирован тестовыми примерами.

Ключевые слова: инволюция, парабола, конечно-разностная схема, оценка устойчивости, краевая задача.